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## On a class of unitary operators on the Bergman space of the right half plane

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**Abstract:** In this paper, we introduce a class of unitary operators defined on the Bergman space  $L_a^2(\mathbb{C}_+)$  of the right half plane  $\mathbb{C}_+$  and study certain algebraic properties of these operators. Using these results, we then show that a bounded linear operator  $S$  from  $L_a^2(\mathbb{C}_+)$  into itself commutes with all the weighted composition operators  $W_a, a \in \mathbb{D}$  if and only if  $\tilde{S}(w) = \langle S b_{\bar{w}}, b_{\bar{w}} \rangle, w \in \mathbb{C}_+$  satisfies a certain averaging condition. Here for  $a = c + id \in \mathbb{D}, f \in L_a^2(\mathbb{C}_+), W_a f = (f \circ t_a) \frac{M'}{M' \circ t_a}, Ms = \frac{1-s}{1+s}, t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id},$  and  $b_{\bar{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\bar{w}} \frac{2\text{Re}w}{(s+w)^2}, w = M\bar{a}, s \in \mathbb{C}_+.$  Some applications of these results are also discussed.

**Key words:** Right half plane, Bergman space, unitary operator, automorphism, Toeplitz operators

### 1. Introduction

Let  $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : \text{Re} s > 0\}$  be the right half plane. Let  $d\tilde{A}(s) = dx dy$  be the area measure. Let  $L^2(\mathbb{C}_+, d\tilde{A})$  be the space of complex-valued, square-integrable, measurable functions on  $\mathbb{C}_+$  with respect to the area measure. Let  $L_a^2(\mathbb{C}_+)$  be the closed subspace [2] of  $L^2(\mathbb{C}_+, d\tilde{A})$  consisting of those functions in  $L^2(\mathbb{C}_+, d\tilde{A})$  that are analytic. The space  $L_a^2(\mathbb{C}_+)$  is referred to as the Bergman space of the right half plane. The functions  $H(s, w) = \frac{1}{(s+\bar{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$  are the reproducing kernel [4] for  $L_a^2(\mathbb{C}_+)$ . Let  $\mathbf{h}_w(s) = \frac{H(s, w)}{\sqrt{H(w, w)}} = \frac{2\text{Re}w}{(s+\bar{w})^2}.$  The functions  $\mathbf{h}_w, w \in \mathbb{C}_+$  are the normalized reproducing kernels for  $L_a^2(\mathbb{C}_+)$ . Let  $L^\infty(\mathbb{C}_+)$  be the space of complex-valued, essentially bounded, Lebesgue measurable functions on  $\mathbb{C}_+.$  Define for  $f \in L^\infty(\mathbb{C}_+), \|f\|_\infty = \text{ess sup}_{s \in \mathbb{C}_+} |f(s)| < \infty.$  The space  $L^\infty(\mathbb{C}_+)$  is a Banach space with respect to the essential supremum norm. For  $\phi \in L^\infty(\mathbb{C}_+),$  we define [6, 8] the Toeplitz operator  $\mathcal{T}_\phi$  from  $L_a^2(\mathbb{C}_+)$  into  $L_a^2(\mathbb{C}_+)$  by  $\mathcal{T}_\phi \{ = \mathcal{P}_+(\phi\{),$  where  $\mathcal{P}_+$  denote the orthogonal projection from  $L^2(\mathbb{C}_+, d\tilde{A})$  onto  $L_a^2(\mathbb{C}_+);$  the multiplication operator  $\mathcal{M}_\phi$  from  $L^2(\mathbb{C}_+, d\tilde{A})$  into  $L^2(\mathbb{C}_+, d\tilde{A})$  by  $(\mathcal{M}_\phi \{)(f) = \phi(f)\{f).$  The big Hankel operator  $\mathcal{H}_\phi$  from  $L_a^2(\mathbb{C}_+)$  into  $(L_a^2(\mathbb{C}_+))^\perp$  is defined by  $\mathcal{H}_\phi \{ = (\mathcal{I} - \mathcal{P}_+)(\phi\{, \{ \in L_{-1}^\infty(\mathbb{C}_+).$  The little Hankel operator  $h_\phi$  is a mapping from  $L_a^2(\mathbb{C}_+)$  into  $\overline{L_a^2(\mathbb{C}_+)}$  defined by  $h_\phi f = \overline{\mathcal{P}_+(\phi f)},$  where  $\overline{\mathcal{P}_+}$  is the orthogonal projection from  $L^2(\mathbb{C}_+, d\tilde{A})$  onto  $\overline{L_a^2(\mathbb{C}_+)} = \{\bar{f} : f \in L_a^2(\mathbb{C}_+)\}.$

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disk in the complex plane  $\mathbb{C}.$  Let  $L^2(\mathbb{D}, dA)$  be the space

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of complex-valued, square-integrable, measurable functions on  $\mathbb{D}$  with respect to the normalized area measure  $dA(z) = \frac{1}{\pi} dx dy$ . Let  $L_a^2(\mathbb{D})$  be the space consisting of those functions of  $L^2(\mathbb{D}, dA)$  that are analytic. The space  $L_a^2(\mathbb{D})$  is a closed subspace of  $L^2(\mathbb{D}, dA)$  and is called the Bergman space of the open unit disk  $\mathbb{D}$ . The sequence of functions  $\{e_n(z)\}_{n=0}^\infty = \{\sqrt{n+1}z^n\}_{n=0}^\infty$  form an orthonormal basis for  $L_a^2(\mathbb{D})$ . Since point evaluation at  $z \in \mathbb{D}$  is a bounded linear functional on the Hilbert space  $L_a^2(\mathbb{D})$ , the Riesz representation theorem implies that there exists a unique function  $K_z$  in  $L_a^2(\mathbb{D})$  such that

$$f(z) = \int_{\mathbb{D}} f(w) \overline{K_z(w)} dA(w).$$

for all  $f$  in  $L_a^2(\mathbb{D})$ . Let  $K(z, w)$  be the function on  $\mathbb{D} \times \mathbb{D}$  defined by

$$K(z, w) = \overline{K_z(w)}.$$

The function  $K(z, w)$  is analytic in  $z$  and co-analytic in  $w$ . Since

$$f(z) = \int_{\mathbb{D}} f(w) K(z, w) dA(w), f \in L_a^2(\mathbb{D}),$$

the function  $K(z, w) = \frac{1}{(1-z\bar{w})^2}$ ,  $z, w \in \mathbb{D}$  and is the reproducing kernel [11] of  $L_a^2(\mathbb{D})$ . For  $a \in \mathbb{D}$ , let  $k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}} = \frac{(1-|a|^2)}{(1-\bar{a}z)^2}$ . The function  $k_a$  is called the normalized reproducing kernel for  $L_a^2(\mathbb{D})$ . It is clear that  $\|k_a\|_2 = 1$ . Important works on the application of the reproducing kernel were obtained by Karaev et al. [9]. These results in reproducing kernel and Berezin symbols are important in operator theory [7]. Let  $P$  denote the orthogonal projection from  $L^2(\mathbb{D}, dA)$  onto  $L_a^2(\mathbb{D})$ . Let  $Aut(\mathbb{D})$  be the Lie group of all automorphisms (biholomorphic mappings) of  $\mathbb{D}$ . We can define for each  $a \in \mathbb{D}$  an automorphism  $\phi_a$  in  $Aut(\mathbb{D})$  such that

- (i)  $(\phi_a \circ \phi_a)(z) = z$ ;
- (ii)  $\phi_a(0) = a, \phi_a(a) = 0$ ;
- (iii)  $\phi_a$  has a unique fixed point in  $\mathbb{D}$ .

In fact,  $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$  for all  $a$  and  $z$  in  $\mathbb{D}$ . An easy calculation shows that the derivative of  $\phi_a$  at  $z$  is equal to  $-k_a(z)$ . It follows that the real Jacobian determinant of  $\phi_a$  at  $z$  is  $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$ . Given  $a \in \mathbb{D}$  and  $f$  any measurable function on  $\mathbb{D}$ , we define a function  $U_a f$  on  $\mathbb{D}$  by  $U_a f(z) = k_a(z) f(\phi_a(z))$ . In this paper, we introduce a class of unitary operators defined on the Bergman space  $L_a^2(\mathbb{C}_+)$  and study certain algebraic properties of these operators. Using these results, we then show that a bounded linear operator  $S$  from  $L_a^2(\mathbb{C}_+)$  into itself commutes with all the weighted composition operators  $W_a, a \in \mathbb{D}$  if and only if  $\tilde{S}(w) = \langle S b_{\bar{w}}, b_{\bar{w}} \rangle, w \in \mathbb{C}_+$  satisfies certain averaging condition. Some applications of these results are also discussed. The organization of this paper is as follows. In §2, we introduce a class of unitary operators  $V_a, a \in \mathbb{D}$  with the help of the automorphisms of  $\mathbb{C}_+$ . We establish certain algebraic properties of these unitary operators, which are also self-adjoint. In §3, we show that a bounded linear operator  $S$  from  $L_a^2(\mathbb{C}_+)$  into itself commutes with all the weighted composition operators  $W_a, a \in \mathbb{D}$  if and only if  $\tilde{S}(w) = \langle S b_{\bar{w}}, b_{\bar{w}} \rangle, w \in \mathbb{C}_+$  satisfies certain averaging condition. Further, in §4, we establish certain applications of the main result of the paper involving multiplication, Toeplitz, and Hankel operators.

**2. The unitary operator  $V_a$**

In this section, we shall introduce the operator  $V_a, a \in \mathbb{D}$  and prove certain elementary properties of the unitary operator  $V_a$ .

Define  $M : \mathbb{C}_+ \rightarrow \mathbb{D}$  by  $Ms = \frac{1-s}{1+s}$ . Then  $M$  is one-one and onto, and  $M^{-1} : \mathbb{D} \rightarrow \mathbb{C}_+$  is given by  $M^{-1}(z) = \frac{1-z}{1+z}$ . Thus  $M$  is its self-inverse. Let  $W : L_a^2(\mathbb{D}) \rightarrow L_a^2(\mathbb{C}_+)$  be defined by  $Wg(s) = \frac{2}{\sqrt{\pi}}g(Ms)\frac{1}{(1+s)^2}$ . Then  $W^{-1} : L_a^2(\mathbb{C}_+) \rightarrow L_a^2(\mathbb{D})$  is given by  $W^{-1}G(z) = 2\sqrt{\pi}G(Mz)\frac{1}{(1+z)^2}$ , where  $Mz = \frac{1-z}{1+z}$ .

**Lemma 2.1** *If  $a \in \mathbb{D}$  and  $a = c + id, c, d \in \mathbb{R}$ , then  $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$  is an automorphism from  $\mathbb{C}_+$  onto  $\mathbb{C}_+$ .*

**Proof** It is not difficult to verify that the map  $t_a : \mathbb{C}_+ \rightarrow \mathbb{C}_+$  is one-one and onto. The lemma follows.  $\square$

**Proposition 2.2** *For  $a \in \mathbb{D}$ , the following hold:*

(i)  $(t_a \circ t_a)(s) = s$ .

(ii)  $t'_a(s) = -l_a(s)$ , where  $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$ .

**Proof** One can verify (i) and (ii) by direct calculation.  $\square$

For  $a \in \mathbb{D}$ , define  $V_a : L_a^2(\mathbb{C}_+) \rightarrow L_a^2(\mathbb{C}_+)$  by  $(V_a g)(s) = (g \circ t_a)(s)l_a(s)$ . In Proposition 2.3, we show that  $V_a$  is a self-adjoint unitary operator that is also an idempotent.

**Proposition 2.3** *For  $a \in \mathbb{D}$ ,*

(i)  $V_a l_a = 1$ .

(ii)  $V_a^{-1} = V_a, V_a^2 = I$ .

(iii)  $V_a$  is self-adjoint.

(iv)  $V_a$  is unitary.

(v)  $V_a P_+ = P_+ V_a$ .

**Proof** We shall first prove (i). If  $a \in \mathbb{D}$ , then by Proposition 2.2,  $t'_a(s) = -l_a(s)$ . Therefore

$$\begin{aligned} (V_a l_a)(s) &= (l_a \circ t_a)(s)l_a(s) \\ &= (-t'_a \circ t_a)(s)l_a(s) \\ &= -(t'_a \circ t_a)(s)l_a(s) \\ &= [-t'_a(t_a(s))]l_a(s) \\ &= -\left[ t'_a \left( \frac{-ids+(1-c)}{(1+c)s+id} \right) \frac{1-|a|^2}{((1+c)s+id)^2} \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 - |a|^2}{\left[ (1 + c) \left( \frac{-ids + (1-c)}{(1+c)s + id} \right) + id \right]^2} \frac{1 - |a|^2}{[(1 + c)s + id]^2} \\
 &= \frac{(1 - |a|^2)(1 - |a|^2)[(1 + c)s + id]^2}{[-ids + 1 - c - idsc + c - c^2 + id(s + cs + id)]^2 [(1 + c)s + id]^2} \\
 &= \frac{(1 - |a|^2)^2}{[-ids + 1 - c - idsc + c - c^2 + ids + idsc - d^2]^2} \\
 &= \frac{(1 - |a|^2)^2}{[1 - c^2 - d^2]^2} \\
 &= \frac{(1 - |a|^2)^2}{[1 - (c^2 + d^2)]^2} \\
 &= \frac{(1 - |a|^2)^2}{(1 - |a|^2)^2} = 1.
 \end{aligned}$$

This proves (i). The assertions in (ii), (iii), and (iv) can be verified by direct calculation. Note that  $V_a$  can also be defined from  $L^2(\mathbb{C}_+)$ . To prove (v), observe that  $V_a(L_a^2(\mathbb{C}_+)) \subset L_a^2(\mathbb{C}_+)$  and  $V_a(L_a^2(\mathbb{C}_+))^\perp \subset (L_a^2(\mathbb{C}_+))^\perp$ . Now let  $f \in L^2(\mathbb{C}_+)$  and  $f = f_1 + f_2$ , where  $f_1 \in L_a^2(\mathbb{C}_+)$  and  $f_2 \in (L_a^2(\mathbb{C}_+))^\perp$ . Hence,

$$\begin{aligned}
 P_+ V_a f &= P_+ V_a (f_1 + f_2) \\
 &= P_J (V_a f_1 + V_a f_2) \\
 &= P_+ V_a f_1 \\
 &= V_a f_1 \\
 &= V_a P_+ f.
 \end{aligned}$$

□

Suppose  $a \in \mathbb{D}$  and  $w = \frac{1-\bar{a}}{1+a} = M\bar{a} \in \mathbb{C}_+$ . Define  $b_{\bar{w}}(s) = \frac{(-1)}{\sqrt{\pi}}(k_a \circ M)(s)M'(s)$ .

**Lemma 2.4** *Let  $a \in \mathbb{D}$ . For  $w_1 \in \mathbb{C}_+$ ,  $V_a b_{\bar{w}_1} = \alpha b_{t_a(w_1)}$  for some  $\alpha \in \mathbb{C}$  such that  $|\alpha| = 1$ .*

**Proof** To prove the lemma, we shall first show that for  $z_1, z_2 \in \mathbb{D}$ ,  $U_{z_1} k_{z_2} = \alpha k_{\phi_{z_1}(z_2)}$  for some complex constant  $\alpha$  such that  $|\alpha| = 1$ . Suppose  $z_1, z_2 \in \mathbb{D}$ . If  $f \in L_a^2(\mathbb{D})$ , then

$$\langle f, U_{z_1} K_{z_2} \rangle = \langle U_{z_1} f, K_{z_2} \rangle = (U_{z_1} f)(z_2) = -(f \circ \phi_{z_1})(z_2) \phi'_{z_1}(z_2) = \langle f, \overline{(-\phi'_{z_1}(z_2))} K_{\phi_{z_1}(z_2)} \rangle. \tag{2.1}$$

Thus  $U_{z_1} K_{z_2} = \overline{(-\phi'_{z_1}(z_2))} K_{\phi_{z_1}(z_2)}$ . Rewriting this in terms of normalized reproducing kernels, we have

$$U_{z_1} k_{z_2} = \alpha k_{\phi_{z_1}(z_2)} \tag{2.2}$$

for some complex constant  $\alpha$ . Since  $U_{z_1}$  is unitary and  $\|k_{z_2}\|_2 = \|k_{\phi_{z_1}(z_2)}\|_2 = 1$ , we obtain that  $|\alpha| = 1$ .

Let  $w_1 \in \mathbb{C}_+$  and define  $w_1 = M\bar{a}_1$ . Since

$$t_a(\bar{w}_1) = \frac{-id\bar{w}_1 + (1 - c)}{(1 + c)\bar{w}_1 + id},$$

we obtain

$$\overline{t_a(\bar{w}_1)} = \frac{idw_1 + (1 - c)}{(1 + c)w_1 - id} = t_{\bar{a}}(w_1).$$

Thus,

$$\begin{aligned} V_a b_{\bar{w}_1} &= WU_a k_{a_1} \\ &= \alpha W k_{\phi_a(a_1)} \\ &= \alpha b_{\bar{l}}, \end{aligned}$$

where

$$\begin{aligned} l &= \overline{M\phi_a(a_1)} \\ &= \overline{M\phi_a(M\bar{w}_1)} \\ &= \overline{t_a(\bar{w}_1)} = t_{\bar{a}}(w_1). \end{aligned}$$

□

**Lemma 2.5** *Let  $a \in \mathbb{D}$ , and  $w = M\bar{a}$ . Then*

$$(i) \quad V_a b_{\bar{w}} = \frac{(-1)}{\sqrt{\pi}} M'.$$

$$(ii) \quad V_a \left( \frac{(-1)}{\sqrt{\pi}} M' \right) = b_{\bar{w}}.$$

**Proof** Let  $a \in \mathbb{D}$ . Then, since  $U_a k_a = 1$  and  $W1 = \frac{(-1)}{\sqrt{\pi}} M'$ , we obtain

$$\begin{aligned} V_a b_{\bar{w}} &= WU_a k_a \\ &= W1 \\ &= \frac{(-1)}{\sqrt{\pi}} M'. \end{aligned}$$

Now, to prove (ii), observe that

$$\begin{aligned} V_a \left( \frac{(-1)}{\sqrt{\pi}} M' \right) &= WU_a 1 \\ &= Wk_a \\ &= b_{\bar{w}}. \end{aligned}$$

Let  $\mathcal{L}(\mathcal{L}_{\bar{1}}^{\xi}(\mathbb{C}_+))$  be the space of all bounded linear operators from  $L_a^2(\mathbb{C}_+)$  into itself. For  $T \in \mathcal{L}(\mathcal{L}_{\bar{1}}^{\xi}(\mathbb{C}_+))$ , define the function  $\tilde{T}$  on  $\mathbb{C}_+$  as  $\tilde{T}(w) = \langle T b_{\bar{w}}, b_{\bar{w}} \rangle$ . □

**Theorem 2.6** *Let  $S, T \in \mathcal{L}(\mathcal{L}_{\bar{1}}^{\xi}(\mathbb{C}_+))$ . If  $\tilde{S}(w) = \tilde{T}(w)$  for all  $w \in \mathbb{C}_+$ , then  $S = T$ .*

**Proof** Let  $\tilde{S}(w) = \tilde{T}(w)$  for all  $w \in \mathbb{C}_+$ . Then, for  $w = M\bar{a}$ , we have

$$\begin{aligned} \langle Sb_{\bar{w}}, b_{\bar{w}} \rangle &= \langle SWk_a, Wk_a \rangle \\ &= \langle W^{-1}SWk_a, k_a \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} \langle Tb_{\bar{w}}, b_{\bar{w}} \rangle &= \langle TWk_a, Wk_a \rangle \\ &= \langle W^{-1}TWk_a, k_a \rangle. \end{aligned}$$

Hence for all  $a \in \mathbb{D}$ ,

$$\langle W^{-1}SWk_a, k_a \rangle = \langle W^{-1}TWk_a, k_a \rangle.$$

This implies

$$\langle (W^{-1}SW - W^{-1}TW)k_a, k_a \rangle = \langle Lk_a, k_a \rangle = 0,$$

where  $L = W^{-1}SW - W^{-1}TW$ . Hence,

$$\langle LK_a, K_a \rangle = K(a, a)\langle Lk_a, k_a \rangle = K(a, a) \cdot 0 = 0.$$

Define  $F(x, y) = \langle LK_{\bar{x}}, K_y \rangle$ . The function  $F$  is holomorphic in  $x$  and  $y$  and  $F(x, y) = 0$  if  $x = \bar{y}$  [5]. It can be verified that such functions must vanish identically. Let  $x = u + iv, y = u - iv$ . Let  $G(u, v) = F(x, y)$ . The function  $G$  is holomorphic and vanishes if  $u$  and  $v$  are real. Hence  $F(x, y) = G(u, v) \equiv 0$ . Thus even  $\langle LK_x, K_y \rangle = 0$  for any  $x, y$ . Since linear combinations of  $K_x, x \in \mathbb{D}$ , are dense in  $L_a^2(\mathbb{D})$  [3], it follows that  $L = 0$ . That is,  $W^{-1}SW = W^{-1}TW$ . Hence  $S = T$ .  $\square$

**Corollary 2.7** Let  $S, T \in \mathcal{L}(\mathcal{L}_{\bar{1}}^{\infty}(\mathbb{C}_+))$ . Suppose for all  $a \in \mathbb{D}$ ,

$$\left\langle (V_aSV_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle = \left\langle (V_aTV_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle.$$

Then  $S = T$ .

**Proof** Let  $a \in \mathbb{D}$ . Then since  $W^{-1} \left( \frac{(-1)}{\sqrt{\pi}} M' \right) = 1$ , hence

$$\begin{aligned} \left\langle (V_aSV_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle &= \left\langle U_a(W^{-1}SW)U_aW^{-1} \left( \frac{(-1)}{\sqrt{\pi}} M' \right), W^{-1} \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle \\ &= \langle U_a(W^{-1}SW)U_a1, 1 \rangle \\ &= \langle (W^{-1}SW)U_a1, U_a1 \rangle \\ &= \langle (W^{-1}SW)k_a, k_a \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\langle (V_aTV_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle &= \left\langle U_a(W^{-1}TW)U_aW^{-1} \left( \frac{(-1)}{\sqrt{\pi}} M' \right), W^{-1} \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle \\ &= \langle U_a(W^{-1}TW)U_a1, 1 \rangle \\ &= \langle (W^{-1}TW)U_a1, U_a1 \rangle \\ &= \langle (W^{-1}TW)k_a, k_a \rangle. \end{aligned}$$

Thus,

$$\left\langle (V_a S V_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle = \left\langle (V_a T V_a) \left( \frac{(-1)}{\sqrt{\pi}} M' \right), \left( \frac{(-1)}{\sqrt{\pi}} M' \right) \right\rangle \text{ for all } a \in \mathbb{D}$$

implies

$$\langle (W^{-1} S W - W^{-1} T W) k_a, k_a \rangle = 0 \text{ for all } a \in \mathbb{D}.$$

Hence,

$$\langle (W^{-1} S W - W^{-1} T W) K_a, K_a \rangle = K(a, a) \langle (W^{-1} S W - W^{-1} T W) k_a, k_a \rangle = K(a, a) \cdot 0 = 0.$$

Proceeding similarly as in Corollary 2.7, we obtain  $W^{-1} S W = W^{-1} T W$ . Hence  $S = T$ . □

### 3. Main result

The operators  $W_a$  are called weighted composition operators on  $L_a^2(\mathbb{C}_+)$ . In this section, we shall show that a bounded linear operator  $S$  from  $L_a^2(\mathbb{C}_+)$  into itself commutes with all the weighted composition operators  $W_a, a \in \mathbb{D}$ , if and only if  $\tilde{S}$  satisfies a certain averaging condition.

**Theorem 3.1** *A bounded linear operator  $S \in \mathcal{L}(L_a^2(\mathbb{C}_+))$  commutes with all the weighted composition operators  $W_a, a \in \mathbb{D}$  if and only if*

$$\tilde{S}(w_1) = \int_{\mathbb{D}} \tilde{S}(t_{\bar{a}}(w_1)) dA(a), \text{ for all } w_1 \in \mathbb{C}_+.$$

**Proof** Suppose

$$\tilde{S}(w_1) = \int_{\mathbb{D}} \tilde{S}(t_{\bar{a}}(w_1)) dA(a), \tag{3.1}$$

for all  $w_1 \in \mathbb{C}_+$ . Then, by Lemma 2.4, there exists a constant  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$  such that for all  $w_1 \in \mathbb{C}_+$ ,

$$\begin{aligned} \langle S b_{\bar{w}_1}, b_{\bar{w}_1} \rangle &= \int_{\mathbb{D}} \langle S b_{\overline{t_{\bar{a}}(w_1)}}, b_{\overline{t_{\bar{a}}(w_1)}} \rangle dA(a) \\ &= \int_{\mathbb{D}} \langle \alpha S V_a b_{\bar{w}_1}, \alpha V_a b_{\bar{w}_1} \rangle dA(a) \\ &= \int_{\mathbb{D}} \langle V_a S V_a b_{\bar{w}_1}, b_{\bar{w}_1} \rangle dA(a) \\ &= \left\langle \left( \int_{\mathbb{D}} V_a S V_a dA(a) \right) b_{\bar{w}_1}, b_{\bar{w}_1} \right\rangle \\ &= \langle \widehat{S} b_{\bar{w}_1}, b_{\bar{w}_1} \rangle, \end{aligned}$$

$$\text{where } \widehat{S} = \int_{\mathbb{D}} V_a S V_a dA(a).$$

Thus, by Theorem 2.6,  $S = \widehat{S}$ . Hence for all  $f, g \in L_a^2(\mathbb{C}_+)$ ,  $\langle S f, g \rangle = \langle \widehat{S} f, g \rangle$ . Thus the equation (3.1) is equivalent to saying that

$$\int_{\mathbb{D}} \langle S V_a f, V_a g \rangle dA(a) = \int_{\mathbb{C}_+} (S f)(w_1) \overline{g(w_1)} d\tilde{A}(w_1)$$



for all  $f, g \in L_a^2(\mathbb{C}_+)$ . Let  $w \in \mathbb{C}_+$  and let  $w = M\bar{a} = \frac{1-\bar{a}}{1+a}$ . Since

$$\begin{aligned} b_{\bar{w}}(s) &= \frac{(-1)}{\sqrt{\pi}} \frac{(1 - |a|^2)}{(1 - \bar{a}Ms)^2} \frac{(-2)}{(1 + s)^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{(1 - |a|^2)}{(1 - \bar{a}Ms)^2} \frac{1}{(1 + s)^2}, \end{aligned}$$

we obtain

$$\begin{aligned} b_{\bar{w}}(\bar{w}) &= \frac{2}{\sqrt{\pi}} \frac{(1 - |a|^2)}{(1 - \bar{a}M\bar{w})^2} \frac{1}{(1 + \bar{w})^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{(1 - |a|^2)}{(1 - |a|^2)^2} \frac{1}{\left(1 + \frac{1-a}{1+a}\right)^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{1}{(1 - |a|^2)} \frac{(1 + a)^2}{4} \\ &= \frac{1}{2\sqrt{\pi}} \frac{(1 + a)^2}{(1 - |a|^2)}. \end{aligned}$$

Thus

$$\begin{aligned} b_{\bar{w}}(s)b_{\bar{w}}(\bar{w}) &= \frac{2}{\sqrt{\pi}} \frac{(1 - |a|^2)}{(1 - \bar{a}Ms)^2} \frac{1}{(1 + s)^2} \frac{1}{2\sqrt{\pi}} \frac{(1 + a)^2}{(1 - |a|^2)} \\ &= \frac{1}{\pi} \frac{1}{(1 - \bar{a}Ms)^2} \frac{(1 + a)^2}{(1 + s)^2} \\ &= \frac{(-1)}{2\pi} \frac{(1 + a)^2}{(1 - \bar{a}Ms)^2} \frac{(-2)}{(1 + s)^2} \\ &= \frac{(-1)}{2\pi} \frac{(1 + a)^2}{(1 - \bar{a}Ms)^2} M' \\ &= B(s, w) \text{ (let)}. \end{aligned}$$

Thus  $b_{\bar{w}}(s) = \frac{B(s,w)}{b_{\bar{w}}(\bar{w})}$  and  $(b_{\bar{w}}(\bar{w}))^2 = B(\bar{w}, w)$ . That is,  $b_{\bar{w}}(s) = \frac{B(s,w)}{b_{\bar{w}}(\bar{w})} = \frac{B(s,w)}{\sqrt{B(\bar{w},w)}}$ . Note that

$$W1 = \frac{(-1)}{\sqrt{\pi}} M' \text{ and therefore } W^{-1} \left( \frac{-M'}{\sqrt{\pi}} \right) = 1.$$

$$\text{That is, } (M' \circ M)M' = (-1)\sqrt{\pi} \frac{(-1)}{\sqrt{\pi}} (M' \circ M)M' = W^{-1} \left( \frac{-M'}{\sqrt{\pi}} \right) = 1.$$

From Lemma 2.5, it follows that

$$(b_{\bar{w}} \circ t_a)l_a = \frac{(-1)}{\sqrt{\pi}} M'.$$

This implies

$$b_{\bar{w}}(l_a \circ t_a) = \frac{(-1)}{\sqrt{\pi}} (M' \circ t_a). \tag{3.2}$$

That is,

$$b_{\bar{w}} \left( \frac{l_a \circ t_a}{\frac{(-1)}{\sqrt{\pi}} (M' \circ t_a)} \right) = 1.$$

Thus  $(-\sqrt{\pi})b_{\bar{w}}\left(\frac{l_a}{M'} \circ t_a\right) = 1$  and therefore  $(-\sqrt{\pi})b_{\bar{w}}[(l_a(M' \circ M)) \circ t_a] = 1$ . Hence

$$b_{\bar{w}}[(-\sqrt{\pi})(l_a \circ t_a)(M' \circ M \circ t_a)] = 1.$$

This implies  $b_{\bar{w}} \in H^\infty(\mathbb{C}_+)$  and  $\frac{1}{b_{\bar{w}}} \in H^\infty(\mathbb{C}_+)$ . Further  $B(s, M\bar{0}) = B(s, M0) = B(s, 1) = \frac{(-1)}{2\pi}M' = \frac{(-1)}{2\pi} \frac{(-2)}{(1+s)^2} = \frac{1}{\pi} \frac{1}{(1+s)^2}$ . Again  $B(Ma, M\bar{0}) = \frac{1}{\pi} \frac{1}{(1+Ma)^2}$  and  $B(M0, M\bar{0}) = \frac{1}{4\pi}$ . Now note that  $W_a f = (f \circ t_a) \frac{M'}{M' \circ t_a}$ . Hence  $W_a S f = S W_a f$  for all  $f \in L_a^2(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$  if and only if

$$[(Sf) \circ t_a] \frac{M'}{M' \circ t_a} = S \left[ (f \circ t_a) \frac{M'}{M' \circ t_a} \right]$$

for all  $a \in \mathbb{D}$  and for all  $f \in L_a^2(\mathbb{C}_+)$ . That is, if and only if,

$$\left[ \frac{(Sf)}{M'} \circ t_a \right] M' = S \left[ \left( \frac{f}{M'} \circ t_a \right) M' \right]$$

for all  $f \in L_a^2(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . Putting  $\frac{f}{b_{\bar{w}}}$  in place of  $f$ , we obtain  $S W_a f = W_a S f$  for all  $f \in L_a^2(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$  if and only if

$$\left[ \left( \frac{1}{M'(w_1)} S \left( \frac{f}{b_{\bar{w}}} \right) \right) \circ t_a \right] M'(w_1) = S \left[ \left( \frac{f}{M' b_{\bar{w}}} \circ t_a \right) M' \right] (w_1) \tag{3.3}$$

for all  $f \in L_a^2(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . Now to prove the necessary part of the theorem, assume that  $S W_a f = W_a S f$  for all  $f \in L_a^2(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . We shall prove that

$$\int_{\mathbb{C}_+} (Sf)(w_1) \overline{g(w_1)} d\tilde{A}(w_1) = \int_{\mathbb{D}} \langle S V_a f, V_a g \rangle dA(a)$$

for all  $f, g \in L_a^2(\mathbb{C}_+)$ . Note that

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{C}_+} (Sf)(z) \overline{g(z)} d\tilde{A}(z) \\ &= \int_{\mathbb{C}_+} \frac{\overline{g(z)} d\tilde{A}(z)}{M'(z)} S \left( \frac{f M'}{\left(\frac{-1}{2\pi}\right) M'} \right) (z) \left( \frac{-1}{2\pi} M' \right) (z) \\ &= \int_{\mathbb{C}_+} \frac{\overline{g(z)} d\tilde{A}(z)}{M'(z)} S \left( \frac{f M'}{B(\cdot, M0)} \right) (z) B(z, M0). \end{aligned}$$

Hence by the mean-value property for harmonic functions [1], we obtain

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{C}_+} \frac{\overline{g(z)} d\tilde{A}(z)}{M'(z)} \int_{\mathbb{D}} S \left( \frac{f M' \sqrt{B(Ma, M\bar{a})}}{B(\cdot, M\bar{a})} \right) (z) \frac{B(z, M\bar{a})}{\sqrt{B(Ma, M\bar{a})}} dA(a) \\ &= \int_{\mathbb{C}_+} \frac{\overline{g(z)} d\tilde{A}(z)}{M'(z)} \int_{\mathbb{D}} S \left( \frac{f M'}{b_{\bar{w}}} \right) (z) b_{\bar{w}}(z) dA(a). \end{aligned}$$

Using Fubini's theorem [10] and using the identity (3.2), we obtain

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S \left( \frac{fM'}{b_{\bar{w}}} \right) (z) b_{\bar{w}}(z) \overline{g(z)} d\tilde{A}(z) \right] dA(a) \\ &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S \left( \frac{fM'}{b_{\bar{w}}} \right) (z) \frac{b_{\bar{w}}}{l_a}(z) \overline{g(z)} l_a(z) d\tilde{A}(z) \right] dA(a) \\ &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S \left( \frac{fM'}{b_{\bar{w}}} \right) (z) b_{\bar{w}}(l_a \circ t_a)(z) \overline{g(z)} l_a(z) d\tilde{A}(z) \right] dA(a) \\ &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) (z) \left( \frac{-1}{\sqrt{\pi}} \right) (M' \circ t_a)(z) \overline{g(z) (l_a \circ t_a)(z)} |l_a(z)|^2 d\tilde{A}(z) \right] dA(a) \\ &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \left[ \left( \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right) \circ t_a \right] (z) \left( \frac{-1}{\sqrt{\pi}} \right) M'(z) \overline{(g \circ t_a)(z)} l_a(z) d\tilde{A}(z) \right] dA(a). \end{aligned}$$

Now observe that by the identity (3.2) we obtain

$$\begin{aligned} SV_a f &= S[(f \circ t_a)l_a] = S \left[ \frac{f \circ t_a}{b_{\bar{w}} \circ t_a} \frac{(-1)}{\sqrt{\pi}} M' \right] \\ &= S \left[ \left( \frac{f}{b_{\bar{w}}} \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] = S \left[ \left( \left( \frac{fM'}{M'b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] \\ &= \left[ \left( \left( \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right]. \end{aligned}$$

This last equality follows from (3.3) since  $SW_a f = W_a Sf$ . Thus for all  $f, g \in L_a^2(\mathbb{C}_+)$ , we obtain

$$\langle Sf, g \rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a).$$

We shall now prove the sufficient part. Suppose  $\langle Sf, g \rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a)$  for all  $f, g \in L_a^2(\mathbb{C}_+)$ . We shall show that  $SW_a = W_a S$  for all  $a \in \mathbb{D}$ . We have already verified that

$$SV_a f = S \left[ \left( \left( \frac{f}{b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right].$$

Hence

$$\begin{aligned} \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a) &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} S \left[ \left( \left( \frac{f}{b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] (w_1) \overline{(g \circ t_a)(w_1)} l_a(w_1) d\tilde{A}(w_1) \right] dA(a) \\ &= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} S \left[ \left( \left( \frac{fM'}{M'b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] (w_1) \overline{(g \circ t_a)(w_1)} l_a(w_1) d\tilde{A}(w_1) \right] dA(a). \end{aligned}$$

Using Fubini's theorem [10], we obtain

$$\int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a) = \int_{\mathbb{C}_+} \left[ \int_{\mathbb{D}} S \left[ \left( \left( \frac{fM'}{M'b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] (w_1) \overline{(g \circ t_a)(w_1)} l_a(w_1) dA(a) \right] d\tilde{A}(w_1). \quad (3.4)$$

Now

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{C}_+} (Sf)(z) \overline{g(z)} d\tilde{A}(z) \\ &= \int_{\mathbb{C}_+} \overline{g(z)} \frac{1}{M'(z)} S \left( \frac{fM'}{M'} \right) (z) M'(z) d\tilde{A}(z) \\ &= \int_{\mathbb{C}_+} \overline{g(z)} \frac{1}{M'(z)} S \left( \frac{fM'}{\left(\frac{-1}{2\pi}\right)M'} \right) (z) \left(\frac{-1}{2\pi}\right) M'(z) d\tilde{A}(z) \\ &= \int_{\mathbb{C}_+} \overline{g(z)} \frac{1}{M'(z)} S \left( \frac{fM'}{B(\cdot, M0)} \right) (z) B(z, M0) d\tilde{A}(z). \end{aligned}$$

By mean value property for harmonic functions [1], we obtain

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{C}_+} \frac{\overline{g(z)} d\tilde{A}(z)}{M'(z)} \int_{\mathbb{D}} S \left( \frac{fM' \sqrt{B(Ma, M\bar{a})}}{B(\cdot, M\bar{a})} \right) (z) \frac{B(z, M\bar{a})}{\sqrt{B(Ma, M\bar{a})}} dA(a) \\ &= \int_{\mathbb{C}_+} \overline{g(z)} d\tilde{A}(z) \int_{\mathbb{D}} \left[ \frac{1}{M'} S \left( \frac{fM'}{\frac{B(\cdot, w)}{\sqrt{B(\bar{w}, w)}}} \right) \right] (z) \frac{B(z, w)}{\sqrt{B(\bar{w}, w)}} dA(a) \\ &= \int_{\mathbb{C}_+} \overline{g(z)} d\tilde{A}(z) \int_{\mathbb{D}} \left[ \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right] (z) b_{\bar{w}}(z) dA(a). \end{aligned}$$

Using Fubini's theorem [10], we obtain

$$\begin{aligned} \langle Sf, g \rangle &= \int_{\mathbb{D}} \int_{\mathbb{C}_+} \left[ \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right] (z) b_{\bar{w}}(z) \overline{g(z)} d\tilde{A}(z) dA(a) \\ &= \int_{\mathbb{D}} \int_{\mathbb{C}_+} \left[ \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \circ t_a \right] (z) (b_{\bar{w}} \circ t_a)(z) \overline{(g \circ t_a)(z)} |l_a(z)|^2 d\tilde{A}(z) dA(a) \\ &= \int_{\mathbb{D}} \int_{\mathbb{C}_+} \left[ \left( \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right) \circ t_a \right] (z) \frac{\left(\frac{-1}{\sqrt{\pi}}\right) M'(z)}{l_a(z)} \overline{(g \circ t_a)(z)} l_a(z) d\tilde{A}(z) dA(a) \\ &= \int_{\mathbb{D}} \int_{\mathbb{C}_+} \left[ \left( \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right) \circ t_a \right] (z) \frac{(-1)}{\sqrt{\pi}} M'(z) \overline{(g \circ t_a)(z)} l_a(z) d\tilde{A}(z) dA(a) \end{aligned} \quad (3.5)$$

From (3.4) and (3.5), it follows that if  $\langle Sf, g \rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a)$  for all  $f, g \in L^2_a(\mathbb{C}_+)$  then

$$S \left[ \left( \left( \frac{fM'}{M'b_{\bar{w}}} \right) \circ t_a \right) \left( \frac{-1}{\sqrt{\pi}} \right) M' \right] = \left[ \left( \frac{1}{M'} S \left( \frac{fM'}{b_{\bar{w}}} \right) \right) \circ t_a \right] \frac{(-1)}{\sqrt{\pi}} M' \quad (3.6)$$

for all  $f \in L^2_a(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ .

Putting  $\frac{f}{M'}$  in place of  $f$  we obtain (3.6), which holds if and only if

$$S \left[ \left( \left( \frac{f}{M'b_{\bar{w}}} \right) \circ t_a \right) \frac{(-1)}{\sqrt{\pi}} M' \right] = \left[ \left( \frac{1}{M'} S \left( \frac{f}{b_{\bar{w}}} \right) \right) \circ t_a \right] \frac{(-1)}{\sqrt{\pi}} M'$$

for all  $f \in L^2_a(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . Thus if  $\langle Sf, g \rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a)$  for all  $f, g \in L^2_a(\mathbb{C}_+)$ , then

$$S \left[ \left( \left( \frac{f}{M'b_{\bar{w}}} \right) \circ t_a \right) M' \right] = \left[ \left( \frac{1}{M'} S \left( \frac{f}{b_{\bar{w}}} \right) \right) \circ t_a \right] M' \tag{3.7}$$

for all  $f \in L^2_a(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ .

By (3.5), the identity (3.7) holds if and only if  $SW_a f = W_a Sf$  for all  $f \in L^2_a(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . Thus we have proved that if  $\langle Sf, g \rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g \rangle dA(a)$  for all  $f, g \in L^2_a(\mathbb{C}_+)$  then  $SW_a f = W_a Sf$  for all  $f \in L^2_a(\mathbb{C}_+)$  and for all  $a \in \mathbb{D}$ . The theorem follows.  $\square$

#### 4. Applications

In this section, we establish certain applications of the main result of the paper involving multiplication, Toeplitz, and Hankel operators. For  $\phi \in L^\infty(\mathbb{C}_+, d\tilde{A})$  and  $T \in \mathcal{L}(\mathcal{L}^2_{-}(\mathbb{C}_+))$ , let  $\hat{\phi}(s) = \int_{\mathbb{D}} \phi(t_a(s)) dA(a)$  and  $\hat{T} = \int_{\mathbb{D}} V_a T V_a dA(a)$ .

**Corollary 4.1** *Let  $\phi \in L^\infty(\mathbb{C}_+)$ . Then the following hold:*

(i)  $\hat{\mathcal{M}}_\phi = \mathcal{M}_{\hat{\phi}}$ .

(ii)  $\hat{\mathcal{T}}_\phi = \mathcal{T}_{\hat{\phi}}$ .

(iii)  $\hat{\mathcal{H}}_\phi = \mathcal{H}_{\hat{\phi}}$ .

**Proof** From Proposition 2.2, it follows that for given  $h \in L^2(\mathbb{C}_+, d\tilde{A})$  and  $g \in L^2(\mathbb{C}_+, d\tilde{A})$  we have

$$\begin{aligned} \langle \hat{\mathcal{M}}_\phi g, h \rangle &= \int_{\mathbb{D}} \langle \phi V_a g, V_a h \rangle dA(a) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s) (V_a g)(s) \overline{(V_a h)(s)} d\tilde{A}(s) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s) (g \circ t_a)(s) l_a(s) \overline{(h \circ t_a)(s)} \overline{l_a(s)} d\tilde{A}(s) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} (l_a \circ t_a)(s) \overline{(l_a \circ t_a)(s)} |l_a(s)|^2 d\tilde{A}(s) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)\overline{h(s)} |(l_a \circ t_a)(s)|^2 |l_a(s)|^2 d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)\overline{h(s)} d\tilde{A}(s) \\
 &= \int_{\mathbb{C}_+} g(s)\overline{h(s)} d\tilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\
 &= \int_{\mathbb{C}_+} \widehat{\phi}(s)g(s)\overline{h(s)} d\tilde{A}(s) = \langle \mathcal{M}_{\widehat{\phi}} \rangle, \langle \cdot \rangle.
 \end{aligned}$$

This proves (i). To prove (ii), let  $h$  and  $g$  in  $L^2_a(\mathbb{C}_+, d\tilde{A})$ . Then since  $(l_a \circ t_a)(s)l_a(s) = s$ , we obtain

$$\begin{aligned}
 \langle \widehat{\mathcal{T}}_{\phi}g, h \rangle &= \int_{\mathbb{D}} \langle V_a \mathcal{T}_{\phi} \mathcal{V}_+ \rangle, \langle \cdot \rangle dA(a) \\
 &= \int_{\mathbb{D}} \langle \mathcal{T}_{\phi} \mathcal{V}_+ \rangle, \mathcal{V}_+ \langle \cdot \rangle dA(a) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle P_+(\phi V_a g), V_a h \rangle d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle \phi V_a g, P_+(V_a h) \rangle d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle \phi V_a g, V_a h \rangle d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(V_a g)(s)\overline{(V_a h)(s)} d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(g \circ t_a)(s)l_a(s)\overline{(h \circ t_a)(s)} \overline{l_a(s)} d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)(l_a \circ t_a)(s)\overline{h(s)} \overline{(l_a \circ t_a)(s)} |l_a(s)|^2 d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)\overline{h(s)}(l_a \circ t_a)(s)\overline{(l_a \circ t_a)(s)} |l_a(s)|^2 d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)\overline{h(s)}|(l_a \circ t_a)(s)|^2 |l_a(s)|^2 d\tilde{A}(s) \\
 &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s)g(s)\overline{h(s)} d\tilde{A}(s) \\
 &= \int_{\mathbb{C}_+} g(s)\overline{h(s)} d\tilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\
 &= \int_{\mathbb{C}_+} \widehat{\phi}(s)g(s)\overline{h(s)} d\tilde{A}(s) \\
 &= \langle \widehat{\phi}g, h \rangle \\
 &= \langle \widehat{\phi}g, P_+h \rangle = \langle P_+(\widehat{\phi}g), h \rangle = \langle \mathcal{T}_{\widehat{\phi}} \rangle, \langle \cdot \rangle.
 \end{aligned}$$

Therefore,  $\widehat{\mathcal{T}}_\phi = \mathcal{T}_{\widehat{\phi}}$ . This proves (ii). Now we shall establish (iii). It is not difficult to see that for  $a \in \mathbb{D}$ ,  $V_a(L_a^2(\mathbb{C}_+)) \subset L_a^2(\mathbb{C}_+)$  and  $V_a((L_a^2(\mathbb{C}_+))^\perp) \subset (L_a^2(\mathbb{C}_+))^\perp$ . Further, from Proposition 2.3, it follows that for  $g \in L_a^2(\mathbb{C}_+)$  and  $h \in (L_a^2(\mathbb{C}_+))^\perp$ , we have

$$\begin{aligned} \langle \widehat{\mathcal{H}}_\phi g, h \rangle &= \int_{\mathbb{D}} \langle V_a \mathcal{H}_\phi \mathcal{V}_{\cdot} \rangle, \langle \rangle [\mathcal{A}(\cdot)] \\ &= \int_{\mathbb{D}} \langle \mathcal{H}_\phi \mathcal{V}_{\cdot} \rangle, \mathcal{V}_{\cdot} \langle \rangle [\mathcal{A}(\cdot)] \\ &= \int_{\mathbb{D}} \langle (I - P_+)(\phi V_a g), V_a h \rangle dA(a) \\ &= \int_{\mathbb{D}} \langle \phi V_a g, (I - P_+)(V_a h) \rangle dA(a) \\ &= \int_{\mathbb{D}} \langle \phi V_a g, V_a h \rangle dA(a) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(V_a g)(s) \overline{(V_a h)(s)} d\tilde{A}(s) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(g \circ t_a)(s) l_a(s) \overline{(h \circ t_a)(s) l_a(s)} d\tilde{A}(s) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) l_a(s) \overline{h(s) l_a(s)} |l_a(s)|^2 d\tilde{A}(s) \\ &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} d\tilde{A}(s) \\ &= \int_{\mathbb{C}_+} g(s) \overline{h(s)} d\tilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\ &= \int_{\mathbb{C}_+} \widehat{\phi}(s) g(s) \overline{h(s)} d\tilde{A}(s) \\ &= \langle \widehat{\phi} g, h \rangle \\ &= \langle \widehat{\phi} g, (I - P_+) h \rangle \\ &= \langle (I - P_+)(\widehat{\phi} g), h \rangle \\ &= \langle H_{\widehat{\phi}} g, h \rangle. \end{aligned}$$

Hence  $\widehat{\mathcal{H}}_\phi = \mathcal{H}_{\widehat{\phi}}$ . □

**Corollary 4.2** *Let  $a \in \mathbb{D}$  and  $\phi \in L^\infty(\mathbb{C}_+)$ . Then the following hold:*

(i)  $V_a \mathcal{M}_\phi \mathcal{V}_{\cdot} = \mathcal{M}_{\phi \circ \sqcup_{\cdot}}$ .

(ii)  $V_a \mathcal{T}_\phi \mathcal{V}_{\cdot} = \mathcal{T}_{\phi \circ \sqcup_{\cdot}}$ .

(iii)  $V_a \mathcal{H}_\phi \mathcal{V}_{-1} = \mathcal{H}_{\phi \circ \sqcup_{-1}}$ .

(iv)  $V_a h_\phi V_a = h_{\phi \circ t_a}$ .

**Proof** We first prove (i). Note that since  $(l_a \circ t_a)(s)l_a(s) = s$ , we have for  $f \in L_a^2(\mathbb{C}_+)$ ,

$$\begin{aligned} V_a \mathcal{M}_\phi \mathcal{V}_{-1} \{ &= V_a \mathcal{M}_\phi [(\{ \circ \sqcup_{-1} \} \uparrow_{-1})] \\ &= V_a [\phi(f \circ t_a)l_a] \\ &= (\phi \circ t_a)f(l_a \circ t_a)l_a \\ &= (\phi \circ t_a)f \\ &= \mathcal{M}_{\phi \circ \sqcup_{-1}} \{. \end{aligned}$$

This proves (i). To prove (ii), let  $f \in L_a^2(\mathbb{C}_+)$ . Then we have

$$\begin{aligned} V_a \mathcal{T}_\phi \mathcal{V}_{-1} \{ &= V_a \mathcal{T}_\phi [(\{ \circ \sqcup_{-1} \} \uparrow_{-1})] \\ &= V_a P_+ [\phi(f \circ t_a)l_a] \\ &= P_+ V_a [\phi(f \circ t_a)l_a] \\ &= P_+ [(\phi \circ t_a)f(l_a \circ t_a)l_a] \\ &= P_+ [(\phi \circ t_a)f] \\ &= \mathcal{T}_{\phi \circ \sqcup_{-1}} \{, \end{aligned}$$

since  $(l_a \circ t_a)(s)l_a(s) = s$ . This proves (ii). Now to establish (iii), let  $f \in L_a^2(\mathbb{C}_+)$ . Then

$$\begin{aligned} V_a \mathcal{H}_\phi \mathcal{V}_{-1} \{ &= V_a \mathcal{H}_\phi [(\{ \circ \sqcup_{-1} \} \uparrow_{-1})] \\ &= V_a (I - P_+) [\phi(f \circ t_a)l_a] \\ &= (I - P_+) V_a [\phi(f \circ t_a)l_a] \\ &= (I - P_+) [(\phi \circ t_a)f(l_a \circ t_a)l_a] \\ &= (I - P_+) [(\phi \circ t_a)f] \\ &= \mathcal{H}_{\phi \circ \sqcup_{-1}} \{. \end{aligned}$$

This proves (iii). To prove (iv), let  $f \in L_a^2(\mathbb{C}_+)$ . Then since  $\overline{P}_+ = J P J$ , where  $J f(s) = f(\overline{s})$  for  $f \in L_a^2(\mathbb{C}_+)$  and  $V_a \overline{P}_+ = \overline{P}_+ V_a$ , we obtain

$$\begin{aligned} V_a h_\phi V_a f &= V_a h_\phi [(f \circ t_a)l_a] \\ &= V_a \overline{P}_+ [\phi(f \circ t_a)l_a] \\ &= \overline{P}_+ V_a [\phi(f \circ t_a)l_a] \\ &= \overline{P}_+ [(\phi \circ t_a)f(l_a \circ t_a)l_a] \\ &= \overline{P}_+ [(\phi \circ t_a)f] \\ &= h_{\phi \circ t_a} f. \end{aligned}$$

□



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