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SADETTİN ERDEM

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On product and golden stuctures and harmonicity

Sadettin ERDEM*
Department of Mathematics, Faculty of Science, Anadolu University, Eskişehir-Turkey

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Abstract: In this work, almost product and almost golden structures are studied. Conditions for those structures being integrable and parallel are investigated. Moreover, the harmonicity of a map between almost product or almost golden manifolds with pure or hyperbolic metric is discussed under certain conditions.

Key words: Golden structure, golden map, pure and hyperbolic metric, product structure, harmonic maps

1. Introduction

Let M^n be a smooth manifold of dimension n with a $(1,1)$-tensor field φ of rank n. Then (*see* [3, $7-9$, 12, 13]) the pair (M,φ) is called *polynomial manifold* provided $\mathcal{U}(\varphi) = 0$ for some polynomial $\mathcal{U}(x)$ over the field of real numbers $\mathbb{R}$. In particular, φ and (M,φ) are respectively called

i) *metallic structure* and *metallic manifold* if $\mathcal{U}(x) = x^2 - \eta x - \delta$ for some positive integers η and δ, so that $\mathcal{U}(\varphi) = \varphi^2 - \eta\varphi - \delta I = 0$.

ii) *almost complex structure* and *almost complex manifold* if $\mathcal{U}(x) = x^2 + 1$, so that $\mathcal{U}(\varphi) = \varphi^2 + I = 0$.

iii) *almost product structure* and *almost product manifold* if $\mathcal{U}(x) = x^2 - 1$. In this case we reserve the letter P for φ. Thus $\mathcal{U}(P) = P^2 - I = 0$.

iv) *almost golden structure* and *almost golden manifold* if $\mathcal{U}(x) = x^2 - x - 1$. In this case we reserve the letter G for φ. Thus $\mathcal{U}(G) = G^2 - G - I = 0$.

Here I denotes the identity tensor field.

Note here that an almost golden manifold (M,G) is in fact a metallic manifold with $\eta = \delta = 1$. Golden structure has been attracting more attention among many geometers (*see* : for example [3, 8, 12, 13]) in the last few years as it is closely related to the golden ratio, which plays an important role in various disciplines such as physics, topology, probability, and field theory (*see* [3, 8] and the references therein).

In this work, we have dealt with almost product and almost golden structures simultaneously as one can be obtained from the other, and provided the following results besides some side ones:

*Correspondence: serdemays@gmail.com
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Let φ denote either almost product structure P or almost golden structure G. To emphasize this we shall be writing $\varphi\,(=P,G)$. On an almost product or an almost golden manifold $(M,\ h,\ \varphi\,(=P,G))$ with pure or hyperbolic metric h, (Definition (2.1)) :

1) By analogy with the result for the paracomplex case, we introduce conditions $P\,(*)$ and $G\,(*)$ (*see page 12,* just before Proposition (2.5)), which, together with the integrability condition of φ, guarantee that φ is parallel (Proposition (2.5)).

2) For the bilinear operator $S_{\varphi}:\Gamma\,(TM)\times\Gamma\,(TM)\to\Gamma\,(TM)$ (see : right after Definition (2.4)) it is shown that vanishing of S_{φ} is equivalent to that of φ being parallel (Proposition (2.6)), unlike the case in which the metric h is hyperbolic, vanishing of S_{φ} does not imply that φ is parallel. Instead, it provides a bigger class whose members are called quasi para-Hermitian manifolds, quasi golden-Hermitian manifolds (Definition (2.5)).

3) We introduced a subclass of $(M,\ h,\ \varphi\,(=P,G))$, namely, a class of semidecomposable product (or golden) Riemannian manifolds (Definition (2.4)) and that was used later on for the harmonicity of a certain map (Theorems (3.1) & (3.2)).

4) By analogy with the concept of an *anti-paraholomorphic map*, a concept of *antigolden map* is introduced (Definition (3. 2)) and later it is used for its harmonicity (Theorems (3.1) & (3.2)).

5) It is shown that being a golden ($resp$: paraholomorphic) map of an almost golden ($resp$: almost product) manifold with a pure metric is no way sufficient for its harmonicity, whereas it is sufficient when the metric is hyperbolic. However, on the same line, an alternative result is provided Theorem (3.1).

6) Finally, (Theorems (3.1) & (3.2)), for a nonconstant map

$$F:(M,\ h,\ \varphi\,(=P,G))\to(N,\ g,\ \varphi\,(=Q,K))$$

(where h and g are hyperbolic), the harmonicity results given in [2, 6, 11] for $\pm\,(P,Q)$-paraholomorphic map F are extended to the cases where

- F is $\pm\,(P,Q)$-paraholomorphic and h is hyperbolic, g is pure.
- F is $\pm\,(G,K)$-golden and h is hyperbolic, g is pure or hyperbolic.

7) Overall, we have managed so far to express the results involving almost golden structures in terms of almost product structures.

2. Definitions and some basic results

The structure $\varphi\,(=P,G)$ on M^n has two distinct real eigenvalues, namely k and $\overline{k}$. Let denote the corresponding eigendistributions by $\mathcal{E}_{(k)}$ and $\mathcal{E}_{(\overline{k})}$.

Note that (see [2, 3, 7 − 9, 12, 13]),

1) $\varphi:TM\to TM$ is an isomorphism.

2) $TM = \mathcal{E}_{(k)} \oplus \mathcal{E}_{\left(\overline{k}\right)}$.

3) For an almost product manifold (M, P) we have

- $k = 1$ and $\overline{k} = -1$.
- $P^2(X) = X, \quad \forall \ X \in \Gamma(TM)$

4) For an almost golden manifold $(M,\ G)$ we have

- $k = \frac{1}{2}\left(1 + \sqrt{5}\right)$ and $\overline{k} = \frac{1}{2}\left(1 - \sqrt{5}\right)$.

Throughout this work we shall be setting

$$\sigma = \frac{1}{2}\left(1 + \sqrt{5}\right) \quad \text{and} \quad \bar{\sigma} = \frac{1}{2}\left(1 - \sqrt{5}\right).$$

Observe that

$$\sigma^2 = \sigma + 1, \quad \bar{\sigma}^2 = \bar{\sigma} + 1 \quad \text{and} \quad \sigma\bar{\sigma} = -1.$$

- $G^2(X) = GX + X, \quad \forall \ X \in \Gamma(TM)$.

5)
- For every almost product structure P, define a P*-associated* $(1,1)$*-tensor field* $G_P = \mathcal{G}$ by

$$G_p = \mathcal{G} = \frac{1}{2}\left(I + \sqrt{5}P\right)$$

- For every almost golden structure G, define a G-associated $(1,1)$-tensor field $P_G = \mathcal{R}$ by

$$P_G = \mathcal{R} = \frac{1}{\sqrt{5}}(2G - I)$$

Note that

i) for every almost product structure P on M, the corresponding $G_p = \mathcal{G}$ is an almost golden structure on M and therefore it will be called a P*-associated almost golden structure.*

ii) for every almost golden structure G on M, the corresponding $P_G = \mathcal{R}$ is an almost product structure on M and therefore it will be called a G*-associated almost product structure.*

iii) we have

$$\mathcal{E}^{\mathcal{G}}_{(\sigma)} = \mathcal{E}^{P}_{(1)} \qquad \textit{and} \qquad \mathcal{E}^{\mathcal{G}}_{(\bar{\sigma})} = \mathcal{E}^{P}_{(-1)}$$

iv) there is a one-to-one correspondence between the set of all almost product structures and the set of all almost golden structures on a manifold M. We shall be calling the pairs $\{P,\ G_P\}$ $\{G,\ P_G\}$) *an associated pair* or *a twin pair.* We also say that $\{P,\ G_P\}$ (or $\{G,\ P_G\}$) are *twins.* It is easy to see that for a given pair of twin structures $\{P,\ G_P\}$, the G_P-associated almost product structure is equal to P, that is,

$$P_{(G_P)} = P.$$

Similarly, for a twin pair $\{G,\ P_G\}$, the P_G-associated almost golden structure is equal to G, that is,

$$G_{(P_G)} = G.$$

v) If P is an almost product structure on M then $\widehat{P} = -P$ is also an almost product structure on M. Observe that P and $\widehat{P}$ have the same set $\{1, -1\}$ of eigenvalues. However, for their corresponding eigendistributions we have

$$\mathcal{E}^{P}_{(1)} = \mathcal{E}^{\widehat{P}}_{(-1)} \qquad \text{and} \qquad \mathcal{E}^{\widehat{P}}_{(1)} = \mathcal{E}^{P}_{(-1)}$$

We shall be calling $\widehat{P}$ *the conjugate almost product structure of* P or *the* P*-conjugate almost product structure.*

vi) If G is an almost golden structure on M then $\widehat{G} = I - G$ is also an almost golden structure on M. Observe that G and $\widehat{G}$ have the same eigenvalues σ and $\bar{\sigma}$. However, for their corresponding eigendistributions we have

$$\mathcal{E}^{G}_{(\sigma)} = \mathcal{E}^{\widehat{G}}_{(\bar{\sigma})}, \qquad \text{and} \qquad \mathcal{E}^{G}_{(\bar{\sigma})} = \mathcal{E}^{\widehat{G}}_{(\sigma)}$$

We shall be calling $\widehat{G}$ *the conjugate almost golden structure of* G or *the* G*-conjugate almost golden structure.*

vii) If $\{P,\ G\}$ is a twin pair then

$$\widehat{G} = G_{\widehat{P}} = \frac{1}{2}\left(I + \sqrt{5}\widehat{(P_G)}\right) = \frac{1}{2}\left(I - \sqrt{5}P_G\right),$$

that is, $\left\{\widehat{P},\ \widehat{G}\right\}$ is also a twin pair. Conversely, if $\left\{\widehat{P},\ \widehat{G}\right\}$ is a twin pair then $G = \frac{1}{2}\left(I + \sqrt{5}P\right)$, that is, $\{P,\ G\}$ is also a twin pair

viii) If $\left\{\widehat{P},\ \widehat{G}\right\}$ is a twin pair then

$$\mathcal{E}^{P}_{(1)} = \mathcal{E}^{\widehat{P}}_{(-1)} = \mathcal{E}^{G}_{(\sigma)} = \mathcal{E}^{\widehat{G}}_{(\bar{\sigma})} \qquad \text{and} \qquad \mathcal{E}^{\widehat{P}}_{(1)} = \mathcal{E}^{P}_{(-1)} = \mathcal{E}^{G}_{(\bar{\sigma})} = \mathcal{E}^{\widehat{G}}_{(\sigma)}.$$

An almost product manifold $(M,\ P)$ is called *an almost paracomplex manifold* if the eigendistributions $\mathcal{E}_{(1)}$, and $\mathcal{E}_{(-1)}$ are of the same rank [2, 9]. An almost golden manifold $(M,\ G)$ is called *an almost para-golden manifold* if the eigendistributions $\mathcal{E}_{(\sigma)}$ and $\mathcal{E}_{(\bar{\sigma})}$ are of the same rank. It is clear from their definitions that *an* almost paracomplex manifold $(M,\ P)$ and an almost para-golden manifold $(M,\ G)$ are necessarily of even dimensions.

Definition (2.1/A) *Let* M *be a smooth manifold together with a* $(1,1)$ *tensor field* $\varphi\,(= P, G)$ *and a Riemannian metric* h *satisfying*

$$h\,(\varphi X,\ Y) = h\,(X,\ \varphi Y)\,; \qquad \forall\ X,\ Y \in \Gamma\,(TM)\,. \qquad *$$

Then

i) $(M,\ h,\ P)$ is called *almost product Riemannian manifold,* [7].

ii) $(M,\ h,\ G)$ is called *almost golden Riemannian manifold,* [7, 8].

We refer to the condition $(*)$ as *the compatibility of* h *and* φ. We also say " h *is pure with respect to* φ " if h *and* φ are compatible, and call h *pure metric (with respect to* φ*).* Note here that the eigendistributions $\mathcal{E}_{(k)}$ and $\mathcal{E}_{\left(\bar{k}\right)}$ are h-orthogonal.

iii) An almost product Riemannian manifold $(M,\ h,\ P)$ and its metric h are also called *almost* **B**-*manifold* and **B**-*metric* respectively if the eigendistributions $\mathcal{E}_{(1)}$ and $\mathcal{E}_{(-1)}$ are of the same rank [12].

iv) An almost golden Riemannian manifold $(M,\ h,\ P)$ is also called an *almost para-golden Riemannian manifold* if the eigendistributions $\mathcal{E}_{(\sigma)}$, and $\mathcal{E}_{(\bar{\sigma})}$ are of the same rank.

Definition (2.1/B) *Let* M *be a smooth manifold together with a* $(1,1)$ *tensor field* $\varphi\,(=P,\ G)$ *and a nondegenerate metric* h *satisfying*

$$h\,(\varphi X,\ Y) = h\,(X,\ \widehat{\varphi} Y)\,; \qquad \forall\ X,\ Y \in \Gamma\,(TM)\,. \tag{$**$}$$

Then

i) $(M,\ h,\ P)$ is called an *almost para-Hermitian manifold* [2].

ii) $(M,\ h,\ G)$ is called an *almost golden-Hermitian manifold.*

In this case, we refer to the conditions $(**)$ as *the hyperbolic compatibility of* h *and* φ. We also say "h is *hyperbolic with respect to* φ" if h *and* φ are hyperbolic compatible, and call h *hyperbolic metric (with respect to* φ).

Note here that the hyperbolic case differs from the pure one. To be precise:

On a manifold $(M,\ h,\ \varphi)$ with a hyperbolic metric h (with respect to φ) one has

1) $h\,(PX,\ Y) = h\left(X,\ \widehat{P}Y\right) = -h\,(X,\ PY)\,; \qquad \forall\ X,\ Y \in \Gamma\,(TM)\,.$ Therefore, we have

$$h\,(PX,\ X) = 0, \qquad \forall\ X \in \Gamma\,(TM)$$

unlike the pure case where, for example,

$$h\,(PX,\ X) = h\,(X,\ X)\,, \qquad \forall\ X \in \Gamma\left(\mathcal{E}^P_{(1)}\right)$$

2) $h\,(GX,\ Y) = h\left(X,\ \widehat{G}Y\right); \quad \forall\ X,\ Y \in \Gamma\,(TM)$. Therefore, we have

$$2h\,(GX,\ X) = h\,(X,\ X) = 2h\left(\widehat{G}X,\ X\right), \qquad \forall\ X \in \Gamma\,(TM)\,.$$

3) $h(X, Y)=0; \quad \forall\, X, Y \in \Gamma\left(\mathcal{E}^{\varphi}_{(k)}\right) \quad or \quad \forall\, X, Y \in \Gamma\left(\mathcal{E}^{\varphi}_{(\bar{k})}\right)$.

That is, hyperbolic metric h is null on the eigendistributions $\mathcal{E}^{\varphi}_{(k)}$ and $\mathcal{E}^{\varphi}_{(\bar{k})}$ (and therefore the hyperbolic metric is necessarily semi-Riemannian whereas the pure metric needs not to be.)

Indeed, let $X, Y \in \Gamma\left(\mathcal{E}^{\varphi}_{(k)}\right)$ then $h(\varphi X, Y) = kh(X, Y)$ and $h(X, \widehat{\varphi}Y) = \bar{k}h(X, Y)$. On the other hand, $h(\varphi X, Y) = h(X, \widehat{\varphi}Y)$ since h is hyperbolic. Thus $kh(X, Y) = \bar{k}h(X, Y)$, which gives $\left(k-\bar{k}\right)h(X, Y) = 0$, so that $h(X, Y) = 0; \ \forall\, X, Y \in \Gamma\left(\mathcal{E}^{\varphi}_{(k)}\right)$. By the same argument we get $h(X, Y) = 0, \ \forall\, X, Y \in \Gamma\left(\mathcal{E}^{\varphi}_{(\bar{k})}\right)$.

Lemma (2.2/A) [2] *Let* (M, h, P) *be an almost para-Hermitian manifold. Then*

i) h is of signature (m, m) on TM, where $2m = \dim M$.

ii) $rank\left(\mathcal{E}^{P}_{(1)}\right) = rank\left(\mathcal{E}^{P}_{(-1)}\right) = m.$

Having given an almost golden manifold (M, h, G) with a hyperbolic metric h, since h is also hyperbolic with respect to the product structure P_{G}, by considering the almost para-Hermitian manifold (M, h, P_{G}) and using the above Lemma, we get:

Lemma (2.2/B) *Let* (M, h, G) *be an almost golden-Hermitian manifold. Then*

i) h is of signature (m, m) on TM, where $2m = \dim M$.

ii) $rank\left(\mathcal{E}^{G}_{(\sigma)}\right) = rank\left(\mathcal{E}^{G}_{(\bar{\sigma})}\right) = rank\left(\mathcal{E}^{\mathcal{R}}_{(1)}\right) = m,$ where $\mathcal{R} = P_{G}$.

Proposition 2.1 *Let an almost product structure* P *and an almost golden structure* G *form a twin pair* $\{P, G\}$ *on a smooth manifold* M. *For a nondegenerate metric* h *on* M *the following statements are equivalent:*

i) h *is pure* $[resp: \ hyperbolic]$ *with respect to* P.

ii) h *is pure* $[resp: \ hyperbolic]$ *with respect to* $\widehat{P}$.

iii) h *is pure* $[resp: \ hyperbolic]$ *with respect to* G.

iv) h *is pure* $[resp: \ hyperbolic]$ *with respect to* $\widehat{G}$.

Proof We are only showing the equivalence of (i) and (iv) as the rest of the cases follow by the similar argument:

Assume (i), then $\forall\ X,\ Y \in \Gamma(TM)$

$$\begin{aligned} h\left(X,\ \widehat{G}Y\right) &= h\left(X,\ \tfrac{1}{2}\left(I+\sqrt{5}\ \widehat{P}\right)Y\right) \\ &= \tfrac{1}{2}h(X,\ Y) + \tfrac{\sqrt{5}}{2}h\left(X,\ \ \widehat{P}Y\right) \\ &= \tfrac{1}{2}h(X,\ Y) - \tfrac{\sqrt{5}}{2}h(X,\ PY) \\ &= \tfrac{1}{2}h(X,\ Y) - \tfrac{\sqrt{5}}{2}h(PX,\ Y) \\ &= \tfrac{1}{2}h(X,\ Y) + \tfrac{\sqrt{5}}{2}h\left(\ \widehat{P}X,\ Y\right) \\ &= h\left(\tfrac{1}{2}\left(I+\sqrt{5}\ \widehat{P}\right)X,\ Y\right) \\ &= h\left(\widehat{G}X,\ Y\right) \end{aligned}$$

Next assume (iv) then $\forall\ X,\ Y \in \Gamma(TM)$

$$\begin{aligned} h(X,\ PY) &= -h\left(X,\ \ \widehat{P}Y\right) \\ &= -h\left(X,\ \tfrac{1}{\sqrt{5}}\left(2\widehat{G}-I\right)Y\right) - \left[-\tfrac{1}{\sqrt{5}}h(X,\ Y) + \tfrac{2}{\sqrt{5}}h\left(\widehat{G}X,\ Y\right)\right] \\ &= -\left[h\left(\tfrac{1}{\sqrt{5}}\left(2\widehat{G}-I\right)X,\ Y\right)\right] \\ &= -h\left(\widehat{P}X,\ Y\right) \\ &= h(PX,\ Y) \end{aligned}$$

We immediately get, from Proposition (2.1), the following $\square$

Proposition 2.2 *Let* ***an almost product structure*** P *and an almost golden structure* G *form a twin pair* $\{P,\ G\}$ *on a smooth manifold* M.

(**A**)*: The following statements are equivalent:*

i) $(M,\ h,\ P)$ *is an almost product Riemannian manifold.*

ii) $\left(M,\ h,\ \widehat{P}\right)$ *is an almost product Riemannian manifold.*

iii) $(M,\ h,\ G)$ *is an almost golden-Riemannian manifold.*

iv) $\left(M,\ h,\ \widehat{G}\right)$ *is an almost golden-Riemannian manifold.*

(**B**)*: The following statements are equivalent*

i) $(M,\ h,\ P)$ *is an almost para-Hermitian manifold.*

ii) $\left(M,\ h,\ \widehat{P}\right)$ *is an almost para-Hermitian manifold.*

iii) $(M,\ h,\ G)$ *is an almost golden-Hermitian manifold.*

iv) $\left(M,\ h,\ \widehat{G}\right)$ *is an almost golden-Hermitian manifold.*

Definition 2.1 *An almost product manifold* $(M,\ P)$ *and an almost golden manifold* $(M,\ G)$ *are said to be twins if* P *and* G *are twins (on the same manifold* M *).*

Remark 2.1 *It is obvious that* $(M,\ P)$ *and* $(M,\ G)$ *are twins if and only if* $\left(M,\ \widehat{P}\right)$ *and* $\left(M,\ \widehat{G}\right)$ *are twins.*

For an almost product (or golden) manifold $(M,\ \varphi)$, φ is said to be *integrable* if its Nijenhuis tensor field $\mathcal{N}_{\varphi}$ vanishes [3, 4].

That is, $\forall\ X,\ Y \in \Gamma(TM)$

$$\mathcal{N}_{\varphi}(X,\ Y) = \varphi^2[X,\ Y] + [\varphi X,\ \varphi Y] - \varphi[\varphi X,\ Y] - \varphi[X,\ \varphi Y] = 0,$$

For an almost product (or golden) manifold $(M,\ \varphi)$ with integrable φ we drop the adjective "*almost*" and then simply call it a *product (* or *golden) manifold.*

Lemma 2.1 [3], *For a twin pair* $\{P,\ G\}$ *on a manifold* M *with any linear connection* $\widetilde{\nabla}$ *one has*

$$5\mathcal{N}_P = 4\mathcal{N}_G \qquad \textit{and} \qquad \sqrt{5}\widetilde{\nabla}P = 2\widetilde{\nabla}G.$$

This lemma gives immediately:

Corollary 2.1 *Let* $\{P,\ G\}$ *be a twin pair on a manifold* M, *then we have:*

P *is integrable if and only if* $\widehat{P}$ *is integrable if and only if* $\widehat{G}$ *is integrable if and only if* G *is integrable.*

Lemma 2.2 $([4],\ Pg: 150-151)$ *For an almost product manifold* $(M,\ P)$,

i) There always exists a linear connection $\bar{\nabla}$ *on* M *with* $\bar{\nabla}P = 0$.

(Note that the condition that $\bar{\nabla}P = 0$, *does not imply the integrability of the almost product structure* P *unless* $\bar{\nabla}$ *is symmetric.)*

ii) For any symmetric linear connection $\check{\nabla}$ *on* M

$$\mathcal{N}_P(X,\ Y) = \left(\check{\nabla}_{PX}P\right)Y - \left(\check{\nabla}_{PY}P\right)X - P\left(\left(\check{\nabla}_X P\right)Y\right) + P\left(\left(\check{\nabla}_Y P\right)X\right)$$

and therefore if $\check{\nabla}P = 0$ *then* P *is integrable.*

iii) If P *is integrable then there always exists a symmetric linear connection* $\overset{s}{\nabla}$ *on* M *with* $\overset{s}{\nabla}P = 0$.

From Corollary (2.1) and Lemma (2.4) one gets:

Corollary 2.2 *Let* $\{P,\ G\}$ *be a twin pair of almost product and almost golden structures on a smooth manifold* $(M,\ h)$ *with a nondegenerate metric* h. *Then for the Levi-Civita connection* ∇ *on* $(M,\ h)$ *one has:*

A) *The following are equivalent:*

i) $\nabla P = 0$.

ii) $\nabla \widehat{G} = 0$.

iii) $\nabla G = 0$.

iv) $\nabla \widehat{P} = 0$.

B) *If* $\nabla P = 0$ *then* P, $\widehat{P}$, $\widehat{G}$, *and* G *are all integrable.*

Remark 2.2 *Note that*

i) the above Corollary is true regardless of whether h *is pure or hyperbolic or neither with respect to* P *(and therefore with respect to* $\widehat{P}$, G, *and* $\widehat{G}$*).*

ii) Integrability of $\varphi\,(= P,\ G)$ *does not imply that* φ *is parallel (with respect to the metric (Levi-Civita) connection).*

Let $(M,\ h,\ \varphi\,(= P,\ G))$ be an almost product (or golden) manifold with a metric h that is pure or hyperbolic with respect to φ. Then due to the above lemma $(2.4)/\,(iii)$, an integrable φ is always parallel with respect to some symmetric connection $\overset{s}{\nabla}$ anyway. However, $\overset{s}{\nabla} h = 0$ need not be true, that is, $\overset{s}{\nabla}$ need not be the Levi-Civita connection. The question here is what extra condition should be imposed so that integrability of φ, together with the imposed condition, guarantees that φ is parallel under the Levi-Civita connection? The answer to this question will differ depending on whether the metric h is pure or hyperbolic with respect to φ.

From here on, unless otherwise stated, the connections involved will be the Levi-Civita ones and denoted by ∇.

I : **The hyperbolic case:** Even though this case is well known for $\varphi = P$, (*see* [2, 11]), we will give an outline to some extent.

Let $(M,\ h,\ \varphi\,(= P,\ G))$ be an almost para-Hermitian manifold (so that h is hyperbolic) with its Levi-Civita connection ∇. Set $\forall X,\ Y \in \Gamma\,(TM)$

$$\Omega_P\,(X,\ Y) = \Omega\,(X,\ Y) = h\,(PX,\ Y)$$

Then Ω_P is a ($P-$associated) 2-form on M and it is called "*fundamental 2-form*" or "*para-Kaehler form*". The exterior differential $d\Omega$ is a 3-form on M, which is given by [5],

$$\begin{aligned} d\Omega\,(X,\ Y,\ Z) =& \nabla_X\,(\Omega\,(Y,\ Z)) - \nabla_Y\,(\Omega\,(X,\ Z)) + \nabla_Z\,(\Omega\,(X,\ Y)) \\ & - \Omega\,([X,\ Y]\,,\ \ Z) - \Omega\,([Y,\ Z]\,,\ \ X) + \Omega\,([X,\ Z]\,,\ \ Y) \end{aligned}$$

which can also be expressed as

$$d\Omega\,(X,\ Y,\ Z) = (\nabla_X \Omega)\,(Y,\ Z) - (\nabla_Y \Omega)\,(X,\ Z) + (\nabla_Z \Omega)\,(X,\ Y)\,. \tag{2.1}$$

Definition 2.2 $\mathbf{A}:$ [11]

i) An almost para-Hermitian manifold $(M,\ h,\ P)$ *is called almost para-Kaehler if its para-Kaehler form* Ω_P *is closed, i.e.* $d\Omega_P = 0$.

ii) An almost para-Kaehler manifold $(M,\ h,\ P)$ *with integrable* P *is called para-Kaehler.*

$\mathbf{B}:$

i) An almost golden-Hermitian manifold $(M,\ h,\ G)$ *is called almost golden-Kaehler if the para-Kaehler form* $\Omega_{\mathcal{R}}$ *is closed, i.e.* $d\Omega_{\mathcal{R}} = 0$, *where* $\mathcal{R} = P_G$ *is the* $G-$ *associated product structure and* $\Omega_{\mathcal{R}}$ *is the* $\mathcal{R}$*-associated 2-form.*

ii) An almost golden-Kaehler manifold $(M,\ h,\ G)$ *with integrable* G *is called golden-Kaehler.*

Proposition 2.3 *Let* $\{P,\ G\}$ *be twin structures on* (M, h) *with a hyperbolic metric* h *with respect to* P *(and therefore with respect to* G*). Then the following statements are equivalent:*

i) The manifold $(M,\ h,\ P)$ *is almost para-Kaehler, that is,* $d\Omega_P = 0$.

ii) The manifold $(M,\ h,\ G)$ *is an almost golden-Kaehler, that is,* $d\Omega_{\mathcal{R}} = 0$.

iii) The manifold $\left(M,\ h,\ \widehat{P}\right)$ *is an almost para-Kaehler, that is,* $d\Omega_{\widehat{P}} = 0$.

iv) The manifold $\left(M, h, \widehat{G}\right)$ *is an almost golden-Kaehler, that is,* $d\Omega_{\widehat{\mathcal{R}}} = 0$.

Proof The result follows from the fact that $\mathcal{R} = P_G = P$ since P and G are twins. □

Lemma 2.3 [9] *Let* $(M,\ h,\ P)$ *be an almost para-Hermitian manifold with its Levi-Civita connection* ∇ *and para-Kaehler form* Ω. *Then the following relation holds:* $\forall X,\ Y,\ Z \in \Gamma(TM)$

$$2h\left(\left(\nabla_X P\right) Y,\ Z\right) + 3d\Omega\left(X,\ Y,\ Z\right) + 3d\Omega\left(X,\ PY,\ PZ\right) + h\left(\mathcal{N}_P\left(Y,\ Z\right),\ PX\right) = 0.$$

Proposition 2.4 *Let* $(M,\ h,\ \varphi\,(= P,\ G))$ *be an almost para-Hermitian or an almost golden-Hermitian manifold.*

$A:$ *Then the following are equivalent:*

i) P *is parallel with respect to the Levi-Civita connection* ∇*, that is* $\nabla P = 0$.

ii) M *is para-Kaehler (that is,* $N_P = 0$ *and* $d\Omega_P = 0$*).*

$B:$ *Then the following are equivalent:*

i) G *is parallel with respect to the Levi-Civita connection* ∇*, that is,* $\nabla G = 0$.

ii) M *is golden-Kaehler (that is* $N_G = 0$ *and* $d\Omega_{\mathcal{R}} = 0$*), where* $R = P_G$ *is the* G*-associated product structure.*

Proof **A** : ([11])

(i) $\Rightarrow (ii)$: Note that

$$\nabla (\Omega (X,\ Y)) = (\nabla\Omega)(X,\ Y) + \Omega (\nabla X,\ Y) + \Omega (X,\ \nabla Y) \tag{2.2}$$

On the other hand, since $\nabla P = 0$ and $\nabla h = 0$, we have

$$\begin{aligned} \nabla (\Omega (X,\ Y)) &= \nabla (h (PX,\ Y)) \\ &= (\nabla h)(PX,\ Y) + h (\nabla (PX),\ Y) + h (PX,\ \nabla Y) \\ &= h (P (\nabla X),\ Y) + h (PX,\ \nabla Y) \\ &= \Omega (\nabla X,\ Y) + \Omega (X,\ \nabla Y) \end{aligned} \tag{2.3}$$

However, then the equalities (2.2) and (2.3) give us that $\nabla\Omega = 0$. Thus, from (2.1), we get $d\Omega_P = 0$. The equality $\mathcal{N}_P = 0$ follows from Lemma (2.3).

(ii) $\Rightarrow (i)$: This follows directly from Lemma (2.3).

B : Since $\{\mathcal{R},\ G\}$ is a twin pair on M, the required equivalence follows from part (A). □

Remark 2.3 *Let J be a $(1,\ 1)$-tensor field with $J^2 = -I$ on a Riemannian manifold $(M,\ g)$, where g is hyperbolic with respect to J and $\widehat{J} = -J$ is the conjugate of J. Then J and $(M,\ g,\ J)$ are called almost complex structure and almost Hermitian manifold respectively. In this case it is well known that Proposition $(2.4)/A$ is also valid when P is replaced by J*

II : **The pure case**:

Let $(M,\ h,\ \varphi\,(= P,\ G))$ be an almost product (or an almost golden) Riemannian manifold (so that h is pure) with its Levi-Civita connection ∇. The so-called "Tachibana operator"

$$\phi_\varphi : \Im_2^0 (M) \to \Im_3^0 (M)$$

from the set of all $(0,2)-$tensor fields into the set of all $(0,3)-$tensor fields over M is given by (*see* [8, 12]) : $\forall\ u \in \Im_2^0 (M)$ and $\forall\ X,\ Y,\ Z \in \Gamma (TM)$

$$(\phi_\varphi u)(X,\ Y,\ Z) = \begin{array}{l} (\varphi X)(u (Y,\ Z)) - X (u (\varphi Y,\ Z)) \\ +u ((\mathcal{L}_Y \varphi) X,\ Z) + u (Y,\ (\mathcal{L}_Z \varphi) X) \end{array}$$

where $\mathcal{L}\varphi$ is the Lie derivative of φ.

In particular, for the pure metric h with respect to φ, the above equality takes the form (*see* [7, 12]) : $\forall\ X,\ Y,\ Z \in \Gamma (TM)$

$$(\phi_\varphi h)(X,\ Y,\ Z) = -h ((\nabla_X \varphi) Y,\ Z) + h ((\nabla_Y \varphi) X,\ Z) + h ((\nabla_Z \varphi) X,\ Y).$$

Now let us define another operator

$$\mathbf{\Psi}_\varphi : \Im_2^0 (M) \to \Im_3^0 (M)$$

by $\forall\ X,\ Y,\ Z \in \Gamma(TM)$

$$(\mathbf{\Psi}_{\varphi} u)(X,\ Y,\ Z) = (\phi_{\varphi} u)(X,\ Y,\ Z) + (\phi_{\varphi} u)(Z,\ Y,\ X)$$

Lemma 2.4 ([7, 12]) *Let* $(M,\ h,\ \varphi(= P,\ G))$ *be an almost product (or an almost golden) Riemannian manifold with its Levi-Civita connection* ∇*. Then*

i) $\forall\ X,\ Y,\ Z \in \Gamma(TM)$

$$(\mathbf{\Psi}_{\varphi} h)(X,\ Y,\ Z) = 2h\left((\nabla_{Y} \varphi) X,\ Z\right)$$

ii) The following are equivalent:

a°) $(\mathbf{\Psi}_{\varphi} h) = 0$

b°) $(\phi_{\varphi} h) = 0$

c°) $\nabla \varphi = 0$

Proposition 2.5 *Let* $(M,\ h,\ \varphi(= P,\ G))$ *be an almost product (or an almost golden) Riemannian manifold. Then*

A : The following are equivalent:

i) P is parallel with respect to the Levi-Civita connection ∇, that is, $\nabla P = 0$.

ii) P is integrable (that is, $\mathcal{N}_P = 0$) and the condition that $\forall\ X,\ Y,\ Z \in \Gamma(TM)$

$$(\mathbf{\Psi}_{P} h)(X,\ Y,\ Z) = (\mathbf{\Psi}_{P} h)(Y,\ X,\ Z) + (\mathbf{\Psi}_{P} h)(PY,\ PX,\ Z), \qquad (P(*))$$

holds.

B : The the following are equivalent:

i) G is parallel with respect to the Levi-Civita connection ∇, that is, $\nabla G = 0$.

ii) G is integrable, that is, $\mathcal{N}_G = 0$ and the condition that $\forall\ X,\ Y,\ Z \in \Gamma(TM)$

$$(\mathbf{\Psi}_{\mathcal{R}} h)(X,\ Y,\ Z) = (\mathbf{\Psi}_{\mathcal{R}} h)(Y,\ X,\ Z) + (\mathbf{\Psi}_{\mathcal{R}} h)(\mathcal{R}Y,\ \mathcal{R}X,\ Z), \qquad (G(*))$$

holds.

Here $\mathcal{R} = P_G$ is the twin product structure of G, so that $\{G,\ \mathcal{R}\}$ form a twin pair.

Proof A :

(i) $\Rightarrow (ii)$: Since $\nabla P = 0$ by the assumption, we have from Lemma (2.4) / (ii) that $\mathcal{N}_P = 0$, and from Lemma (2.6) the condition $P(*)$ follows.

(ii) $\Rightarrow (i)$: From Lemma (2.4) / (ii) we have: $\forall\ X,\ Y,\ Z \in \Gamma(TM)$

$$\begin{aligned}\mathcal{N}_P(X,\ Y) &= (\nabla_{PX}P)Y-(\nabla_{PY}P)X-P((\nabla_X P)Y)+P((\nabla_Y P)X)\\ &= (\nabla_{PX}P)Y-(\nabla_{PY}P)X+(\nabla_X P)(PY)-(\nabla_Y P)(PX)\end{aligned}$$

Hence

$$\begin{aligned}h(\mathcal{N}_P(X,\ Y),\ Z) &= h((\nabla_{PX}P)Y,\ Z)-h((\nabla_{PY}P)X,\ Z)\\ &+h((\nabla_X P)(PY),\ Z)-h((\nabla_Y P)(PX),\ Z)\end{aligned}$$

That is,

$$\begin{aligned}&h(\mathcal{N}_P(X,\ Y),\ Z)-h((\nabla_X P)(PY),\ Z)\\ &= h((\nabla_{PX}P)Y,\ Z)-h((\nabla_{PY}P)X,\ Z)-h((\nabla_Y P)(PX),\ Z).\end{aligned}$$

Then using Lemma (2.6) we get $\forall\ X,\ Y,\ Z\in\Gamma(TM)$

$$\begin{aligned}&2\{h(\mathcal{N}_P(X,\ Y),\ Z)-h((\nabla_X P)(PY),\ Z)\}\\ &= (\boldsymbol{\Psi}_P h)(Y,\ PX,\ Z)-(\boldsymbol{\Psi}_P h)(X,\ PY,\ Z)-(\boldsymbol{\Psi}_P h)(PX,\ Y,\ Z)\quad .\end{aligned}$$

Exchanging X with Y this equation reads:

$$\begin{aligned}&2\{h(\mathcal{N}_P(Y,\ X),\ Z)-h((\nabla_Y P)(PX),\ Z)\}\\ &= (\boldsymbol{\Psi}_P h)(X,\ PY,\ Z)-(\boldsymbol{\Psi}_P h)(Y,\ PX,\ Z)-(\boldsymbol{\Psi}_P h)(PY,\ X,\ Z)\quad .\end{aligned}$$

Then putting Y for PY in the last equation (doing this does not alter the equation since P is an isomorphism) we get $\forall\ X,\ Y,\ Z\in\Gamma(TM)$

$$\begin{aligned}&2\{h(\mathcal{N}_P(PY,\ X),\ Z)-h((\nabla_{PY}P)(PX),\ Z)\}\\ &= (\boldsymbol{\Psi}_P h)(X,\ Y,\ Z)-(\boldsymbol{\Psi}_P h)(PY,\ PX,\ Z)-(\boldsymbol{\Psi}_P h)(Y,\ X,\ Z)\quad .\end{aligned}$$

However, then under the assumptions that $\mathcal{N}_P=0$ and the condition $P(*)$ holds, the last equation gives us that

$$h((\nabla_{PY}P)(PX),\ Z)=0,\ \ \forall\ X,\ Y,\ Z\in\Gamma(TM),$$

which means that $\nabla P=0$.

B :

(i) $\Rightarrow(ii)$: By the assumption, $\nabla G=0$ and therefore $\nabla\mathcal{R}=0$. Thus by part (A) above, we get $\mathcal{N}_{\mathcal{R}}=0$, and therefore $\mathcal{N}_G=0$ by Lemma (2.3). Moreover, by part (A), we get that the condition $G(*)$ holds.

(ii) $\Rightarrow(i)$: By the assumption, $\mathcal{N}_G=0$ and therefore $\mathcal{N}_{\mathcal{R}}=0$ and the condition $G(*)$ holds. Thus by part (A), we get $\nabla\mathcal{R}=0$, and therefore $\nabla G=0$ by Corollary (2.2).

□

For an almost product (or an almost golden) manifold $(M,\ h,\ \varphi)$ with a pure or hyperbolic metric h with respect to φ, and with its Levi-Civita connection ∇, the divergence $\mathrm{div}\varphi$ of φ is given by [5]

$$\mathrm{div}\varphi=\sum_{i=1}^{m}h_{ii}(\nabla_{e_i}\varphi)e_i.$$

Here $\{e_1,...,e_m\}$ is a local orthonormal frame field for $\Gamma(TM)$ and $h_{ii}=h(e_i,\ e_i)$.

Definition 2.3 $(\mathbf{A})$: *An almost product Riemannian manifold* $(M,\ h,\ P)$ *with its Levi-Civita connection* ∇ *is called*

i) locally product Riemannian manifold if P *is integrable* [7].

ii) almost decomposable product Riemannian manifold if $P(*)$ *holds.*

iii) locally decomposable product Riemannian manifold if P *is integrable and* $P(*)$ *holds (that is,* P *is parallel)* [7]. *In particular, if* $(M,\ h,\ P)$ *is a* $\mathbf{B}$*-manifold (resp : almost* $\mathbf{B}$*-manifold) holding the condition* $P(*)$ *then it is also called para-holomorphic* $\mathbf{B}$*-manifold*[12] *(resp : almost para-holomorphic* B*-manifold). Note here that by virtue of Proposition* (2.5), *if* $(M,\ h,\ P)$ *is a para-holomorphic* $\mathbf{B}$*-manifold then* $\nabla P = 0$, *i.e.* P *is parallel.*

iv) Semidecomposable product Riemannian manifold if $\mathrm{div} P = 0$

$(\mathbf{B})$: *An almost golden Riemannian manifold* $(M,\ h,\ G)$ *with its Levi-Civita connection* ∇, *is called*

i) locally golden Riemannian manifold if G *is integrable* [7].

ii) almost decomposable golden Riemannian manifold if $G(*)$ *holds.*

iii) locally decomposable golden Riemannian manifold if G *is integrable and* $G(*)$ *holds (that is,* G *is parallel)* [7].

iv) Semidecomposable golden Riemannian manifold if $\mathrm{div} G = 0$.

Define a bilinear map [6],

$$S_\varphi : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

on a manifold $(M,\ h,\ \varphi(=P,\ G))$ *with the Levi-Civita connection* ∇ *by*

$$S_\varphi(X,\ Y) = (\nabla_X \varphi) Y + \varphi\left(\nabla_{\varphi X} \varphi\right) Y \qquad \forall\ X,\ Y \in \Gamma(TM).$$

Lemma 2.5 $(\mathbf{A})$: *For* $\varphi = P$ *and* $\forall\ X,\ Y \in \Gamma(TM)$ *we have*

i)

$$S_P(X,\ Y) = (\nabla_X P) Y - (\nabla_{PX} P)(PY).$$

ii)

$$P(S_P(X,\ Y)) = -S_P(X,\ PY) = S_P(PX,\ Y).$$

iii)

$$S_P(X,\ Y) = 0,\ \ \forall\ X,\ Y \in \Gamma\left(\mathcal{E}_{(1)}\right) \ \ \text{and} \ \ S_P(X,\ Y) = 0,\ \forall\ X,\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right).$$

$(\mathbf{B})$: *For* $\varphi = G$ *and* $\forall\ X,\ Y \in \Gamma(TM)$ *we have*

i)

$$S_G(X,\ Y) = (\nabla_X G) Y - (\nabla_{GX} G)(GY) + (\nabla_{GX} G) Y.$$

ii)

$$S_G(X,\ Y) = 0, \quad \forall\ X,\ Y \in \Gamma\left(\mathcal{E}_{(\sigma)}\right) \quad \textit{and} \quad S_G(X,\ Y) = 0, \quad \forall\ X,\ Y \in \Gamma\left(\mathcal{E}_{(\bar{\sigma})}\right).$$

Proof Using the fact that

$$P((\nabla P)X) = -\ (\nabla P)(PX) \quad \text{and} \quad G((\nabla G)Y) = -(\nabla G)(GY) + (\nabla G)Y$$

we get $A/(i)$ and $B/(i)$. Next, $A/(ii)$ and $A/(iii)$ are easy. For $B/(ii)$ let $X,\ Y \in \Gamma\left(\mathcal{E}_{(\sigma)}\right)$, then

$$\begin{aligned} S_G(X,\ Y) &= (\nabla_X G)Y - (\nabla_{GX} G)(GY) + (\nabla_{GX} G)Y \\ &= (\nabla_X G)Y - \sigma^2 (\nabla_X G)(Y) + \sigma(\nabla_X G)Y \\ &= \left(1 - \sigma^2 + \sigma\right)(\nabla_X G)Y = 0, \quad \textit{since} \quad \sigma^2 = 1 + \sigma. \end{aligned}$$

By the same argument we also get that

$$S_G(X,\ Y) = 0, \qquad \forall\ X,\ Y \in \Gamma\left(\mathcal{E}_{(\bar{\sigma})}\right),$$

which completes the proof. □

Lemma 2.6 (*c.f* [6]) *On an almost product Riemannian or an almost para Hermitian manifold* $(M,\ h,\ P)$, *the following statements are equivalent:*

i) $S_P(X,\ Y) = 0 \qquad \forall\ X,\ Y \in \Gamma(TM)$.

ii) $S_P(X,\ PX) = 0 \qquad \forall\ X \in \Gamma(TM)$.

iii) $S_P(X,\ X) = 0 \qquad \forall\ X \in \Gamma(TM)$.

Proposition 2.6 *For an almost product Riemannian manifold* $M_P = (M,\ h,\ P)$ *and an almost golden Riemannian manifold* $M_G = (M,\ h,\ G)$,

$(\mathbf{A})$: *on* M_P

i) $\nabla P = 0$ *if and only if* $S_P = 0$ *if and only if* $(\mathbf{\Psi}_P h) = 0$ *if and only if* $(\phi_P h) = 0$.

ii) $\nabla G = 0$ *if and only if* $S_G = 0$ *if and only if* $(\mathbf{\Psi}_G h) = 0$ *if and only if* $(\phi_G h) = 0$.

$(\mathbf{B})$: *If* M_P *and* M_G *are twin manifolds then the following are equivalent:*

i) $\nabla P = 0$ *on* M_P.

ii) $S_P = 0$ *on* M_P.

iii) $\nabla G = 0$ *on* M_G.

iv) $S_G = 0$ *on* M_G.

v) $\nabla \widehat{G} = 0 \quad on \quad M_{\widehat{G}}.$

vi) $S_{\widehat{P}} = 0 \quad on \quad M_{\widehat{P}}.$

vii) $\nabla \widehat{P} = 0 \quad on \quad M_{\widehat{P}}.$

Proof $(\mathbf{A})$:

(i) If $\nabla P = 0$ then obviously $S_P = 0.$

Conversely, assume that $S_P = 0$. Then for $X \in \Gamma\left(\mathcal{E}_{(1)}\right)$ and $Y \in \Gamma\left(\mathcal{E}_{(-1)}\right)$

$$S_P\left(Y,\ X\right) = \left(\nabla_X P\right)Y - \left(\nabla_{PX} P\right)PY = 2\left(\nabla_X P\right)Y = 0,$$

which gives

$$\left(\nabla_X P\right)Y = 0; \qquad \forall\ X \in \Gamma\left(\mathcal{E}_{(1)}\right) \text{ and } \forall\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right). \tag{2.4}$$

By a similar argument we get

$$\left(\nabla_Y P\right)X = 0; \qquad \forall\ X \in \Gamma\left(\mathcal{E}_{(1)}\right) \text{ and } \forall\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right). \tag{2.5}$$

From (2.4) we get

$$\left(\nabla_X P\right)Y = \nabla_X\left(PY\right) - P\left(\nabla_X Y\right) = -\nabla_X Y - P\left(\nabla_X Y\right) = 0.$$

Thus,

$$\nabla_X Y \in \Gamma\left(\mathcal{E}_{(-1)}\right) \qquad \forall\ X \in \Gamma\left(\mathcal{E}_{(1)}\right) \text{ and } \forall\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right). \tag{2.6}$$

On the other hand, $\forall\ X,\ Z \in \Gamma\left(\mathcal{E}_{(1)}\right)$ and $\forall\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right)$

$$X\left(h\left(Y, Z\right)\right) = h\left(\nabla_X Y,\ Z\right) + h\left(Y,\ \nabla_X Z\right) = 0, \qquad \text{since} \quad h\left(Y, Z\right) = 0.$$

Using (2.6), this gives $h\left(Y,\ \nabla_X Z\right) = 0, \quad \forall\ Y \in \Gamma\left(\mathcal{E}_{(-1)}\right)$ and therefore $\nabla_X Z \in \Gamma\left(\mathcal{E}_{(1)}\right)$. However, then

$$P\left(\nabla_X Z\right) = \nabla_X Z = \nabla_X\left(PZ\right)$$

which gives

$$\left(\nabla_X P\right)Z = 0; \qquad \forall\ X,\ Z \in \Gamma\left(\mathcal{E}_{(1)}\right) \tag{2.7}$$

By a similar argument we also get

$$\left(\nabla_X P\right)Z = 0; \qquad \forall\ X,\ Z \in \Gamma\left(\mathcal{E}_{(-1)}\right) \tag{2.8}$$

However, then (2.4), (2.5), (2.7), and (2.8) give us that $\nabla P = 0$, i.e. P is parallel. The rest of the statements in (i) will follow from Lemma (2.6).

(ii) This will follow by mimicking the arguments used in (i).

$(\mathbf{B})$: Now, observing that $S_P = -S_{\widehat{P}}$, $\nabla P = -\nabla \widehat{P}$, and $\nabla G = -\frac{\sqrt{5}}{2}\nabla \widehat{P} = -\nabla \widehat{G}$, together with the part (A), proofs of the statements (i) to (vii) in part B will easily follow. □

For an almost para-Hermitian manifold $M_P = (M,\ h,\ P)$ and an almost golden Hermitian manifold $M_G = (M,\ h,\ G)$ (note that here the metric is hyperbolic with respect to the indicated structures rather than pure) we do not have Proposition $(2.6/A)$ type of results. Instead, some conditions on the operator S_φ, with $\varphi\,(= P,\ G)$ induce some extra subclasses of those manifolds. To be precise:

Definition 2.4 $(\mathbf{A})$: *An almost para-Hermitian manifold* $(M,\ h,\ P)$ *with its Levi-Civita connection* ∇ *is said to be,* [6],

i) nearly para-Kaehler if $(\nabla_X P)\, X = 0, \quad \forall\ X \in \Gamma\,(TM)$

ii) quasi para-Kaehler if $S_P = 0.$

iii) semi para-Kaehler if $\sum_{i=1}^{m} h_{ii} S_P\,(e_i,\ e_i) = 0$ *(equivalently,* $\operatorname{div}(P) = 0$*), where* $\{e_1, ..., e_m;\ Pe_1, ..., Pe_m\}$ *is a local orthonormal frame field for* $\Gamma\,(TM)$ *and* $h_{ii} = h\,(e_i,\ e_i)$.

$(\mathbf{B})$: *An almost golden-Hermitian manifold* $(M,\ h,\ G)$ *with its Levi-Civita connection* ∇ *is said to be*

i) nearly golden-Kaehler if $(\nabla_X G)\, X = 0, \quad \forall\ X \in \Gamma\,(TM)$

ii) quasi golden-Kaehler if $S_{\mathcal{R}} = 0,$ *where* $\mathcal{R} = P_G$, G*-associated product structure.*

iii) semi golden-Kaehler if $\sum_{i=1}^{m} h_{ii} S_{\mathcal{R}}\,(e_i,\ e_i) = 0$ *(equivalently,* $\operatorname{div}(\mathcal{R}) = 0$ *or* $\operatorname{div}(G) = 0$*) where* $\{e_1, ..., e_m;\ \mathcal{R}e_1, ..., \mathcal{R}e_m\}$ *is a local orthonormal frame field for* $\Gamma\,(TM)$ *and* $h_{ii} = h\,(e_i,\ e_i)$.

3. Harmonicity

Definition 3.1 *A distribution* D *over a (semi) Riemannian manifold* $(M,\ h)$ *with its Levi-Civita connection* ∇, *is said to be*

i) (c.f. [10]*) Vidal if*

$$\nabla_X X \in D, \quad \forall\ X \in \Gamma\,(D).$$

ii) [1] *critical if*

$$\sum_{i=1}^{n} h_{ii} \nabla_{v_i} v_i \in D$$

If the restriction $h\,|_D$ of h to D is positive (or negative) definite then the critical distribution D is also called *minimal*. Here $\{v_1, ..., v_n\}$ is a local orthonormal frame field for D and $h_{ii} = h\,(v_i,\ v_i)$.

Remark 3.1 *1) For an almost product (or an almost golden) Riemannian manifold* $(M,\ h,\ \varphi\,(= P,\ G))$

i) every Vidal distribution is critical.

ii) if φ *is parallel then the eigendistributions* $\mathcal{E}_{(k)}$ *and* $\mathcal{E}_{(\bar{k})}$ *of* φ *are both Vidal and therefore they are minimal. Here* $k=1$, $\bar{k}=-1$ *for* $\varphi=P$ *and* $k=\sigma$, $\bar{k}=\bar{\sigma}$ *for* $\varphi=G$

iii) the eigendistributions $\mathcal{E}_{(1)}$ *and* $\mathcal{E}_{(-1)}$ *of* P *(resp :* $\mathcal{E}_{(\sigma)}$ *and* $\mathcal{E}_{(\bar{\sigma})}$ *of* G *) are both minimal if and only if* $\mathrm{div}P=0$, *that is,* $(M,\ h,\ P)$ *is semidecomposable product Riemannian (resp :* $\mathrm{div}G=0$, *that is,* $(M,\ h,\ G)$ *semidecomposable golden Riemannian) manifold*

2) For an almost product (or an almost golden) manifold $(M,\ h,\ \varphi\,(=P,\ G))$ *with a pure or hyperbolic metric* h, *if* $\{P,\ G\}$ *is a twin pair then the following are equivalent:*

i) $\mathrm{div}P=0$

ii) $\mathrm{div}G=0$

Lemma 3.1 *Let* $F:(M,\ \varphi_{_M})\to(N,\ \varphi_{_N})$ *be a smooth map with its differential map* $F_*:TM\to TN$, *where* $\varphi\,(=P,\ G)$.

i) If $F_*\circ P_M=G_N\circ F_*$ *or* $F_*\circ G_M=P_N\circ F_*$ *then* F *is constant.*

ii) If any one of the following

- $F_*\circ P_M=\widehat{G}_N\circ F_*$,
- $F_*\circ \widehat{P}_M=\widehat{G}_N\circ F_*$,
- $F_*\circ \widehat{P}_M=G_N\circ F_*$,
- $F_*\circ G_M=\widehat{P}_N\circ F_*$,
- $F_*\circ \widehat{G}_M=\widehat{P}_N\circ F_*$,
- $F_*\circ \widehat{G}_M=P_N\circ F_*$

holds then F *is constant*

Proof The statement (i) is treated in ([13], Theorem 7&8). The argument used in [13] works for all the cases in (ii) □

Definition 3.2 *A smooth map* $F:(M,\ \varphi_{_M})\to(N,\ \varphi_{_N})$ *with its differential map* $dF=F_*:TM\to TN$ *is said to be*

i) ([2, 6]), $(P_M,\ P_N)$*-paraholomorphic,* [*resp :* $(P_M,\ P_N)$ *-anti-paraholomorphic*] *if*

$$F_*\circ P_M=P_N\circ F_*,\qquad\qquad \Big[resp:\ F_*\circ P_M=\widehat{P}_N\circ F_*=-P\circ F_*\Big]$$

ii)

- [13], $(G_M,\ G_N)$*-golden if*

$$F_*\circ G_M=G_N\circ F_*$$

- $(G_M,\ G_N)$*-antigolden if*

$$F_* \circ G_M = \widehat{G}_N \circ F_* = I_N - G_N F_*,$$

where $\widehat{\varphi}\left(=\widehat{P},\ \widehat{G}\right)$ *is the conjugate of* φ.

We shall be writing $\pm(P_M,\ P_N)$*-paraholomorphic to mean either* $(P_M,\ P_N)$*-paraholomorphic or* $(P_M,\ P_N)$*-anti-paraholomorphic. Similarly, we shall be writing* $\pm(G_M,\ G_N)$*-golden to mean either* $(G_M,\ G_N)$*-golden or* $(G_M,\ G_N)$*-antigolden.*

Note that since $\widehat{\widehat{\varphi}} = \varphi$, *we have:*

If a map $F : (M,\ \varphi_M) \to (N,\ \varphi_N)$ *is* $(P_M,\ P_N)$*-paraholomorphic then it is* $\left(P_M,\ \widehat{P}_N\right)$*-anti-paraholomorphic as a map* $F : (M,\ P_M) \to \left(N,\ \widehat{P}_N\right)$*, and if* $(G_M,\ G_N)$*-golden then it is* $\left(G_M,\ \widehat{G}_N\right)$*-antigolden as a map* $F : (M,\ G_M) \to \left(N,\ \widehat{G}_N\right)$*. Conversely, if a map* $F : (M,\ \varphi_M) \to (N,\ \varphi_N)$ *is* $(P_M,\ P_N)$*-anti-paraholomorphic then it is* $\left(P_M,\ \widehat{P}_N\right)$*-paraholomorphic as a map* $F : (M,\ P_M) \to \left(N,\ \widehat{P}_N\right)$*, and if* $(G_M,\ G_N)$*-antigolden then it is* $\left(G_M,\ \widehat{G}_N\right)$*-golden as a map* $F : (M,\ G_M) \to \left(N,\ \widehat{G}_N\right)$*.*

Proposition 3.1 *For twin pairs* $\{P_M,\ G_M\}$ *and* $\{P_N,\ G_N\}$ *let*

$$F : (M,\ \varphi_M(= P_M,\ G_M)) \to (N,\ \varphi_N(= P_N,\ G_N))$$

be a smooth map. Then the following statements are equivalent:

i) F *is* $(P_M,\ P_N)$*-paraholomorphic* $[resp : (P_M,\ P_N) - \text{anti-paraholomorphic}]$.

ii) F *is* $(G_M,\ G_N)$*-golden* $[resp : (G_M,\ G_N)\text{-antigolden}]$.

Proof

$$F \text{ is } (G_M,\ G_N)\text{-antigolden}$$

$\Longleftrightarrow$

$$F_* \circ G_M = \widehat{G}_N \circ F_*$$

$\Longleftrightarrow$

$$F_* \circ (tI + rP_M) = \left(tI + r\widehat{P}_N\right) \circ F_*$$

$\Longleftrightarrow$

Since $\{P_M,\ G_M\}$ is a twin pair and then so is $\left\{\widehat{P}_M,\ \widehat{G}_M\right\}$, so that

$$G_M = tI + rP_M \text{ and } \widehat{G}_M = tI + r\widehat{P}_M, \quad \text{where} \quad t = \frac{1}{2}, \quad r = \frac{\sqrt{5}}{2}$$

$\Longleftrightarrow$

$$tF_* + r(F_* \circ P_M) = tF_* + r\left(\widehat{P}_N \circ F_*\right)$$

$\Longleftrightarrow$

$$F_* \circ P_M = \widehat{P}_N \circ F_*$$

$\Longleftrightarrow$

$$F \quad \text{is} \quad (P_M,\ P_N)\text{-}\textit{anti-paraholomorphic}$$

The rest of the cases can be shown similarly. □

Let $F : (M,\ h) \to (N,\ g)$ be a smooth map between (semi) Riemannian manifolds. *The second fundamental form*

$$\nabla F_* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TN)$$

of F is given by $\forall\ X,\ Y \in \Gamma(TM)$

$$(\nabla F_*)(X,\ Y) = \overset{N}{\nabla}_{(F_*X)}(F_*Y) - F_*\left(\overset{M}{\nabla}_X Y\right),$$

where $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$ are the Levi-Civita connections on M and N respectively. Note that the map ∇F_* is bilinear and symmetric (*see* [1,5]).

For a given distribution D over a (*semi*) Riemannian manifold $(M,\ h)$, the *D-tension field* $\mathcal{T}_D(F)$ of $F : (M,\ h) \to (N,\ g)$ is given by (*c.f.* [1,5, 6])

$$\mathcal{T}_D(F) = \sum_{i,j=1}^{s} h^{ij}(\nabla F_*)(e_i,\ e_j) \in \Gamma(TN), \tag{3.1}$$

where $\{e_1, ..., e_s\}$ is a local frame field for D and $\left(h^{ij}\right) = (h_{ij})^{-1}$, $h_{ij} = h(e_i,\ e_j)$. In particular, if $\{e_1, ..., e_s\}$ is a local h-orthonormal frame field for D then the expression (3.1) takes the form

$$\mathcal{T}_D(F) = \sum_{i=1}^{s} h^{ii}(\nabla F_*)(e_i,\ e_i) \in \Gamma(TN) \tag{3.2}$$

In the cases where $D = TM$, we simply write $\mathcal{T}(F)$ for $\mathcal{T}_{TM}(F)$ and call it the "*tension field of* F"

Definition 3.3 (*c.f.* [1, 5]) *A smooth map* $F : (M,\ h) \to (N,\ g)$ *is said to be harmonic* [*resp* : *D-harmonic*] *if its tension field* [*resp* : *D-tension field*] *vanishes. In particular,*

- *for a map* $F : (M,\ h,\ P) \to (N,\ g)$ *from an almost product Riemannian manifold* M, *if* $D = \overset{M}{\mathcal{E}}_{(1)}\left[resp : D = \overset{M}{\mathcal{E}}_{(-1)}\right]$ *then D-harmonic F is also called plus-eigen harmonic* [*resp* : *minus-eigen harmonic*].
- *for a map* $F : (M,\ h,\ G) \to (N,\ g)$ *from an almost golden Riemannian manifold* M, *if* $D = \overset{M}{\mathcal{E}}_{(\sigma)}\left[resp : D = \overset{M}{\mathcal{E}}_{(\bar{\sigma})}\right]$ *then D-harmonic F is also called plus-eigen harmonic [resp: minus-eigenharmonic].*

Proposition 3.2 *For almost product manifolds* $(M,\ h,\ P)$, $(N,\ g,\ Q)$ *with pure or hyperbolic metric* h *with respect to* P *and pure or hyperbolic metric* g *with respect to* Q, *let* $F:(M,\ h,\ P)\to(N,\ g,\ Q)$ *be a* $\pm(P,\ Q)$*-paraholomorphic map. Then for every local section* $X,\ Y\in\Gamma(TM)$,

$$\begin{aligned}(\nabla F_*)(PX,\ PY) &= (\nabla F_*)(X,\ Y)\\ &+\left(\overset{N}{\nabla}_{QX\ '}Q\right)Y\ '-\left(\overset{N}{\nabla}_{Y\ '}Q\right)(QX\ ')\\ &-F_*\left[\left(\overset{M}{\nabla}_{PX}\ P\right)Y-\left(\overset{M}{\nabla}_{Y}P\right)(PX)\right]\end{aligned}\tag{3.3}$$

In particular,

$$\begin{aligned}(\nabla F_*)(PX,\ PX) &= (\nabla F_*)(X,\ X)+S_Q(QX\ ',X\ ')\\ &\quad -F_*[S_P(PX\ ,X)]\\ &= (\nabla F_*)(X,\ X)+Q\{S_Q(X\ ',X\ ')-\lambda F_*[S_P(X\ ,X)]\},\end{aligned}\tag{3.4}$$

where $X\ '=F_*X,$ $Y\ '=F_*Y,$ *and* $\lambda=1$ *when* F *is* $(P,\ Q)$*-paraholomorphic,* $\lambda=-1$ *when* F *is* $(P,\ Q)$*-antiparaholomorphic.*

Proof Let F be a $(P,\ Q)$-paraholomorphic map so that $F_*\circ P=Q\circ F_*$ then

$$\begin{aligned}(\nabla F_*)(X,\ PY) &= \overset{N}{\nabla}_{X\ '}(PY)'-F_*\left(\overset{M}{\nabla}_X(PY)\right)\\ &= \overset{N}{\nabla}_{X\ '}Q(Y')-F_*\left(\overset{M}{\nabla}_X(PY)\right)\\ &= \left(\overset{N}{\nabla}_{X\ '}Q\right)Y\ '+Q\left(\overset{N}{\nabla}_{X\ '}Y\ '\right)\\ &\quad -F_*\left[\left(\overset{M}{\nabla}_X P\right)Y+P\left(\overset{M}{\nabla}_X Y\right)\right]\\ &= Q\left[\overset{N}{\nabla}_{X\ '}Y\ '-F_*\left(\overset{M}{\nabla}_X Y\right)\right]\\ &\quad +\left(\overset{N}{\nabla}_{X\ '}Q\right)Y\ '-F_*\left[\left(\overset{M}{\nabla}_X P\right)Y\right]\end{aligned}$$

Thus we get

$$(\nabla F_*)(X,\ PY)=Q[(\nabla F_*)(X,\ Y)]+\left(\overset{N}{\nabla}_{X\ '}Q\right)Y\ '-F_*\left[\left(\overset{M}{\nabla}_X P\right)Y\right]$$

However, then this gives us

$$\begin{aligned}(\nabla F_*)(PX,\ PY) &= Q[(\nabla F_*)(PX,\ Y)]+\left(\overset{N}{\nabla}_{(PX)'}Q\right)Y\ '\\ &\quad -F_*\left[\left(\overset{M}{\nabla}_{PX}P\right)Y\right]\\ &= Q[(\nabla F_*)(PX,\ Y)]+\left(\overset{N}{\nabla}_{Q(X\ ')}Q\right)Y\ '\\ &\quad -F_*\left[\left(\overset{M}{\nabla}_{PX}P\right)Y\right]\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}(\nabla F_*)(Y,\ PX) &= Q[(\nabla F_*)(Y,\ X)]\\ &\quad +\left(\overset{N}{\nabla}_{Y\ '}Q\right)X\ '-F_*\left[\left(\overset{M}{\nabla}_Y P\right)X\right]\end{aligned}\tag{3.6}$$

Using (3.6) in (3.5) and the symmetry of ∇F_*, we get

$$\begin{aligned}(\nabla F_*)(PX,\ PY) &= (\nabla F_*)(X,\ Y) \\ &\quad + \left(\overset{N}{\nabla}_{Q(X')}Q\right)Y' + Q\left(\left(\overset{N}{\nabla}_{Y'}Q\right)X'\right) \\ &\quad - F_*\left[\left(\overset{M}{\nabla}_{PX}P\right)Y + P\left(\left(\overset{M}{\nabla}_{Y}P\right)X\right)\right]\end{aligned} \tag{3.7}$$

Finally using the fact that $P\circ(\nabla P) = -(\nabla P)\circ P$ in equation (3.7) we get the desired result (3.3). □

In particular, since

$$\begin{aligned}S_Q(QX',X') &= \left(\overset{N}{\nabla}_{QX'}Q\right)X' - \left(\overset{N}{\nabla}_{X'}Q\right)(QX') = Q\left(S_Q(X',X')\right) \\ S_P(PX,X) &= \left(\overset{M}{\nabla}_{PX}P\right)X - \left(\overset{M}{\nabla}_{X}P\right)(PX) = P\left(S_P(X,X)\right)\end{aligned}$$

we get the equation (3.4), that is,

$$\begin{aligned}(\nabla F_*)(PX,\ PX) &= (\nabla F_*)(X,\ X) + S_Q(QX',X') - F_*\left[S_P(PX,X)\right] \\ &= (\nabla F_*)(X,\ X) + Q\left\{S_Q(X',X') - F_*\left[S_P(X,X)\right]\right\}.\end{aligned}$$

For the case where F is $(P,\ Q)$-anti-paraholomorphic, the same argument works so that the required result (3.4) follows.

For a pair of almost product [*resp*: almost golden] manifolds $(M,\ h,\ P)$, $(N,\ g,\ Q)$ [*resp*: $(M,\ h,\ G)$, $(N,\ g,\ K)$] with pure or hyperbolic metric h with respect to P [*resp*: G] and pure or hyperbolic metric g with respect to Q [*resp*: K] we set the following conditions:

- **(I)** *$(M,\ h,\ P)$ is a para-Kaehler manifold and $(N,\ g,\ Q)$ is either a para-Kaehler manifold or a locally decomposable product Riemannian manifold.*
- **(II)** *$(M,\ h,\ P)$ is a locally decomposable product Riemannian manifold and $(N,\ g,\ Q)$ is either a para-Kaehler manifold or a locally decomposable product Riemannian manifold.*
- **(III)** *$(M,\ h,\ P)$ is a quasi para-Kaehler manifold and $(N,\ g,\ Q)$ is either quasi para Kaehler manifold or a locally decomposable product Riemannian manifold.*
- **(IV)** *$(M,\ h,\ P)$ is a locally decomposable product Riemannian manifold and $(N,\ g,\ Q)$ is either a quasi para-Kaehler manifold or a locally decomposable product Riemannian manifold.*

Corollary 3.1 *For a map $F:(M,\ h,\ P)\to(N,\ g,\ Q)$*

i) let F be $\pm(P,\ Q)$-paraholomorphic between manifolds holding the condition **(I)** *or* **(II)** *then for every local section $X,\ Y\in\Gamma(TM)$,*

$$(\nabla F_*)(PX,\ PY) = (\nabla F_*)(X,\ Y)$$

ii) let F be $\pm(P,\ Q)$-paraholomorphic between manifolds holding the condition **(III)** *or* **(IV)** *then for every local section $X\in\Gamma(TM)$,*

$$(\nabla F_*)(PX,\ PX) = (\nabla F_*)(X,\ X) \tag{3.8}$$

Proof

i) Since $\overset{N}{\nabla}Q = 0$ and $\overset{M}{\nabla}P = 0$ in the case (**I**) or (**II**), the result follows from Proposition (3.2).

ii) Since $S_Q(QX', X') = 0$ and $S_P(PX, X) = 0$ in the case (**III**) or (**IV**), the result follows from Proposition (3.2).

$\square$

Theorem (3.1/A) : *Let $F : (M, h, P) \to (N, g, Q)$ be a $\pm(P, Q)$-paraholomorphic map from a semidecomposable product Riemannian manifold M into either an **almost** product Riemannian manifold or an **almost** para-Hermitian manifold N with Vidal eigendistributions $\overset{N}{\mathcal{E}}_{(1)}$ and $\overset{N}{\mathcal{E}}_{(-1)}$ of Q. Then the following statements are equivalent:*

i) F is harmonic

ii) F is plus-eigen harmonic and minus-eigen harmonic

Proof $(ii) \Rightarrow (i)$: This is obvious.

$(i) \Rightarrow (ii)$: Let $\{u_1, ..., u_s\}$ and $\{v_1, ..., v_t\}$ be local orthonormal frame fields for $\overset{M}{\mathcal{E}}_{(1)}$ *and* $\overset{M}{\mathcal{E}}_{(-1)}$ respectively. Then observe that

$a°)$ Since $\mathrm{div}P = 0$ and therefore $\overset{M}{\mathcal{E}}_{(1)}$ and $\overset{M}{\mathcal{E}}_{(-1)}$ are both minimal distributions,

$$u = \sum_{i=1}^{s} \overset{M}{\nabla}_{u_i} u_i \in \Gamma\left(\overset{M}{\mathcal{E}}_{(1)}\right) \qquad \text{and} \qquad v = \sum_{i=1}^{t} \overset{M}{\nabla}_{v_i} v_i \in \Gamma\left(\overset{M}{\mathcal{E}}_{(-1)}\right)$$

$b°)$ Since F is $\pm(P, Q)$-paraholomorphic, $(a°)$ gives us

$$F_*(u) \quad \text{and} \quad u'_i = F_*(u_i) \in \Gamma\left(\overset{N}{\mathcal{E}}_{(c)}\right), \quad \forall\, i = 1, ..., s$$

and

$$F_*(v), \quad v'_i = F_*(v_i) \in \Gamma\left(\overset{N}{\mathcal{E}}_{(-c)}\right), \quad \forall\, i = 1, ..., t$$

$c°)$ Since N is Vidal, $(b°)$ gives us

$$\overset{N}{\nabla}_{u'_i} u'_i \in \Gamma\left(\overset{N}{\mathcal{E}}_{(c)}\right), \quad \forall\, i = 1, ..., s \qquad \text{and} \qquad \overset{N}{\nabla}_{v'_i} v'_i \in \Gamma\left(\overset{N}{\mathcal{E}}_{(-c)}\right), \quad \forall\, i = 1, ..., t$$

and therefore

$$\sum_{i=1}^{s} \overset{N}{\nabla}_{u'_i} u'_i \in \Gamma\left(\overset{N}{\mathcal{E}}_{(c)}\right) \qquad \text{and} \qquad \sum_{i=1}^{t} \overset{N}{\nabla}_{v'_i} v'_i \in \Gamma\left(\overset{N}{\mathcal{E}}_{(-c)}\right)$$

Hence, from $(c°)$, we have

$$\mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(1)}\right)}(F) = \sum_{i=1}^{s} (\nabla F_*)(u_i,\, u_i) = \sum_{i=1}^{s} \left[\overset{N}{\nabla}_{u'_i} u'_i - F_*\left(\overset{M}{\nabla}_{u_i} u_i\right)\right] \in \Gamma\left(\overset{N}{\mathcal{E}}_{(c)}\right)$$

and

$$\mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(-1)}\right)}(F) = \sum_{i=1}^{t} (\nabla F_*)(v_i,\ v_i) = \sum_{i=1}^{t} \left[\overset{N}{\nabla}_{v'_i} v'_i - F_* \left(\overset{M}{\nabla}_{v_i} v_i \right) \right] \in \Gamma\left(\overset{N}{\mathcal{E}}_{(-c)} \right)$$

Here $c = 1$, when F is $(P,\ Q)$-paraholomorphic and $c = -1$, when F is $(P,\ Q)$-anti-paraholomorphic. However, then, since $\overset{N}{\mathcal{E}}_{(1)} \cap \overset{N}{\mathcal{E}}_{(-1)} = \{0\}$, we have that $\mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(1)}\right)}(F)$ and $\mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(-1)}\right)}(F)$ are linearly independent. Thus, by the fact that $\{u_1, ..., u_s;\ v_1, ..., v_t\}$ is a local orthonormal frame field for TM and

$$\mathcal{T}(F) = \mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(1)}\right)}(F) + \mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(-1)}\right)}(F),$$

we have $\mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(1)}\right)}(F) = 0 = \mathcal{T}_{\left(\overset{M}{\mathcal{E}}_{(-1)}\right)}(F)$ by the assumption that $\mathcal{T}(F) = 0$. □

Corollary (3.2/A) *Let $F : (M,\ h,\ P) \to (N,\ g,\ Q)$ be a $\pm(P,\ Q)$-paraholomorphic map from a locally decomposable product Riemannian manifold M into either a locally decomposable product Riemannian manifold or nearly para-Kaehler (in particular, para-Kaehler) manifold N. Then the following statements are equivalent:*

i) F *is harmonic.*

ii) F *is plus-eigen harmonic and minus-eigen harmonic.*

Proof By Remark (3.1) one gets that

$a°$) for every locally decomposable product Riemannian manifold M, the eigendistributions $\overset{M}{\mathcal{E}}_{(1)}$ and $\overset{M}{\mathcal{E}}_{(-1)}$ are both minimal.

$b°$) for every nearly para-Kaehler manifold N, the eigendistributions $\overset{N}{\mathcal{E}}_{(1)}$ and $\overset{N}{\mathcal{E}}_{(-1)}$ are also both Vidal.

Thus the equivalence of (i) and (ii) follows from the observations $(a°)$, $(b°)$ and Theorem $(2.1/A)$. □

Theorem (3.1/B) : Let $F : (M,\ h,\ G) \to (N,\ g,\ K)$ be a $\pm(G,\ K)$-golden map from a semidecomposable golden Riemannian manifold M into either an **almost** golden Riemannian manifold or an **almost** golden-Hermitian manifold N with Vidal eigendistributions $\overset{N}{\mathcal{E}}_{(\sigma)}$ and $\overset{N}{\mathcal{E}}_{(\bar{\sigma})}$ of K. Then the following statements are equivalent:

i) F *is harmonic.*

ii) F *is plus-eigen harmonic and minus-eigen harmonic.*

Proof Let P_G and Q_K denote the twin product structures of G and K respectively. Then by Remark (3.1)/(2) and Proposition (3.1) the hypothesis of this theorem becomes equivalent to the hypothesis of Theorem $(3.1/A)$, namely:

"Let $F:(M,\ h,\ P_G)\to(N,\ g,\ Q_K)$ be a $\pm(P_G,\ Q_K)$-paraholomorphic map from a semidecomposable product Riemannian manifold $(M,\ h,\ P_G)$ into either an **almost** product Riemannian manifold or an **almost** para-Hermitian manifold $(N,\ g,\ Q_K)$ with Vidal eigendistributions $\overset{N}{\mathcal{E}}_{(1)}$ and $\overset{N}{\mathcal{E}}_{(-1)}$ of Q_K."

Hence the required conclusion of the theorem follows from Theorem $(3.1/A)$. □

From Theorem $(3.1/B)$ we immediately get.

Corollary (3.2/B) : *Let $F:(M,\ h,\ G)\to(N,\ g,\ K)$ be a $\pm(G,\ K)$-golden map from a locally decomposable golden Riemannian manifold M into either a locally decomposable golden Riemannian manifold or nearly golden-Kaehler (in particular, golden-Kaehler) manifold N. Then the following statements are equivalent:*

i) F *is harmonic.*

ii) F *is plus-eigen harmonic and minus-eigen harmonic.*

Proposition 3.3 *Let $F:(M,\ h,\ P)\to(N,\ g,\ Q)$ be a $\pm(P,\ Q)$-paraholomorphic map from an almost para-Hermitian manifold $(M,\ h,\ P)$ into an almost para-Hermitian manifold or an almost product Riemannian manifold $(N,\ g,\ Q)$. Then the tension field $\mathcal{T}\ (F)$ of F takes the form*

$$\begin{aligned}\mathcal{T}\ (F)\quad &=\sum_{i=1}^{m}\{h(e_i,\ e_i)(\nabla F_*)(e_i,\ e_i)+h(Pe_i,\ Pe_i)(\nabla F_*)(Pe_i,\ Pe_i)\}\\ &=-Q\left\{\sum_{i=1}^{m}h_{ii}S_Q(e_i',\ e_i')-\lambda F_*(\mathrm{div}(P))\right\},\end{aligned}$$

where $\{e_1,...,e_m,Pe_1,...,Pe_m\}$ is a local orthonormal frame field for TM and $\lambda=1$ when F is $(P,\ Q)$-paraholomorphic, $\lambda=-1$ when F is $(P,\ Q)$-anti-paraholomorphic and $h_{ii}=h(e_i,\ e_i)$, $e_i'=F_*(e_i)$.

Proof For an orthonormal frame field $\{e_1,...,e_m,Pe_1,...,Pe_m\}$ for TM we have, by definition,

$$\begin{aligned}\mathcal{T}\ (F)\quad &=\sum_{i=1}^{m}\{h(e_i,\ e_i)(\nabla F_*)(e_i,\ e_i)+h(Pe_i,\ Pe_i)(\nabla F_*)(Pe_i,\ Pe_i)\}\\ &=\sum_{i=1}^{m}h_{ii}\{(\nabla F_*)(e_i,\ e_i)-(\nabla F_*)(Pe_i,\ Pe_i)\}\end{aligned}\tag{3.9}$$

On the other hand, from Proposition (3.2), we have

$$(\nabla F_*)(Pe_i,\ Pe_i)=(\nabla F_*)(e_i,\ e_i)+Q\{S_Q(e_i{}',e_i')-\lambda F_*[S_P(e_i\ ,e_i)]\}$$

Replacing this into (3.8) we get

$$\begin{aligned}\mathcal{T}\ (F)\quad &=-Q\left\{\sum_{i=1}^{m}h_{ii}[S_Q(e_i',\ e_i')-\lambda F_*(S_P(e_i\ ,e_i))]\right\}\\ &=-Q\left\{\sum_{i=1}^{m}h_{ii}[S_Q(e_i',\ e_i')]-\lambda F_*\left(\sum_{i=1}^{m}h_{ii}S_P(e_i\ ,e_i)\right)\right\}\\ &=-Q\left\{\sum_{i=1}^{m}h_{ii}S_Q(e_i',\ e_i')-\lambda F_*(\mathrm{div}(P))\right\}\end{aligned}$$

□

Theorem (3.2/A) : *Let* $F:(M,\ h,\ P)\rightarrow(N,\ g,\ Q)$ *be a* $\pm(P,\ Q)$*-paraholomorphic map from an almost para-Hermitian manifold* $(M,\ h,\ P)$ *into an almost para-Hermitian manifold or an almost product Riemannian manifold* $(N,\ g,\ Q)$. *If either*

i) [2, 6, 11], $(M,\ h,\ P)$ *is a semi para-Kaehler manifold and* $(N,\ g,\ Q)$ *is a quasi para-Kaehler manifold, or*

ii) $(M,\ h,\ P)$ *is a semi para-Kaehler manifold and* $(N,\ g,\ Q)$ *is a locally decomposable product Riemannian manifold,*

then F *is harmonic.*

Proof For a local orthonormal frame field $\{e_1,...,e_m,Pe_1,...,Pe_m\}$ for TM we have, by Proposition (3.3) that

$$\mathcal{T}\,(F)\quad=-Q\left\{\sum_{i=1}^{m}h_{ii}S_Q\left(e_i',\ e_i'\right)-\lambda F_*\left(\operatorname{div}\left(P\right)\right)\right\}.$$

However, then, $\lambda F_*\left(\operatorname{div}\left(P\right)\right)=0$ since $(M,\ h,\ P)$ is semi para-Kaehler and $S_Q\left(e_i',\ e_i'\right)=0$ since $(N,\ g,\ Q)$ is either quasi para-Kaehler or locally decomposable product Riemannian. Hence harmonicity of F follows. □

Theorem (3.2/B) : *For a* $\pm(G,\ K)$*-golden map* $F:(M,\ h,\ G)\rightarrow(N,\ g,\ K)$ *from an almost golden-Hermitian manifold* $(M,\ h,\ G)$ *into an almost golden-Hermitian manifold or an almost golden Riemannian manifold* $(N,\ g,\ K)$, *if either*

i) $(M,\ h,\ G)$ *is a semi golden-Kaehler manifold and* $(N,\ g,\ K)$ *is a quasi golden-Kaehler manifold, or*

ii) $(M,\ h,\ G)$ *is a semi golden-Kaehler manifold and* $(N,\ g,\ K)$ *is a locally decomposable golden Riemannian manifold,*

then F *is harmonic.*

Proof Let P_G and Q_K denote the twin product structures of G and K respectively. Then by Lemma (2.1) and Remark $(3.1)/2$ the hypothesis of this theorem becomes equivalent to the hypothesis of Theorem $(3.2/A)$, namely:

"Let $F:(M,\ h,\ P_G)\rightarrow(N,\ g,\ Q_K)$ be a $\pm(P_G,\ Q_K)$-holomorphic map from a semi Kaehler manifold $(M,\ h,\ P_G)$ into either a quasi para-Kaehler manifold or a locally decomposable product Riemannian manifold." □

Then the harmonicity of F follows from Theorem $(3.2/A)$.

Remark 3.2 *For a nonconstant map*

$$F:(M,\ h,\ \varphi\,(=P,G))\rightarrow(N,\ g,\ \psi\,(=Q,K))$$

with a pure metric h *(with respect to* φ*), the* $\pm$ *paraholomorphicity of* F *(or* F *being a* $\pm$*golden) is not much help for the harmonicity of* F*. The best results we seem to get are Theorems* $(3.1)/A$ *and* $(3.1)/B$*. On the other hand, when* h *is hyperbolic, then* $\pm$ *paraholomorphicity of* F *(or* F *being a* $\pm$ *golden) gives its harmonicity under certain conditions as Theorems* $(3.2)/A$ *and* $(3.2)/B$ *state. On these lines we provide the following example:*

Example 3.1 *On* $\mathbb{R}^2$ *for* $X=(x_1,\ x_2)\,,\quad Y=(y_1,\ y_2)\in\Gamma\left(T\mathbb{R}^2\right),$ *define*

$$h\left(X\ ,Y\right)=\sum_{i=1}^{2}x_iy_i \qquad \text{and} \quad P\left(X\right)=\left(x_1,\ -x_2\right), \qquad G\left(X\right)=\left(\sigma x_1,\ \bar{\sigma}x_2\right).$$

Then $\left(\mathbb{R}^2,\ h,\ P\right)$ *becomes a locally decomposable product Riemannian manifold and* $\left(\mathbb{R}^2,\ h,\ G\right)$ *becomes a locally decomposable golden Riemannian manifold. Moreover,* $\left(\mathbb{R}^2,\ h,\ P\right)$ *and* $\left(\mathbb{R}^2,\ h,\ G\right)$ *are twin manifolds as* $\{P,\ G\}$ *form a twin pair on* $\mathbb{R}^2$*. Let* $f:\left(\mathbb{R}^2,\ h,\ \varphi\left(=P,G\right)\right)\to\left(\mathbb{R}^2,\ h,\ \varphi\left(=P,G\right)\right)$ *be defined by*

$$f\left(s,\ t\right)=\left(s,\ e^t\right).$$

Observe that

- f *is* (P,P)*-paraholomorphic and also* (G,G)*-golden and yet*
- f *is not harmonic since* $\mathcal{T}(f)=\frac{\partial^2 f}{\partial s^2}+\frac{\partial^2 f}{\partial t^2}=(0,\ 0)+(0,\ e^t)=(0,\ e^t)\,.$

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