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Frequency independent solvability of surface scattering problems

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Abstract: We address the problem of *frequency independent solvability* of high-frequency scattering problems in the exterior of two-dimensional smooth, compact, strictly convex obstacles. Precisely, we show that if the leading term in the asymptotic expansion of the surface current is incorporated into the integral equation formulations of the scattering problem, then appropriate modifications of both the "frequency-adapted Galerkin boundary element methods" and the "Galerkin boundary element methods based on frequency dependent changes of variables" we have recently developed yield frequency independent approximations. Moreover, for any direct integral equation formulation of the scattering problem, we show that the error can be tuned to decrease *at any desired rate* with increasing frequency, if sufficiently many terms in the aforementioned asymptotic expansion are incorporated into the solution strategy.

Key words: High frequency scattering, integral equations, frequency independent solutions

1. Introduction

In this manuscript, we address the problem of *frequency independent solvability* of exterior sound-soft scattering problems

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \backslash K \\ u = -u^{\text{inc}}, & \text{on } \partial K \\ \lim\limits_{|x|\to\infty} |x|^{1/2} \left(\partial_{|x|} - ik\right) u(x,k) = 0 \end{cases} \tag{1}$$

for compact strictly convex obstacles K with smooth boundary ∂K, and plane-wave incidences $u^{\text{inc}}(x,k) = e^{ik\alpha\cdot x}$ with unit direction α.

Indeed, it is well known that the number of degrees of freedom (NDF) associated with classical numerical schemes for the solution of problem (1) increases at least linearly with increasing *wavenumber* k and this, in return, limits their applicability in high-frequency simulations (cf. the survey [3]). The recent practice is therefore to resort to integral equation reformulations of the scattering problem (1) as they allow the incorporation of asymptotic behavior of the unknown into the problem formulation, and thereby provide significant savings in computational times. In more detail, when the integral equation reformulation is based on the *direct approach*, the *scattered field* u is expressed as a *single-layer potential*

$$u(x,k) = -\int_{\partial K} \Phi_k(x,y)\, \eta(y,k)\, d\sigma(y)$$

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(where $\Phi_k(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)$ is the fundamental solution of the Helmholtz equation and $H_0^{(1)}$ is the Hankel function of the first kind and order zero), and this shifts the problem to the determination of the *normal derivative of the total field*

$$\eta(x,k) = \partial_{\mathbf{n}(x)}(u(x,k) + u^{\mathrm{inc}}(x,k)),$$

which is known as the *surface current* in electromagnetism [5]. Therefore, in single-scattering simulations (such as those associated with convex obstacles), η naturally inherits the oscillations present in the incident field of radiation, which gives rise to the ansatz

$$\eta(x,k) = e^{ik\alpha\cdot x}\,\eta^{\mathrm{slow}}(x,k), \qquad x \in \partial K \tag{2}$$

that represents η as a highly oscillatory exponential weighed by the *amplitude* η^{slow}. Previous attempts [2, 6–8, 12, 13, 15] geared towards the *frequency independent computation* of the surface current η (for compact strictly convex obstacles) were all based on different interpretations of the Melrose–Taylor asymptotics [17]

$$\eta^{\mathrm{slow}}(x,k) \sim \sum_{p,q\geq 0} a_{p,q}(x,k) = \sum_{p,q\geq 0} k^{2/3-2p/3-q}\, b_{p,q}(x)\,\Psi^{(p)}(k^{1/3}Z(x)) \tag{3}$$

(see Theorem 5 below for details on this expansion) for the efficient numerical discretization of uniquely solvable linear integral equations for η that take the form of an operator equation

$$\mathcal{R}_k\eta = f \tag{4}$$

in $L^2(\partial K)$. To date, however, no *rigorous* method exists that displays the capability of producing *frequency independent solutions* to integral equations (4) in the sense that η can be approximated within any prescribed numerical accuracy utilizing an NDF *independent* of k. In this paper, we prove that appropriate modifications of the Galerkin boundary element methods we have recently developed [7, 8] display this capability provided sufficiently many terms (depending on the operator $\mathcal{R}_k$; e.g., only the leading term) in the Melrose–Taylor asymptotics (3) are incorporated into the integral equations (4).

The very first algorithm concerning the frequency independent solution of integral equations (4) is due to Bruno et al. [2], which was based on Nyström discretizations together with extensions of the method of stationary phase along with frequency dependent changes of variables in regions where η^{slow} possesses boundary layers. In the same spirit, collocation methods were later developed by Giladi [13] based on (nonrigorous) geometrical theory of diffraction and Huybrechs and Vandewalle [15] utilizing the method of steepest descents for the evaluation of highly oscillatory integrals. The first numerical analysis associated with a p-version Galerkin boundary element implementation of these algorithms was presented by Domínguez et al. [6], which has provided an upper of $\mathcal{O}(k^{1/9})$ in NDF necessary to attain any prescribed numerical accuracy independently of k. We note, however, that the methods in [2, 6, 15] are asymptotic rather than convergent as they approximate the solution in the deep shadow regions $\partial K^{DS} = \{x \in \partial K : \alpha\cdot\mathbf{n}(x) \gg k^{-1/3}\}$ simply by zero.

It is worth mentioning that, for the related (but distinct) problem of high-frequency scattering by polygons, the methods of [4, 14] provide almost frequency-independent solvability, with the error in the Galerkin solution uniformly bounded in k if the NDF grows like $\log k$ as $k \to \infty$.

Motivated with these observations, we have recently demonstrated [7, 8] that representation (2) when combined with the Melrose–Taylor asymptotics (3) allows the generation of Galerkin approximation spaces for the *almost* frequency independent solution of integral equations (4). Specifically, in connection with the Galerkin boundary element methods developed in [7, 8], we proved that it is sufficient to increase the NDF as $\mathcal{O}(k^{\epsilon})$ (for any $\epsilon > 0$) with increasing k to maintain any prescribed accuracy independently of k. These results, however, are based on a fine balance between the NDF and the parameter ϵ as the former has to increase linearly with decreasing ϵ.

In this paper, we show that this restriction can be completely eliminated once the leading term $a_{0,0}$ in the asymptotic expansion (3) is incorporated into the integral equation (4) when $\mathcal{R}_k$ is either the *combined field integral operator* (CFIO) or the *star combined integral operator* (SCIO) [3, 18] (definitions of these integral operators and the associated integral equations are provided in Appendix A). More precisely, in this case, we show that appropriate modifications of our Galerkin boundary element methods [7, 8] reduce the aforementioned need for $\mathcal{O}(k^{\epsilon})$ increase to $\mathcal{O}(1)$. Thus the developments herein address the long standing theoretical problem of *frequency independent solvability* of integral equation (4) for CFIO and SCIO.

More generally, given an integral operator $\mathcal{R}_k$ for which the *stability constant* C_k/c_k (the ratio of the continuity and coercivity constants) of the associated sesquilinear form $\mathcal{B}_k(\mu,\eta) = \langle \mu, \mathcal{R}_k \eta \rangle$ on $L^2(\partial K) \times L^2(\partial K)$ behaves like k^{δ} as $k \to \infty$ for some $\delta > 0$, we address the following question:

Question 1 *Given $r > 0$, what is the least number of terms $a_{p,q}$ in the Melrose–Taylor asymptotics* (3) *that needs to be incorporated into the integral equation* (4) *so as to guarantee that the numerical error associated with any given NDF behaves like $\mathcal{O}(k^{-r})$ as $k \to \infty$?*

Thus the theoretical developments in this paper go beyond the frequency independent, i.e. $\mathcal{O}(1)$, solvability of integral equations (4). Indeed, as we shall see in §3, the answer to Question 1 lies in the determination of the slowest increasing sequence of finite sets $\{\mathcal{F}_\ell\}_{\ell \in \mathbb{Z}_+} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ having the property that the derivatives of the remainder $\rho_\ell^{\text{slow}} = \eta^{\text{slow}} - \sigma_\ell^{\text{slow}}$ associated with $\sigma_\ell^{\text{slow}} = \sum_{(p,q)\in\mathcal{F}_\ell} a_{p,q}$ has the mildest blow up with increasing wavenumber k (cf. Remark 11).

The paper is organized as follows. In §2, we first present the details underlying the Melrose–Taylor asymptotics (3), and characterize Hörmander classes and provide sharp estimates on the derivatives of the terms $a_{p,q}$ in (3). We then use these estimates to construct the aforementioned sequence $\{\mathcal{F}_\ell\}_{\ell \in \mathbb{Z}_+}$, and also clarify the recursive relations among the remainders ρ_ℓ^{slow}. In §3, we use these results to provide the rigorous answer to Question 1 (cf. Corollary 12).

2. Asymptotic analysis and derivative estimates

Let us begin with considering a compact strictly convex set $K \subset \mathbb{R}^2$ with a smooth boundary ∂K, in which case, any arc length parametrization γ of ∂K is naturally periodic with period $P = |\partial K|$. For definiteness, we assume that γ is directed in the counterclockwise orientation and $\alpha \cdot \mathbf{n}(\gamma(0)) = 1$, where $\mathbf{n}$ is the outward unit normal. These choices guarantee that if $0 < t_1 < t_2 < P$ are the preimages of the *shadow boundary points*

$$\gamma\left(\{t_1, t_2\}\right) = \partial K^{SB} = \{x \in \partial K : \alpha \cdot \mathbf{n}(x) = 0\},$$

then the *illuminated* and *shadow regions* are given by

$$\gamma\left((t_1, t_2)\right) = \partial K^{IL} = \{x \in \partial K : \alpha \cdot \mathbf{n}(x) < 0\}$$

and

$$\gamma\left((t_2, t_1+P)\right) = \partial K^{SR} = \{x \in \partial K : \alpha \cdot \mathbf{n}(x) > 0\}.$$

In what follows, we write $\eta(s,k)$ for $\eta(x,k) = \eta(\gamma(s),k)$ etc. so that, in particular, the factorization (2) takes on the form

$$\eta(s,k) = e^{ik\alpha\cdot\gamma(s)}\eta^{\text{slow}}(s,k). \tag{5}$$

To present the details of the Melrose–Taylor asymptotics (3) in this setting, we recall the definitions of Hörmander classes and asymptotic expansions of functions defined on $[0,P]\times(0,\infty)$ (see e.g. [9] and the references therein).

Notation 2 *Throughout the paper we write* $A \lesssim_{a,b,\ldots} B$ *to mean* $0 \le A \le CB$ *for a positive constant* $C = C(a,b,\ldots)$.

Definition 3 (Hörmander classes) *The Hörmander class* $S^{\theta}_{\xi,\zeta}([0,P]\times(0,\infty))$ *of order* $\theta \in \mathbb{R}$ *and type* $\xi,\zeta \in [0,1]$ *is the collection of all complex-valued functions* $a(s,k) \in C^\infty([0,P]\times(0,\infty))$ *having the property that, for all* $n,m \in \mathbb{Z}_+$ *and all* $k_0 > 0$, *there holds*

$$|D_s^n D_k^m a(s,k)| \lesssim_{n,m,k_0} (1+k)^{\theta-\xi m+\zeta n}, \qquad (s,k) \in [0,P]\times[k_0,\infty).$$

Definition 4 (asymptotic expansions) *Suppose* $a_j(s,k) \in S^{\vartheta_j}_{\xi,\zeta}([0,P]\times(0,\infty))$ *for* $j \in \mathbb{Z}_+$ *and* $\lim_{j\to\infty}\vartheta_j = -\infty$, *and set* $\theta_j = \max_{i\ge j}\vartheta_i$. *In this case, a function* $a(s,k) \in S^{\theta_0}_{\xi,\zeta}([0,P]\times(0,\infty))$ *is said to have the asymptotic expansion*

$$a(s,k) \sim \sum_{j\ge 0} a_j(s,k)$$

provided that, for every $j \in \mathbb{Z}_+$,

$$a(s,k) - \sum_{i<j} a_i(s,k) \in S^{\theta_j}_{\xi,\zeta}([0,P]\times(0,\infty)).$$

Note that this definition is invariant under rearrangements of the sequence $\{a_j\}$.

With these definitions the exact sense of asymptotic expansion (3), shown to hold *in a vicinity of the shadow boundary points* by Melrose and Taylor [17], which we later extended to the *entire boundary* [9], reads as follows:

Theorem 5 *[9, Corollary 2.1] The amplitude* $\eta^{\text{slow}}(s,k)$ *in* (5) *belongs to the Hörmander class* $S^1_{2/3,1/3}([0,P]\times(0,\infty))$ *and admits the asymptotic expansion*

$$\eta^{\text{slow}}(s,k) \sim \sum_{p,q\ge 0} a_{p,q}(s,k) = \sum_{p,q\ge 0} k^{2/3-2p/3-q}\, b_{p,q}(s)\, \Psi^{(p)}(k^{1/3}Z(s)),$$

where $b_{p,q}$ *are* P*-periodic complex-valued* C^∞ *functions,* Z *is a* P*-periodic real-valued* C^∞ *function that is positive on* $(t_1,t_2) = \gamma^{-1}(\partial K^{IL})$, *negative on* $(t_2, t_1+P) = \gamma^{-1}(\partial K^{SR})$, *and vanishes precisely to first order* *on* $\{t_1, t_2\} = \gamma^{-1}(\partial K^{SB})$. *Finally,* Ψ *is a complex-valued* C^∞ *function such that, for all* $N, n \in \mathbb{Z}_+$,

$$D_\tau^n \Big\{ \Psi(\tau) - \sum_{j=0}^{N-1} c_j \tau^{1-3j} \Big\} = \mathcal{O}(\tau^{1-3N-n}) \qquad \text{as } \tau \to \infty \tag{6}$$

and

$$D_\tau^n \Psi(\tau) = \mathcal{O}(\tau^{-N}) \qquad \text{as } \tau \to -\infty. \tag{7}$$

The Hörmander classes of $a_{p,q}$ that remain implicit in Theorem 5 are clarified in the next Lemma, which also provides sharp estimates on their derivatives. These estimates will be used in the constructions that follow.

Lemma 6 *For any* $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, *we have* $a_{p,q} \in S^{\vartheta(p,q)}_{2/3,1/3}([0,P] \times (0,\infty))$ *with*

$$\vartheta(p,q) = \begin{cases} 1-q, & p = 0, \\ \frac{2}{3} - \frac{2p}{3} - q, & p \geq 1. \end{cases}$$

Moreover, given $k_0 > 0$ *and* $n, p, q \in \mathbb{Z}_+$, *the estimate*

$$|D_s^n a_{p,q}(s,k)| \lesssim_{n,p,q,k_0} k^{2/3-2p/3-q} \Big(k^{1/3} + \sum_{j=2}^{n} k^{j/3} (1 + k^{1/3} |\omega(s)|)^{-j-2} \Big) \tag{8}$$

holds for all $(s,k) \in [0,P] \times [k_0, \infty)$ *with* $\omega(s) = (s - t_1)(t_2 - s)$.

Proof Given $n, m, p, q \in \mathbb{Z}_+$, an appeal to Lemma 14 in Appendix B entails

$$|D_s^n D_k^m \, a_{p,q}(s,k)| \lesssim_{n,m,p,q} k^{2/3-2p/3-q-m} \sum_{j=0}^{n+m} k^{j/3} |\Psi^{(j+p)}(k^{1/3} Z(s))| \tag{9}$$

for all $(s,k) \in [0,P] \times (0,\infty)$. On the other hand, equations (6) and (7) imply that

$$|\Psi^{(n)}(\tau)| \lesssim_n \begin{cases} (1 + |\tau|), & n = 0, \\ \quad 1, & n = 1, \\ (1 + |\tau|)^{-n-2}, & n \geq 2 \end{cases} \tag{10}$$

holds for all $\tau \in \mathbb{R}$.

When $p = 0$, use of (10) in (9) entails

$$\begin{aligned} |D_s^n D_k^m \, a_{0,q}(s,k)| &\lesssim_{n,m,q} k^{2/3-q-m} \begin{cases} k^{1/3}, & n+m = 0,1 \\ k^{1/3} + \sum_{j=2}^{n+m} k^{j/3} (1 + k^{1/3} |Z(s)|)^{-j-2}, & n+m \geq 2 \end{cases} \\ &\lesssim_{n,m,q} k^{1-q-m} \begin{cases} 1, & n+m = 0,1 \\ 1 + \sum_{j=2}^{n+m} k^{(j-1)/3}, & n+m \geq 2 \end{cases} \end{aligned}$$

for all $(s,k) \in [0,P] \times (0,\infty)$. Therefore, given $k_0 > 0$, we have

$$|D_s^n D_k^m \, a_{0,q}(s,k)| \lesssim_{n,m,q,k_0} (1+k)^{(1-q)-2m/3+n/3}$$

for all $(s,k) \in [0,P] \times [k_0,\infty)$. This shows that $a_{0,q} \in S^{\vartheta(0,q)}_{2/3,1/3}$.

When $p \geq 1$, using (10) in (9), we get

$$|D_s^n D_k^m \, a_{p,q}(s,k)| \lesssim_{n,m,p,q} k^{2/3-2p/3-q-m} \Big(A_p + \sum_{j=j_p}^{n+m} k^{j/3} (1 + k^{1/3}|Z(s)|)^{-j-p-2} \Big)$$

for all $(s,k) \in [0,P] \times (0,\infty)$, where $A_1 = 1$, $j_1 = 1$, and $A_p = 0$, $j_p = 0$ $(p \geq 2)$. Given $k_0 > 0$, we therefore have

$$\begin{aligned} |D_s^n D_k^m \, a_{p,q}(s,k)| &\lesssim_{n,m,p,q} k^{2/3-2p/3-q-m} \sum_{j=0}^{n+m} k^{j/3} \\ &\lesssim_{n,m,p,q,k_0} (1+k)^{(2/3-2p/3-q)-2m/3+n/3} \end{aligned}$$

for all $(s,k) \in [0,P] \times [k_0,\infty)$. This establishes $a_{p,q} \in S^{\vartheta(p,q)}_{2/3,1/3}$ for $p \geq 1$.

Finally, given $k_0 > 0$, careful use of (10) in (9) shows that

$$|D_s^n a_{p,q}(s,k)| \lesssim_{n,p,q,k_0} k^{2/3-2p/3-q} \begin{cases} k^{1/3} + \sum_{j=2}^{n} k^{j/3}(1+k^{1/3}|Z(s)|)^{-j-2}, & p = 0, \\ 1 + \sum_{j=1}^{n} k^{j/3}(1+k^{1/3}|Z(s)|)^{-(j+1)-2}, & p = 1, \\ \sum_{j=0}^{n} k^{j/3}(1+k^{1/3}|Z(s)|)^{-(j+p)-2}, & p \geq 2, \end{cases}$$

holds for all $(s,k) \in [0,P] \times [k_0,\infty]$ and this, in return, implies

$$|D_s^n a_{p,q}(s,k)| \lesssim_{n,p,q,k_0} k^{2/3-2p/3-q} \Big(k^{1/3} + \sum_{j=2}^{n} k^{j/3}(1+k^{1/3}|Z(s)|)^{-j-2} \Big).$$

Since Z vanishes precisely to first order (as stated in Theorem 5) at the points t_1 and t_2, we therefore get the estimate (8). $\square$

We note on account of estimate (8) that if $\sigma^{\text{slow}}_{\mathcal{F}} = \sum_{(p,q)\in\mathcal{F}} a_{p,q}$ for a finite set $\mathcal{F} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$, then part of the derivatives $D_s^n \rho^{\text{slow}}_{\mathcal{F}}$ of the remainder $\rho^{\text{slow}}_{\mathcal{F}} = \eta^{\text{slow}} - \sigma^{\text{slow}}_{\mathcal{F}}$ that depends on $\mathcal{F}$ is dominated by $k^{-\mu(\mathcal{F})}$, where

$$\mu(\mathcal{F}) = \min\{2p/3 + q : (p,q) \in (\mathbb{Z}_+ \times \mathbb{Z}_+)\backslash\mathcal{F}\}.$$

In order to maximize this minimum with the choice of the smallest set $\mathcal{F} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$, upon noting that

$$\{2p/3 + q : p,q \in \mathbb{Z}_+\} = \{0\} \cup \{\ell/3 : \ell = 2,3,\ldots\},$$

which follows from considering the three different cases $(j = 0,1,2)$ $p = 3r + j$ for $r \in \mathbb{Z}_+$, we introduce the following definition.

Definition 7 *Set*

$$\nu(0) = 0 \qquad \textit{and} \qquad \nu(\ell) = \frac{\ell+1}{3} \quad (\ell \geq 1)$$

and, for $\ell \in \mathbb{Z}_+$, *define*

$$\sigma_\ell^{\text{slow}} = \sum_{(p,q)\in\mathcal{F}_\ell} a_{p,q}$$

with

$$\mathcal{F}_\ell = \{(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 2p/3 + q < \nu(\ell)\}.$$

As we just clarified, the choice of the set $\mathcal{F}_\ell$ relates to the least number of terms $a_{p,q}$ that needs to be known in the Melrose–Taylor asymptotics so as to guarantee that the derivatives of the remainder $\rho_\ell^{\text{slow}} = \eta^{\text{slow}} - \sigma_\ell^{\text{slow}}$ have the mildest dependency on the increase in the wavenumber k. In this connection, the first main result of the paper reads as follows:

Theorem 8 *For any* $\ell \in \mathbb{Z}_+$, *the remainder* $\rho_\ell^{\text{slow}}(s,k)$ *belongs to* $S^{1-\nu(\ell)}_{2/3,1/3}([0,P]\times(0,\infty))$. *Moreover, given* $k_0 > 0$ *and* $n \in \mathbb{Z}_+$, *there holds*

$$|D_s^n \rho_\ell^{\text{slow}}(s,k)| \lesssim_{\ell,n,k_0} k^{-\nu(\ell)}\Big(k + \sum_{j=4}^{n+2} W(s,k)^{-j}\Big) \tag{11}$$

for all $(s,k) \in [0,P]\times[k_0,\infty]$, *where* $W(s,k) = k^{-1/3} + |\omega(s)|$ *and* $\omega(s) = (s-t_1)(t_2-s)$.

Proof Since $\eta^{\text{slow}} \in S^1_{2/3,1/3}$ and $a_{p,q} \in S^{\vartheta(p,q)}_{2/3,1/3}$, given $\ell \in \mathbb{Z}_+$, Definition 4 entails $\rho_\ell^{\text{slow}} = \eta^{\text{slow}} - \sigma_\ell^{\text{slow}} = \eta^{\text{slow}} - \sum_{(p,q)\in\mathcal{F}_\ell} a_{p,q} \in S^{\theta_\ell}_{2/3,1/3}$ with $\theta_\ell = \max\{\vartheta(p,q) : (p,q) \in (\mathbb{Z}_+\times\mathbb{Z}_+)\backslash\mathcal{F}_\ell\}$. Since $\theta_\ell \le \max\{1-2p/3-q : (p,q) \in (\mathbb{Z}_+\times\mathbb{Z}_+)\backslash\mathcal{F}_\ell\} \le 1-\nu(\ell)$, we thus get $\rho_\ell^{\text{slow}} \in S^{1-\nu(\ell)}_{2/3,1/3}$.

As for the estimate (11), given $n \in \mathbb{Z}_+$, let $m = \max\{1,n\}$. Since

$$\rho_\ell^{\text{slow}} = \rho_{\ell+m}^{\text{slow}} + \sum_{(p,q)\in\mathcal{F}_{\ell+m}\backslash\mathcal{F}_\ell} a_{p,q}$$

and $\rho_{\ell+m}^{\text{slow}} \in S^{1-\nu(\ell+m)}_{2/3,1/3}$, an appeal to Lemma 6 entails

$$|D_s^n \rho_\ell^{\text{slow}}(s,k)| \lesssim_{\ell,n,k_0} (1+k)^{1-\nu(\ell+m)+n/3} + \sum_{(p,q)\in\mathcal{F}_{\ell+m}\backslash\mathcal{F}_\ell} k^{2/3-2p/3-q}\Big(k^{1/3} + \sum_{j=2}^{n} k^{j/3}(1+k^{1/3}|\omega(s)|)^{-j-2}\Big).$$

Since $\nu(\ell+m) - n/3 = \frac{\ell+1}{3} + \frac{m-n}{3} \ge \nu(\ell)$, and $2p/3 + q \ge \nu(\ell)$ for $(p,q) \in \mathcal{F}_{\ell+m}\backslash\mathcal{F}_\ell$, this yields

$$\begin{aligned}|D_s^n \rho_\ell^{\text{slow}}(s,k)| &\lesssim_{\ell,n,k_0} (1+k)^{1-\nu(\ell)} + k^{2/3-\nu(\ell)} \sum_{(p,q)\in\mathcal{F}_{\ell+m}\backslash\mathcal{F}_\ell} \Big(k^{1/3} + \sum_{j=2}^{n} k^{j/3}(1+k^{1/3}|\omega(s)|)^{-j-2}\Big) \\ &\lesssim_{\ell,n,k_0} k^{2/3-\nu(\ell)}\Big(k^{1/3} + \sum_{j=2}^{n} k^{j/3}(1+k^{1/3}|\omega(s)|)^{-j-2}\Big),\end{aligned}$$

which is equivalent to (11). □

We end this section by exploiting the recursive relations among the remainders ρ_ℓ^{slow}.

Theorem 9 *While* $\rho_0^{\text{slow}} = \eta^{\text{slow}}$ *and* $\rho_1^{\text{slow}} = \rho_0^{\text{slow}} - a_{0,0}$, *for* $\ell \geq 1$, *we have*

$$\rho_{2\ell}^{\text{slow}} = \rho_{2\ell-1}^{\text{slow}} - \sum_{r=0}^{\lfloor \frac{\ell}{3} \rfloor} a_{\ell-3r,2r} \qquad \textit{and} \qquad \rho_{2\ell+1}^{\text{slow}} = \rho_{2\ell}^{\text{slow}} - \sum_{r=0}^{\lfloor \frac{\ell-1}{3} \rfloor} a_{\ell-1-3r,2r+1}.$$

Proof The identities for $\ell = 0$ and $\ell = 1$ are obvious. For $\ell' \geq 2$, we have

$$\rho_{\ell'}^{\text{slow}} = \rho_{\ell'-1}^{\text{slow}} - \sum_{(p,q) \in \mathcal{F}_{\ell'} \backslash \mathcal{F}_{\ell'-1}} a_{p,q}$$

with

$$\mathcal{F}_{\ell'} \backslash \mathcal{F}_{\ell'-1} = \{(p,q) \in \mathbb{Z}_+^2 : \nu(\ell'-1) \leq 2p/3 + q < \nu(\ell')\} = \{(p,q) \in \mathbb{Z}_+^2 : 2p + 3q = \ell'\}.$$

Accordingly, if $(p,q) \in \mathcal{F}_{2\ell} \backslash \mathcal{F}_{2\ell-1}$ with $\ell \geq 1$, then q must be even, say $q = 2r$ with $r \in \mathbb{Z}_+$, in which case $p = \ell - 3r$; since $p, r \in \mathbb{Z}_+$, we get $0 \leq r \leq \lfloor \ell/3 \rfloor$ as the range of r. This shows that $\mathcal{F}_{2\ell} \backslash \mathcal{F}_{2\ell-1} \subset \{(\ell-3r, 2r) : 0 \leq r \leq \lfloor \ell/3 \rfloor\}$; the reverse inclusion is obvious. If $(p,q) \in \mathcal{F}_{2\ell+1} \backslash \mathcal{F}_{2\ell}$ with $\ell \geq 1$, then q must be odd, say $q = 2r+1$ with $r \in \mathbb{Z}_+$, in which case $p = \ell - 1 - 3r$; since $p, r \in \mathbb{Z}_+$, we get $0 \leq r \leq \lfloor (\ell-1)/3 \rfloor$ as the range of r. Therefore, we similarly obtain $\mathcal{F}_{2\ell+1} \backslash \mathcal{F}_{2\ell} = \{(\ell-1-3r, 2r+1) : 0 \leq r \leq \lfloor (\ell-1)/3 \rfloor\}$. □

3. Applications to Galerkin boundary element methods

In this section, assuming that $\sigma_\ell^{\text{slow}}$ in Definition 7 is available for some $\ell \in \mathbb{Z}_+$, we consider the numerical approximation of the surface current η through appropriate modifications of the Galerkin boundary element methods we have recently developed [7, 8]. To this end, given $d \in \mathbb{Z}_+$, we construct approximations $\hat{\eta}_{\ell,d}$ to η in the form

$$\hat{\eta}_{\ell,d} = \sigma_\ell + \hat{\rho}_{\ell,d},$$

where $\sigma_\ell(s,k) = e^{ik\alpha\cdot\gamma(s)} \sigma_\ell^{\text{slow}}(s,k)$ and $\hat{\rho}_{\ell,d}$ is the solution of the Galerkin equation

$$\langle \hat{\mu}, \mathcal{R}_k \hat{\rho}_{\ell,d} \rangle = \langle \hat{\mu}, f_\ell \rangle, \qquad \text{for all } \hat{\mu} \in \hat{X} \tag{12}$$

with $f_\ell = f - \mathcal{R}_k \sigma_\ell$, $\mathcal{R}_k$ is the integral operator in equation (4), and the finite dimensional Galerkin approximation space $\hat{X} \subset L^2([0,P])$ is either $\mathcal{A}_d^m$ (*frequency-adapted Galerkin approximation space based on algebraic polynomials*) as developed in [8] or $\mathcal{A}_d^{\mathcal{C}}$ (*Galerkin approximation space based on algebraic polynomials and frequency dependent changes of variables*) as proposed in [7] (the definitions of these spaces are given in Appendix C). In this connection, the second main result of the paper reads as follows.

Theorem 10 *Suppose there exists* $k_0 \geq 1$ *such that the sesquilinear form* $\mathcal{B}_k(\mu,\eta) = \langle \mu, \mathcal{R}_k \eta \rangle$ *on* $L^2([0,P]) \times L^2([0,P])$ *associated with the integral operator* $\mathcal{R}_k$ *is continuous with a continuity constant* C_k *and coercive with a coercivity constant* c_k *for all* $k > k_0$. *Then, given* $d \in \mathbb{Z}_+$, *we have*

$$\frac{\|\eta - \hat{\eta}_{\ell,d}\|}{\|\eta\|} \lesssim_{\ell,n,m} \frac{C_k}{c_k} \frac{1}{k^{\nu(\ell)}} \frac{m\,(1 + k^{\frac{n}{6m+3} - \frac{1}{2}})}{d^n} \qquad \textit{if } \hat{X} = \mathcal{A}_d^m \tag{13}$$

and

$$\frac{\|\eta - \hat{\eta}_{\ell,d}\|}{\|\eta\|} \lesssim_{\ell,n} \frac{C_k}{c_k} \frac{1}{k^{\nu(\ell)}} \frac{(\log k)^{n+1/2}}{d^n} \qquad \text{if } \hat{X} = \mathcal{A}_d^{\mathcal{C}} \tag{14}$$

for all $n \in \{0, \dots, d+1\}$ *and all* $k > k_0$.

Proof Note that if $\rho_\ell(s,k) = e^{ik\alpha\cdot\gamma(s)} \rho_\ell^{\text{slow}}(s,k)$, then, by construction, we have $\rho_\ell = \eta - \sigma_\ell$. Since the operator $\mathcal{R}_k : L^2([0,P]) \to L^2([0,P])$ is linear and invertible, ρ_ℓ is the unique solution of the equation

$$\mathcal{R}_k \rho_\ell = \mathcal{R}_k \eta - \mathcal{R}_k \sigma_\ell = f - \mathcal{R}_k \sigma_\ell = f_\ell$$

or equivalently the associated weak formulation

$$\langle \mu, \mathcal{R}_k \rho_\ell \rangle = \langle \mu, f_\ell \rangle, \qquad \text{for all } \mu \in L^2([0,P]).$$

Since $\mathcal{B}_k$ is continuous and coercive, it follows from Céa's lemma [3] that the Galerkin equation (12) possesses a unique solution $\hat{\rho}_{\ell,d}$ and (note that $\eta - \hat{\eta}_{\ell,d} = \rho_\ell - \hat{\rho}_{\ell,d}$)

$$\|\eta - \hat{\eta}_{\ell,d}\| = \|\rho_\ell - \hat{\rho}_{\ell,d}\| \leq \frac{C_k}{c_k} \inf_{\hat{\mu} \in \hat{X}} \|\rho_\ell - \hat{\mu}\|.$$

Since the spaces $\hat{X} = \mathcal{A}_d^m$ and $\hat{X} = \mathcal{A}_d^{\mathcal{C}}$ are of the form $\hat{X} = e^{ik\alpha\cdot\gamma(s)} \tilde{X}$ (see Appendix C), the preceding estimate implies

$$\|\eta - \hat{\eta}_{\ell,d}\| \leq \frac{C_k}{c_k} \inf_{\tilde{\mu} \in \tilde{X}} \|\rho_\ell^{\text{slow}} - \tilde{\mu}\|. \tag{15}$$

When $\ell = 0$, aside from highly nontrivial additional technicalities, the estimation of the infimum on the right-hand side of this inequality was based on bounds on the derivatives of $\rho_0^{\text{slow}} = \eta^{\text{slow}}$ (see e.g. [8, Theorem 5], which provides estimate (11) above only for $\ell = 0$) for $\hat{X} = \mathcal{A}_d^m$ in [8, Theorem 1 & Corollary 1] and $\hat{X} = \mathcal{A}_d^{\mathcal{C}}$ in [7, Theorem 3 & Corollary 4]. Use of the more general estimate (11) above (that is valid for all $\ell \in \mathbb{Z}_+$) in the proofs of these theorems delivers the estimates in (13) and (14). □

Remark 11 *The importance of the bounds on the derivatives of* ρ_ℓ^{slow} *derived in Theorem 8 is their essential role in estimating the infimum on the right-hand-side of inequality* (15). *For details, we refer to [7, 8].*

The relevance of Theorem 10 stems from the fact that while, in general, the stability constant C_k/c_k is unbounded as $k \to \infty$, the term $k^{-\nu(\ell)}$ in estimates (13) and (14) can be chosen to balance it so as to yield *frequency independent numerical approximations* to the surface current η. Indeed, in this connection, if $\mathcal{R}_k$ is CFIO (respectively SCIO), then the sesquilinear form $\mathcal{B}_k$ is continuous for $k > 0$ and coercive for $k \gg 1$ (respectively for $k > 0$) with $C_k/c_k = \mathcal{O}(k^{1/3})$ as $k \to \infty$ (for the CFIO this is a consequence of the bounds on the single- and double-layer potentials in [10, Theorems 4.29 and 4.32]—see also [11, Theorem 1.4]—and for the SCIO this is a consequence of [10, Theorems 4.29 and 4.32] combined with the bounds in [11, Theorem 1.9]—see also [11, Remark 4.2]—). In regards to integral operators with similar properties, it is therefore relevant to record the following consequence of Theorem 10, which also provides the answer to Question 1.

Corollary 12 *Under the assumptions of Theorem 10, if $C_k/c_k = \mathcal{O}(k^\delta)$ as $k \to \infty$, then*

(a) Given $n \geq 2$, if $\hat{X} = \mathcal{A}_d^m$ with $m = \lfloor \frac{n+1}{3} \rfloor$, then for $d \in \mathbb{N}$ and $k > k_0$

$$\frac{\|\eta - \hat{\eta}_{\ell,d}\|}{\|\eta\|} \lesssim_{\ell,n} k^{\delta-\nu(\ell)} \frac{1}{d^n}. \tag{16}$$

(b) Given $n \in \mathbb{N}$, if $\hat{X} = \mathcal{A}_d^{\mathcal{C}}$, then, for any $\epsilon > 0$, $d \in \mathbb{N}$ and $k > k_0$

$$\frac{\|\eta - \hat{\eta}_{\ell,d}\|}{\|\eta\|} \lesssim_{\ell,n,\epsilon} k^{\delta-\nu(\ell)+\epsilon} \frac{1}{d^n}. \tag{17}$$

Proof That (16) holds for $d \geq n-1$ follows from (13) upon noting that $m = \lfloor (n+1)/3 \rfloor$ can be incorporated into the constant of the inequality, and $(C_k/c_k)k^{-\nu(\ell)} = \mathcal{O}(k^{\delta-\nu(\ell)})$ as $k \to \infty$, and $\frac{n}{6m+3} - \frac{1}{2} \leq 0$ for $m \geq \lfloor \frac{n+1}{3} \rfloor$; choosing a larger constant, if necessary, it follows that (16) holds for all $d \in \mathbb{N}$. Similarly, (17) follows from (14) upon noting that $(C_k/c_k)k^{-\nu(\ell)+\epsilon} = \mathcal{O}(k^{\delta-\nu(\ell)+\epsilon})$ and $k^{-\epsilon}(\log k)^{n+1/2} = o(1)$ as $k \to \infty$. $\square$

For any fixed $k > k_0$, while Corollary 12(a) demonstrates that the approximations $\hat{\eta}_{\ell,d}$ based upon $\mathcal{A}_d^m$ can be tuned to converge to the surface current η at any order $n \geq 2$ simply by choosing $m = \lfloor \frac{n+1}{3} \rfloor$, Corollary 12(b) displays that those based on $\mathcal{A}_d^{\mathcal{C}}$ converge *spectrally* (i.e. at any order $n \in \mathbb{N}$). Moreover, in both cases, given $r > 0$, the relative error decreases as $\mathcal{O}(k^{-r})$ with increasing k when $\delta - \nu(\ell) < -r$, and this provides the rigorous answer to Question 1. In particular, the condition $\delta - \nu(\ell) < 0$ guarantees the rigorous *frequency independent solvability* of integral equations (4).

Remark 13 *As noted above, if $\mathcal{R}_k$ is CFIO or SCIO, then $C_k/c_k = \mathcal{O}(k^\delta)$ with $\delta = 1/3$ as $k \to \infty$ so that $\delta - \nu(1) = -1/3 < 0$. Therefore, in these cases, the knowledge of $\sigma_1^{\text{slow}} = a_{0,0}$ is sufficient to ensure the frequency independent solvability of the integral equation (4). Indeed, if $0 < \varepsilon_1, \varepsilon_2 \ll 1$, then [17, Equation (9.30)]*

$$a_{0,0}(s,k) = 2ik\, \alpha \cdot \mathbf{n}(\gamma(s)), \qquad s \in (t_1 + \varepsilon_1, t_2 - \varepsilon_1) \tag{18}$$

and, for $j = 1, 2$ [17, Corollary 9.31],

$$a_{0,0}(s,k) = k^{2/3} \frac{\alpha \cdot \mathbf{n}(\gamma(s))}{Z(s)} \Psi(k^{1/3} Z(s)) \qquad s \in (t_j - \varepsilon_2, t_j + \varepsilon_2). \tag{19}$$

(see [17] for definitions of Ψ and Z, and [1] for practical approximations of these functions). As noted in [6], employing any C^∞ continuation of the function $b_{0,0}(s,k) = \frac{\alpha \cdot \mathbf{n}(\gamma(s))}{Z(s)}$ into the shadow region (i.e. $s \in (t_2 + \varepsilon_2/2, P + t_1 - \varepsilon_2/2)$), it can be assumed that expression (19) is valid in $(t_2 - \varepsilon_2, P + t_1 + \varepsilon_2)$ (note here that we are using periodicity), and a C^∞ representation of $a_{0,0}$ in $[0, P]$ can then be obtained by matching expressions (18) and (19) using a standard C^∞ partition of unity argument.

4. Conclusions

In this paper, we addressed the problem of rigorous frequency independent solvability of integral equations (4) based on appropriate modifications of the Galerkin boundary element methods we have recently developed in [7, 8]. Moreover, for any direct integral equation formulation of the scattering problem, we demonstrated how the least number of terms $a_{p,q}$ in the Melrose–Taylor asymptotics (3) must be chosen to be incorporated into integral equations (4) so as to guarantee the decay of the relative error with increasing k as $\mathcal{O}(k^{-r})$ for any positive r. The derivation of explicit expressions of $a_{p,q}$ for $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \backslash \{(0,0)\}$ in the asymptotic expansions of the amplitude and related numerical implementations are left for future work.

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Appendix

A. Integral operators and associated integral equations

If $K \subset \mathbb{R}^2$ is a compact domain with Lipschitz boundary and connected complement, then the combined field integral operator (CFIO) $\mathcal{R}_k$ and the associated right-hand side f_k in equation (4) are given by [3, 18]

$$\mathcal{R}_k = \frac{1}{2}\, I + \mathcal{D}'_k - i\delta_k \mathcal{S}_k \quad \text{and} \quad f_k = \frac{\partial u^{\text{inc}}}{\partial \nu} - i\delta_k u^{\text{inc}},$$

where

$$\mathcal{S}_k \eta(x) = \int_{\partial K} \Phi_k(x,y)\, \eta(y)\, ds(y), \qquad x \in \partial K$$

is the acoustic single-layer integral operator,

$$\mathcal{D}'_k \eta(x) = \int_{\partial K} \frac{\partial \Phi_k(x,y)}{\partial \mathbf{n}(x)}\, \eta(y)\, ds(y), \qquad x \in \partial K$$

is its normal derivative, $\delta_k \in \mathbb{R} \backslash \{0\}$ is the coupling parameter usually taken to be equal to k, and the incident field u^{inc} is assumed to satisfy the Helmholtz equation in $\mathbb{R}^2$.

On the other hand, if $K \subset \mathbb{R}^2$ is a bounded, star-shaped Lipschitz domain, and β is the position of the vector relative to an origin from which K is star-shaped, then the star combined integral operator (SCIO) $\mathcal{R}_k$ and the associated right-hand side f_k in equation (4) are given by [3, 18]

$$\mathcal{R}_k = (\beta \cdot \mathbf{n})\Big(\frac{1}{2}\, I + \mathcal{D}'_k\Big) + \beta \cdot \nabla_\Gamma \mathcal{S}_k - i\delta_k \mathcal{S}_k \quad \text{and} \quad f_k = \beta \cdot \nabla u^{\text{inc}} - i\delta_k u^{\text{inc}},$$

where $\delta_k = k|x| + \frac{i}{2}$ is the coupling parameter, $\nabla_\Gamma \mathcal{S}_k$ is the operator specified by

$$\nabla_\Gamma \mathcal{S}_k \eta(x) = \int_{\partial K} \Big(\nabla_x \Phi_k(x,y) - \mathbf{n}(x) \frac{\partial \Phi_k(x,y)}{\partial \mathbf{n}(x)}\Big) \eta(y)\, ds(y), \qquad x \in \partial K,$$

and the incident field u^{inc} is assumed to satisfy the Helmholtz equation in $\mathbb{R}^2$.

B. Derivative estimates

Lemma 14 *Let $a(s,k) = k^\theta\, b(s)\, \varphi(k^\omega Z(s))$, where b, φ, and Z are smooth functions, b and Z are periodic, and $\theta \in \mathbb{R}\backslash\mathbb{N}$ and $\omega \in \mathbb{R}\backslash\mathbb{Z}_+$. Then*

$$|D_s^n D_k^m\, a(s,k)| \lesssim_{n,m} k^{\theta - m} \sum_{j=0}^{n+m} k^{j\omega} |\varphi^{(j)}(k^\omega Z(s))|$$

for all $n, m \in \mathbb{Z}_+$ and all $k > 0$.

Proof We first suppose that $\theta, \omega \in \mathbb{R}\backslash\mathbb{Z}_+$; in which case, Leibniz's rule applied twice yields

$$D_s^n D_k^m\, a(s,k) = \sum_{n'=0}^{n} \sum_{m'=0}^{m} \binom{n}{n'} \binom{m}{m'} (D_s^{n-n'} b(s))(D_k^{m-m'} k^\theta)(D_s^{n'} D_k^{m'} \varphi(k^\omega Z(s))$$

for all $n, m \in \mathbb{Z}_+$. It follows that

$$|D_s^n D_k^m a(s,k)| \lesssim_{n,m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} k^{\theta+m'-m} |D_s^{n'} D_k^{m'} \varphi(k^\omega Z(s))|$$

holds for all $k > 0$. Next we recall Faa di Bruno's formula [16] for the derivatives of a composition

$$D^n f(g(t)) = \sum \left\{ (D^{n'} f)(g(t)) \prod_{j=1}^{n} \frac{j}{n'_j!} \left(\frac{D^j g(t)}{j!} \right)^{n'_j} \middle| \, n' = \sum_{j=1}^{n} n'_j, \, n = \sum_{j=1}^{n} j n'_j, \, n'_j \geq 0 \right\},$$

which, for convenience, we shall write as

$$D^n f(g(t)) = \sum_{\{n'\}} (D^{n'} f)(g(t)) \prod_{j=1}^{n} \frac{j}{n'_j!} \left(\frac{D^j g(t)}{j!} \right)^{n'_j}.$$

Therefore, with $C_j = \prod_{\ell=0}^{j-1} (\omega - \ell)$, we have

$$\begin{aligned} D_k^{m'} \varphi(k^\omega Z) &= \sum_{\{m''\}} \varphi^{(m'')}(k^\omega Z) \prod_{j=1}^{m'} \frac{j}{m''_j!} \left(\frac{D_k^j (k^\omega Z)}{j!} \right)^{m''_j} \\ &= \sum_{\{m''\}} \varphi^{(m'')}(k^\omega Z) \prod_{j=1}^{m'} \frac{j \, Z^{m''_j}}{m''_j!} \left(\frac{C_j \, k^{\omega-j}}{j!} \right)^{m''_j} \\ &= \sum_{\{m''\}} \varphi^{(m'')}(k^\omega Z) \, Z^{m''} \, k^{\omega m'' - m'} \prod_{j=1}^{m'} \frac{j}{m''_j!} \left(\frac{C_j}{j!} \right)^{m''_j}, \end{aligned}$$

which, on account of Leibniz's rule, entails

$$D_s^{n'} D_k^{m'} \varphi(k^\omega Z) = \sum_{n''=0}^{n'} \binom{n'}{n''} \sum_{\{m''\}} (D_s^{n''} \varphi^{(m'')}(k^\omega Z))(D_s^{n'-n''}(Z^{m''})) k^{\omega m'' - m'} \prod_{j=1}^{m'} \frac{j}{m''_j!} \left(\frac{C_j}{j!} \right)^{m''_j}.$$

This gives

$$|D_s^{n'} D_k^{m'} \varphi(k^\omega Z)| \lesssim_{n',m'} \sum_{n''=0}^{n'} \sum_{\{m''\}} k^{\omega m'' - m'} |D_s^{n''} \varphi^{(m'')}(k^\omega Z)|, \tag{20}$$

which, in return, implies

$$\begin{aligned} |D_s^n D_k^m a(s,k)| &\lesssim_{n,m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} k^{\theta+m'-m} \sum_{n''=0}^{n'} \sum_{\{m''\}} k^{\omega m'' - m'} |D_s^{n''} \varphi^{(m'')}(k^\omega Z)| \\ &\lesssim_{n,m} k^{\theta-m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} \sum_{n''=0}^{n'} \sum_{\{m''\}} k^{\omega m''} |D_s^{n''} \varphi^{(m'')}(k^\omega Z)|. \end{aligned} \tag{21}$$

Similar calculations using Faa di Bruno's formula entails

$$\begin{aligned} D_s^{n''}\varphi^{(m'')}(k^\omega Z) &= \sum_{\{n'''\}} \varphi^{(n'''+m'')}(k^\omega Z) \prod_{j=1}^{n''} \frac{j}{n_j'''!} \left(\frac{D_s^j (k^\omega Z)}{j!} \right)^{n_j'''} \\ &= \sum_{\{n'''\}} \varphi^{(n'''+m'')}(k^\omega Z) \prod_{j=1}^{n''} \frac{j\, k^{\omega n_j'''}}{n_j'''!} \left(\frac{D_s^j Z}{j!} \right)^{n_j'''} \\ &= \sum_{\{n'''\}} \varphi^{(n'''+m'')}(k^\omega Z)\, k^{\omega n'''} \prod_{j=1}^{n''} \frac{j}{n_j'''!} \left(\frac{D_s^j Z}{j!} \right)^{n_j'''} \end{aligned}$$

and thus

$$|D_s^{n''}\varphi^{(m'')}(k^\omega Z)| \lesssim_{n''} \sum_{\{n'''\}} k^{\omega n'''} |\varphi^{(n'''+m'')}(k^\omega Z)|. \tag{22}$$

Using this in (21), we finally obtain

$$\begin{aligned} |D_s^n D_k^m\, a(s,k)| &\lesssim_{n,m} k^{\theta-m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} \sum_{n''=0}^{n'} \sum_{\{m''\}} \sum_{\{n'''\}} k^{\omega(n'''+m'')} |\varphi^{(n'''+m'')}(k^\omega Z)| \\ &\lesssim_{n,m} k^{\theta-m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} k^{\omega(n'+m')} |\varphi^{(n'+m')}(k^\omega Z)| \\ &\lesssim_{n,m} k^{\theta-m} \sum_{j=0}^{n+m} k^{j\omega} |\varphi^{(j)}(k^\omega Z)| \end{aligned}$$

as claimed.

In case $\theta = 0$, by Leibniz's rule we have

$$D_s^n D_k^m\, a(s,k) = \sum_{n'=0}^{n} \binom{n}{n'} (D_s^{n-n'} b(s))(D_s^{n'} D_k^m \varphi(k^\omega Z(s)))$$

for all $n, m \in \mathbb{Z}_+$ so that

$$|D_s^n D_k^m\, a(s,k)| \lesssim_n \sum_{n'=0}^{n} |D_s^{n'} D_k^m \varphi(k^\omega Z(s))|$$

holds for all $k > 0$. Using (20), we therefore get

$$|D_s^n D_k^m\, a(s,k)| \lesssim_{n,m} \sum_{n'=0}^{n} \sum_{n''=0}^{n'} \sum_{\{m'\}} k^{\omega m' - m} |D_s^{n''} \varphi^{(m')}(k^\omega Z(s))|$$

which, on account of (22), yields

$$|D_s^n D_k^m\, a(s,k)| \lesssim_{n,m} \sum_{n'=0}^{n} \sum_{n''=0}^{n'} \sum_{\{m'\}} \sum_{\{n'''\}} k^{\omega(n'''+m')-m} |\varphi^{(n'''+m')}(k^\omega Z(s))|.$$

Therefore

$$
\begin{aligned}
|D_s^n D_k^m \, a(s,k)| &\lesssim_{n,m} \sum_{n'=0}^{n} \sum_{m'=0}^{m} k^{\omega(n'+m')-m} |\varphi^{(n'+m')}(k^\omega Z(s))| \\
&\lesssim_{n,m} k^{-m} \sum_{j=0}^{n+m} k^{j\omega} |\varphi^{(j)}(k^\omega Z(s))|
\end{aligned}
$$

completing the proof. □

C. Galerkin approximation spaces

C.1. Frequency-adapted Galerkin approximation spaces

Given $m \in \mathbb{N}$, set

$$
\epsilon_j = \frac{1}{3} \, \frac{2m-2j+1}{2m+1}, \qquad j = 1, \ldots, m.
$$

For $j = 1, \ldots, m-1$, define the *illuminated transition* and *shadow transition* intervals as

$$
\begin{aligned}
I_{IT_1}^j &= [t_1 + \xi_1 k^{-1/3+\epsilon_{j+1}}, t_1 + \xi_1 k^{-1/3+\epsilon_j}] \\
I_{IT_2}^j &= [t_2 - \xi_2 k^{-1/3+\epsilon_j}, t_2 - \xi_2 k^{-1/3+\epsilon_{j+1}}] \\
I_{ST_1}^j &= [t_1 - \zeta_1 k^{-1/3+\epsilon_j}, t_1 - \zeta_1 k^{-1/3+\epsilon_{j+1}}] \\
I_{ST_2}^j &= [t_2 + \zeta_2 k^{-1/3+\epsilon_{j+1}}, t_2 + \zeta_2 k^{-1/3+\epsilon_j}];
\end{aligned}
$$

note that these intervals are redundant if $m = 1$. Further define the *shadow boundary* and *illuminated/shadow regions* as

$$
\begin{aligned}
I_{SB_1} &= [t_1 - \zeta_1 k^{-1/3+\epsilon_m}, t_1 + \xi_1 k^{-1/3+\epsilon_m}] \\
I_{SB_2} &= [t_2 - \xi_1 k^{-1/3+\epsilon_m}, t_2 + \zeta_2 k^{-1/3+\epsilon_m}] \\
I_{IL} &= [t_1 + \xi_1 k^{-1/3+\epsilon_1}, t_2 - \xi_2 k^{-1/3+\epsilon_1}] \\
I_{DS} &= [t_2 + \zeta_2 k^{-1/3+\epsilon_1}, P + t_1 - \zeta_1 k^{-1/3+\epsilon_1}].
\end{aligned}
$$

Here $\xi_1, \xi_2, \zeta_1, \zeta_2 > 0$, $t_1 - \xi_1 < t_2 - \xi_2$, $t_2 + \zeta_2 < P + t_1 - \zeta_1$, and they are chosen so that the above intervals are nondegenerate for all $k > 1$. Denoting these $4m$ intervals as $[a_j, b_j]$, given $d \in \mathbb{Z}_+$, the $4m(d+1)$ dimensional *frequency-adapted Galerkin approximation space based on algebraic polynomials* is then defined as [8]

$$
\mathcal{A}_d^m = \bigoplus_{j=1}^{4m} \mathbb{1}_{[a_j, b_j]} \; e^{ik\,\alpha\cdot\gamma} \; \mathbb{P}_d,
$$

where $\mathbb{P}_d$ is the space of algebraic polynomials of degree at most d.

C.2. Galerkin approximation spaces based on changes of variables

For $j = 1, 2$, let $\xi_j, \xi'_j, \zeta_j, \zeta'_j > 0$ be such that

$$t_1 + \xi_1 \le t_1 + \xi'_1 = t_2 - \xi'_2 \le t_2 - \xi_2$$

and

$$t_2 + \zeta_2 \le t_2 + \zeta'_2 = P + t_1 - \zeta'_1 \le P + t_1 - \zeta_1.$$

For $k > 1$, define the *illuminated transition* and *shadow transition* intervals as

$$\begin{aligned} I_{IT_1} &= [t_1 + \xi_1 k^{-1/3}, t_1 + \xi'_1] \qquad & I_{IT_2} &= [t_2 - \xi'_2, t_2 - \xi_2 k^{-1/3}] \\ I_{ST_1} &= [t_1 - \zeta'_1, t_1 - \zeta_1 k^{-1/3}] \qquad & I_{ST_2} &= [t_2 + \zeta_2 k^{-1/3}, t_2 + \zeta'_2] \end{aligned}$$

and the *shadow boundary* intervals as

$$I_{SB_1} = [t_1 - \zeta_1 k^{-1/3}, t_1 + \xi_1 k^{-1/3}] \qquad I_{SB_2} = [t_2 - \xi_2 k^{-1/3}, t_2 + \zeta_2 k^{-1/3}].$$

Given $d \in \mathbb{Z}_+$, the $6(d+1)$ dimensional *Galerkin approximation space based on algebraic polynomials and frequency dependent changes of variables* is then defined as [7]

$$\mathcal{A}_d^{\mathcal{C}} = \bigoplus_{j=1}^{6} \mathbb{1}_{[a_j,b_j]}\, e^{ik\,\alpha\cdot\gamma}\, \hat{\mathbb{P}}_j^{\mathcal{C}},$$

where

$$\hat{\mathbb{P}}_j^{\mathcal{C}} = \begin{cases} \mathbb{P}_d \circ \phi^{-1}, & \text{if } [a_j, b_j] \text{ is a transition region,} \\ \mathbb{P}_d, & \text{otherwise.} \end{cases}$$

Here ϕ is the change of variables on the transition intervals given by

$$\phi(s) = \begin{cases} t_1 + \varphi(s)\, k^{\psi(s)}, & s \in I_{IT_1}, \\ t_2 - \varphi(s)\, k^{\psi(s)}, & s \in I_{IT_2}, \\ t_1 - \varphi(s)\, k^{\psi(s)}, & s \in I_{ST_1}, \\ t_2 + \varphi(s)\, k^{\psi(s)}, & s \in I_{ST_2}, \end{cases}$$

with

$$\varphi(s) = \begin{cases} \xi_1 + (\xi'_1 - \xi_1)\, \dfrac{s - a_1}{b_1 - a_1}, & s \in I_{IT_1}, \\ \xi'_2 + (\xi_2 - \xi'_2)\, \dfrac{s - a_2}{b_2 - a_2}, & s \in I_{IT_2}, \\ \zeta'_1 + (\zeta_1 - \zeta'_1)\, \dfrac{s - a_3}{b_3 - a_3}, & s \in I_{ST_1}, \\ \zeta_2 + (\zeta'_2 - \zeta_2)\, \dfrac{s - a_4}{b_4 - a_4}, & s \in I_{ST_2}, \end{cases} \quad \text{and} \quad \psi(s) = -\frac{1}{3} \begin{cases} \dfrac{b_1 - s}{b_1 - a_1}, & s \in I_{IT_1}, \\ \dfrac{s - a_2}{b_2 - a_2}, & s \in I_{IT_2}, \\ \dfrac{s - a_3}{b_3 - a_3}, & s \in I_{ST_1}, \\ \dfrac{b_4 - s}{b_4 - a_4}, & s \in I_{ST_2}. \end{cases}$$