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MÜBARİZ TAPDIGOĞLU GARAYEV

MEHMET GÜRDAL

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Remarks on the zero Toeplitz product problem in the Bergman and Hardy spaces

Mübariz Tapdıgöglü GARAYEV¹, Mehmet GÜRDAL^{2,*}

¹Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

²Department of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University, Isparta, Turkey

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Abstract: In this article, we are interested in the zero Toeplitz product problem: for two symbols $f, g \in L^\infty(\mathbb{D}, dA)$, if the product $T_f T_g$ is identically zero on $L_a^2(\mathbb{D})$, then can we claim T_f or T_g is identically zero? We give a particular solution of this problem. A new proof of one particular case of the zero Toeplitz product problem in the Hardy space $H^2(\mathbb{D})$ is also given.

Key words: Toeplitz operator, Bergman space, Hardy space, zero Toeplitz product, Berezin symbol

1. Introduction

Let $dA(\lambda) = \frac{1}{\pi} dx dy$ denote the Lebesgue area measure on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized so that the measure of \mathbb{D} equals 1. The Bergman space $L_a^2 := L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on \mathbb{D} that are also in $L^2(\mathbb{D}, dA)$. It is well known that $L_a^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$.

Let $P : L^2(\mathbb{D}, dA) \rightarrow L_a^2(\mathbb{D})$ be the Bergman orthogonal projector. P is an integral operator represented by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\lambda)}{(1 - \bar{\lambda}z)^2} dA(\lambda).$$

For $f \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_f with symbol f is the operator on $L_a^2(\mathbb{D})$ defined by $T_f h = P(fh)$ for $h \in L_a^2(\mathbb{D})$. It is not difficult to see that T_f is the zero operator if and only if the symbol f is zero almost everywhere (see, for instance, [1, p. 203]).

The Hardy space $H^2 = H^2(\mathbb{D})$ is defined as the space of all analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which the norm

$$\|f\|_2 = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \right)^{1/2}$$

*Correspondence: gurdalmehmet@sdu.edu.tr

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is finite. The reproducing kernel of H^2 is the function

$$k_{H^2,\lambda}(z) = \frac{1}{1 - \bar{\lambda}z}.$$

For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ we have $\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$.

For a function $\varphi \in L^\infty(\mathbb{T})$ the corresponding Toeplitz operator T_φ on H^2 is defined by

$$T_\varphi f = P_+ \varphi f, \quad f \in H^2,$$

where $P_+ : L^2(\mathbb{T}) \rightarrow H^2$ is the Riesz projector.

In this article, we are interested in the zero Toeplitz product problem:

for two symbols $f, g \in L^\infty(\mathbb{D}, dA)$, if the product $T_f T_g$ is identically zero on $L_a^2(\mathbb{D})$, then can we claim T_f or T_g is identically zero?

This is a nontrivial problem and the answer is not known. Here we give a particular answer to this question. We also give a new proof for the zero Toeplitz product problem in the Hardy space H^2 in one particular case, which does not use a deep result of Brown and Halmos [7] for the product of two Toeplitz operators on H^2 (see also Stroethoff [16], Aleman and Vukotić [3], and Guediri [11]). In general, this article is motivated mainly by the following conjecture raised by Čučkovič in his paper [8]:

Conjecture 1 *Let $f, g \in L^\infty(\mathbb{D}, dA)$ with g harmonic. Then $T_f T_g = 0$ on $L_a^2(\mathbb{D})$ has only a trivial solution, i.e. $T_f = 0$.*

In [1], Ahern and Čučkovič solved the zero-product problem for two Toeplitz operators with harmonic symbols. Some particular results are also proved in [13] and [14].

Before giving our results, let us introduce some necessary definitions and notations.

For $\lambda \in \mathbb{D}$, the Bergman reproducing kernel is the function $k_{L_a^2,\lambda}(z) \in L_a^2$ such that

$$f(\lambda) = \langle f, k_{L_a^2,\lambda} \rangle$$

for every $f \in L_a^2$. The normalized Bergman reproducing kernel $\widehat{k}_{L_a^2,\lambda}$ is the function $\frac{k_{L_a^2,\lambda}}{\|k_{L_a^2,\lambda}\|_2}$. (It is well known that $\widehat{k}_{L_a^2,\lambda}(z) = \frac{1-|\lambda|^2}{(1-\bar{\lambda}z)^2}$.) Here, as elsewhere in this article, the norm $\|\cdot\|_2$ and the inner product $\langle \cdot, \cdot \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. The set of bounded linear operators on L_a^2 is denoted by $\mathcal{B}(L_a^2)$.

For $T \in \mathcal{B}(L_a^2)$, the Berezin symbol (or the Berezin transform) of T is the complex-valued function \widetilde{T} on \mathbb{D} defined by (see [6])

$$\widetilde{T}(\lambda) := \langle T \widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \rangle, \quad \lambda \in \mathbb{D}.$$

Often the behavior of the Berezin transform of an operator provides important information about the operator.

The Berezin transform \widetilde{f} of a function $f \in L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_f on L_a^2 . In other words, $\widetilde{f} := \widetilde{T}_f$. Since $\widetilde{T}_f(\lambda) = \langle T_f \widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \rangle = \langle P \left(f \widehat{k}_{L_a^2,\lambda} \right), \widehat{k}_{L_a^2,\lambda} \rangle =$

$\langle f\widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \rangle$, we obtain the formula

$$\widetilde{f}(\lambda) = \int_{\mathbb{D}} \left| \widehat{k}_{L_a^2,\lambda}(z) \right|^2 f(z) dA(z) = \int_{\mathbb{D}} \frac{(1 - |\lambda|^2)^2}{|1 - \bar{\lambda}z|^4} f(z) dA(z).$$

2. The results

Recall that every bounded harmonic function equals its Berezin transform. The converse also holds, so a function in $L^\infty(\mathbb{D}, dA)$ equals its Berezin transform if and only if it is harmonic; for this deep result see Engliš [9] and Ahern et al. [2].

Now we can formulate and prove our results.

Theorem 1 *Let $f, g \in L^\infty(\mathbb{D}, dA)$. The following are true:*

- (a) *If $\dim \overline{\text{Range}(T_g)}^\perp < +\infty$ and $T_f T_g = 0$ on $L_a^2(\mathbb{D})$, then $T_f = 0$.*
- (b) *If g is harmonic with $g(e^{it}) \neq 0$ for almost all $t \in [0, 2\pi)$, $\text{Range}(T_f)$ is closed, and $T_f T_g = 0$, then $T_f = 0$.*

Proof (a) Since $T_f T_g = 0$ on $L_a^2(\mathbb{D})$, we have that $\overline{\text{Range}(T_g)} \subset \ker(T_f)$. Hence, $\ker(T_f)^\perp \subset \overline{\text{Range}(T_g)}^\perp$, and thus $\text{Range}(T_f^*) \subset \overline{\text{Range}(T_g)}^\perp$; that is, $\text{Range}(T_{\bar{f}}) \subset \overline{\text{Range}(T_g)}^\perp$. By using the condition of the theorem, from this we assert that $T_{\bar{f}}$ is a finite rank Toeplitz operator on $L_a^2(\mathbb{D})$. However, by the well-known Luecking theorem [15], the only finite rank Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$ is the zero operator, which implies that $T_{\bar{f}} = 0$, and hence $T_f = 0$. This proves (a).

(b) Let $T_f T_g = 0$. Then obviously $\widetilde{T_f T_g} = 0$; that is, $\langle T_f T_g \widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \rangle = 0$ for all $\lambda \in \mathbb{D}$. Then, by considering that $\widetilde{g} = g$, we have

$$\begin{aligned} 0 &= \langle T_f (T_g \widehat{k}_{L_a^2,\lambda} - \widetilde{T_g}(\lambda) \widehat{k}_{L_a^2,\lambda}), \widehat{k}_{L_a^2,\lambda} \rangle + \langle T_f (\widetilde{T_g}(\lambda) \widehat{k}_{L_a^2,\lambda}), \widehat{k}_{L_a^2,\lambda} \rangle \\ &= \langle T_g \widehat{k}_{L_a^2,\lambda} - g(\lambda) \widehat{k}_{L_a^2,\lambda}, T_f^* \widehat{k}_{L_a^2,\lambda} \rangle + g(\lambda) \widetilde{T_f}(\lambda), \end{aligned}$$

and hence

$$\widetilde{f}(\lambda) g(\lambda) = \widetilde{T_f}(\lambda) g(\lambda) = - \langle T_{g-g(\lambda)} \widehat{k}_{L_a^2,\lambda}, T_f^* \widehat{k}_{L_a^2,\lambda} \rangle$$

for all λ in \mathbb{D} . Since g is a bounded harmonic function, by using a result of Axler and Zheng [5, Corollary 3.7] and the Cauchy–Schwarz inequality, we obtain from the latter identity that

$$\left| \widetilde{f}(\lambda) g(\lambda) \right| \leq \|f\|_\infty \left\| T_{g-g(\lambda)} \widehat{k}_{L_a^2,\lambda} \right\|_2 \rightarrow 0 \text{ as } \lambda \rightarrow \partial\mathbb{D}$$

nontangentially at almost every point of $\partial\mathbb{D}$. This implies that $\lim_{\text{nontangentially}} \left| \widetilde{f}(\lambda) g(\lambda) \right| = 0$, and hence

$\lim_{\text{nontangentially}} \left(\widetilde{f}(\lambda) g(\lambda) \right) = 0$ for a.e. $t \in [0, 2\pi)$. However, since by hypothesis g is a nonzero harmonic

bounded function such that $g(e^{it}) \neq 0$ for a.e. $t \in [0, 2\pi)$, we deduce that the nontangential boundary value $\tilde{f}(e^{it})$ of the function \tilde{f} exists for a.e. $t \in [0, 2\pi)$ and $\tilde{f}(e^{it}) = 0$ for a.e. $t \in [0, 2\pi)$. From this, by applying the Axler–Zheng theorem (see [4, Theorem 2.2]), we assert that T_f is a compact operator on L^2_a . Therefore, since by hypothesis the range of T_f is closed, we have that T_f is a finite rank operator. Hence, by Luecking’s theorem [15] we conclude that T_f must be zero, which proves (b). The theorem is proven. \square

The arguments used in the proof of (b) allow us also to prove the following well-known theorem [7] by using a different method.

Theorem 2 *Let $f, g \in L^\infty(\mathbb{T})$ be two functions such that $f(e^{it}) \neq 0$ and $g(e^{it}) \neq 0$ almost everywhere on \mathbb{T} , and T_f, T_g be two associated Toeplitz operators on the Hardy space H^2 . Then $T_f T_g \neq 0$.*

Proof Suppose that $T_f T_g = 0$. It is well known that (see Engliš [10]) $\widetilde{T_f} = \tilde{f}$, $\widetilde{T_g} = \tilde{g}$. By hypothesis,

$$f(\xi)g(\xi) \neq 0 \tag{1}$$

for almost all $\xi \in \mathbb{T}$. On the other hand, we have from $T_f T_g = 0$ that $\widetilde{T_f T_g} = 0$; that is,

$$\langle T_f T_g \widehat{k}_{H^2, \lambda}, \widehat{k}_{H^2, \lambda} \rangle = 0$$

for all $\lambda \in \mathbb{D}$. Then, as in the proof of (b) in Theorem 1, we have that

$$\widetilde{T_f}(\lambda)\widetilde{T_g}(\lambda) = -\langle T_g \widehat{k}_{H^2, \lambda} - \tilde{g}(\lambda)\widehat{k}_{H^2, \lambda}, T_f \widehat{k}_{H^2, \lambda} \rangle$$

and hence

$$\tilde{f}(\lambda)\tilde{g}(\lambda) = -\langle T_g \widehat{k}_{H^2, \lambda} - \tilde{g}(\lambda)\widehat{k}_{H^2, \lambda}, T_f \widehat{k}_{H^2, \lambda} \rangle$$

for all $\lambda \in \mathbb{D}$. From this, by using the known fact that (see Engliš [10, Theorem 6] and Karaev [12, Lemma 1.1]) $\|T_g \widehat{k}_{H^2, \lambda} - \tilde{g}(\lambda)\widehat{k}_{H^2, \lambda}\| \rightarrow 0$ as $\lambda \rightarrow \mathbb{T}$ radially at almost every point of \mathbb{T} , we obtain that

$$|\tilde{f}(\lambda)\tilde{g}(\lambda)| \leq \|f\|_\infty \|T_g \widehat{k}_{H^2, \lambda} - \tilde{g}(\lambda)\widehat{k}_{H^2, \lambda}\| \rightarrow 0 \tag{2}$$

as $\lambda \rightarrow \mathbb{T}$ radially at almost every point of \mathbb{T} . Since \tilde{f} and \tilde{g} are harmonic, by Fatou’s theorem they have boundary values at almost every point of \mathbb{T} . It follows from the relation (2) by passing to the upper limit, as in the proof of Theorem 1, (b), that $\tilde{f}(\lambda)\tilde{g}(\lambda) \rightarrow 0$ as $\lambda \rightarrow \mathbb{T}$ radially, and hence $f(\xi)g(\xi) = 0$ for almost every $\xi \in \mathbb{T}$. This contradicts (1) and proves the theorem. \square

We remark that, of course, Theorem 2 can also be stated under the condition that $f(\xi)g(\xi) \neq 0$ almost everywhere on \mathbb{T} .

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References

- [1] Ahern P, Čučković Ž. A theorem of Brown-Halmos type for Bergman space Toeplitz operators. *J Funct Anal* 2001; 187: 200-210.
- [2] Ahern P, Flores M, Rudin W. An invariant volume-mean-value property. *J Funct Anal* 1993; 111: 380-397.
- [3] Aleman A, Vukotić D. Zero products of Toeplitz operators. *Duke Math J* 2009; 148: 373-403.
- [4] Axler S, Zheng D. Compact operators via the Berezin transform. *Indiana U Math J* 1998; 47: 387-400.
- [5] Axler S, Zheng D. The Berezin transform on the Toeplitz algebra. *Stud Math* 1998; 127: 113-136.
- [6] Berezin F. Covariant and contravariant symbols of operators. *Izv Akad Nauk SSSR Ser Mat* 1972; 36: 1134-1167.
- [7] Brown A, Halmos PR. Algebraic properties of Toeplitz operators. *J Reine Angew Math* 1963/1964; 213: 89-102.
- [8] Čučković Ž. Berezin versus Mellin. *J Math Anal Appl* 2003; 287: 234-343.
- [9] Engliš M. Functions invariant under the Berezin transform. *J Funct Anal* 1994; 121: 223-254.
- [10] Engliš M. Toeplitz operators and the Berezin transform on H^2 . *Linear Algebra Appl* 1995; 223/224: 171-204.
- [11] Guediri H. New function theoretic proofs of Brown-Halmos theorems. *Arab J Math Sci* 2007; 13: 15-26.
- [12] Karaev MT. On the Riccati equations. *Monatsh Math* 2008; 155: 161-166.
- [13] Le T. Finite rank products of Toeplitz operators in several complex variables. *Integr Equat Oper Th* 2009; 63: 547-555.
- [14] Louhichi I, Rao NV, Yousef A. Two questions on products of Toeplitz operators on the Bergman space. *Complex Anal Oper Th* 2009; 3: 881-889.
- [15] Luecking DH. Finite rank Toeplitz operators on the Bergman space. *P Am Math Soc* 2007; 136: 1717-1723.
- [16] Stroethoff K. Algebraic properties of Toeplitz operators on the Hardy space via the Berezin transform. *Contemp Math* 1999; 232: 313-319.