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## Multiplier and approximation theorems in Smirnov classes with variable exponent

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**Abstract:** Let  $G \subset \mathbb{C}$  be a bounded Jordan domain with a rectifiable Dini-smooth boundary  $\Gamma$  and let  $G^- := \text{ext } \Gamma$ . In terms of the higher order modulus of smoothness the direct and inverse problems of approximation theory in the variable exponent Smirnov classes  $E^{p(\cdot)}(G)$  and  $E^{p(\cdot)}(G^-)$  are investigated. Moreover, the Marcinkiewicz and Littlewood–Paley type theorems are proved. As a corollary some results on the constructive characterization problems in the generalized Lipschitz classes are presented.

**Key words:** Variable exponent Smirnov classes, direct and inverse theorems, Faber series, Lipschitz classes, Littlewood–Paley theorems, Marcinkiewicz theorems

### 1. Introduction

Let  $G \subset \mathbb{C}$  be a bounded Jordan domain, bounded by a rectifiable Jordan curve  $\Gamma$ , and let  $G^- := \text{Ext } \Gamma$ . Let also  $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$ ,  $\mathbb{D} := \text{Int } \mathbb{T}$  and  $\mathbb{D}^- := \text{Ext } \mathbb{T}$ .

The variable exponent Lebesgue space  $L^{p(\cdot)}(\Gamma)$  for a given Lebesgue measurable variable exponent  $p(z) \geq 1$  on  $\Gamma$  is defined as the set of the Lebesgue measurable functions  $f$ , such that  $\int_{\Gamma} |f(z)|^{p(z)} |dz| < \infty$ . If  $p^+ := \text{ess sup}_{z \in \Gamma} p(z) < \infty$ , then it becomes a Banach space, equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(z)/\lambda|^{p(z)} |dz| \leq 1 \right\} < \infty.$$

In the case of  $p(\cdot) \equiv p$  it turns to the classical Lebesgue space  $L^p(\Gamma)$ .

If  $\Gamma := \mathbb{T}$ , then we obtain the space  $L^{p(\cdot)}(\mathbb{T})$  with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf \left\{ \lambda > 0 : \int_0^{2\pi} |f(e^{it})/\lambda|^{p(e^{it})} dt \leq 1 \right\}.$$

A function  $f$  analytic in  $G$  is said to be of the Smirnov class  $E^p(G)$  if there exists a sequence of rectifiable Jordan curves  $(\gamma_n)$  in  $G$ , tending to the boundary  $\Gamma$  in the sense that  $\gamma_n$  eventually surrounds each compact

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subdomain of  $G$ , such that

$$\int_{\gamma_n} |f(z)|^p |dz| \leq M < \infty, \quad 1 \leq p < \infty.$$

Each function  $f \in E^p(G)$  has [8, pp. 419–438] nontangential boundary values almost everywhere (*a.e.*) on  $\Gamma$  and the boundary function belongs to  $L^p(\Gamma)$ . The Smirnov class  $E^p(G^-)$  is defined similarly. The sets

$$E^{p(\cdot)}(G) := \left\{ f \in E^1(G) : f \in L^{p(\cdot)}(\Gamma) \right\},$$

$$E^{p(\cdot)}(G^-) := \left\{ f \in E^1(G^-) : f \in L^{p(\cdot)}(\Gamma) \right\}$$

are called the variable exponent Smirnov classes of analytic functions in  $G$  and  $G^-$ , respectively. Equipped with the norm

$$\|f\|_{E^{p(\cdot)}(G)} := \|f\|_{L^{p(\cdot)}(\Gamma)}, \quad \|f\|_{E^{p(\cdot)}(G^-)} := \|f\|_{L^{p(\cdot)}(\Gamma)}$$

we make  $E^{p(\cdot)}(G)$  and  $E^p(G^-)$  into the Banach spaces.

Throughout this work we suppose that  $f(\infty) = 0$  as soon as  $f \in E^{p(\cdot)}(G^-)$ .

Let  $\mathcal{E}$  be the segment  $[0, 2\pi]$  or a Jordan rectifiable curve  $\Gamma$  and let  $p(\cdot) : \mathcal{E} \rightarrow \mathbb{R}^+ := [0, \infty)$  be a Lebesgue measurable function, which is defined on  $\mathcal{E}$  such that

$$1 \leq p_- := \operatorname{ess\,inf}_{z \in \mathcal{E}} p(z) \leq \operatorname{ess\,sup}_{z \in \mathcal{E}} p(z) =: p^+ < \infty. \tag{1}$$

**Definition 1** We say that  $p(\cdot) \in \mathcal{P}(\mathcal{E})$ , if  $p(\cdot)$  satisfies the conditions (1) and the inequality

$$|p(z_1) - p(z_2)| \ln \left( \frac{|\mathcal{E}|}{|z_1 - z_2|} \right) \leq c(p), \quad \forall z_1, z_2 \in \mathcal{E}, \quad z_1 \neq z_2$$

with a positive constant  $c(p)$ , where  $|\mathcal{E}|$  is the Lebesgue measure of  $\mathcal{E}$ .

If  $p(\cdot) \in \mathcal{P}(\mathcal{E})$  and  $p_- > 1$ , Then we say that  $p(\cdot) \in \mathcal{P}_0(\mathcal{E})$ .

Let  $g$  be a continuous function and let

$$\omega(g, t) := \sup_{|t_1 - t_2| \leq t} |g(t_1) - g(t_2)|, \quad t > 0$$

be its modulus of continuity.

**Definition 2** Let  $\Gamma$  be a smooth Jordan curve and let  $\theta(s)$  be the angle between the tangent and the positive real axis expressed as a function of arclength  $s$ . If  $\theta$  has a modulus of continuity  $\omega(\theta, s)$ , satisfying the Dini-smooth condition

$$\int_0^\delta [\omega(\theta, s) / s] ds < \infty, \quad \delta > 0,$$

Then we say that  $\Gamma$  is a Dini-smooth curve.

The set of Dini-smooth curves is denoted by  $\mathfrak{D}$ .

We suppose that  $\varphi$  and  $\varphi_1$  are the conformal mappings of  $G^-$  and  $G$  onto  $\mathbb{D}^-$ , respectively, and normalized by the following conditions:

$$\varphi(\infty) = \infty, \lim_{z \rightarrow \infty} \varphi(z)/z > 0 \quad \text{and} \quad \varphi_1(0) = \infty, \lim_{z \rightarrow 0} z\varphi_1(z) > 0.$$

Let  $\psi$  and  $\psi_1$  be the inverse mappings of  $\varphi$  and  $\varphi_1$ , respectively. The pairs  $(\varphi, \varphi_1)$  and  $(\psi, \psi_1)$  have continuous extensions to  $\Gamma$  and  $\mathbb{T}$ , respectively. Their derivatives  $(\varphi', \varphi_1')$  and  $(\psi', \psi_1')$  have definite nontangential boundary values *a.e.* on  $\Gamma$  and  $\mathbb{T}$ , which are integrable on  $\Gamma$  and  $\mathbb{T}$ , respectively [8, p. 419–438].

For  $f \in L^{p(\cdot)}(\Gamma)$ ,  $p \in \mathcal{P}(\Gamma)$ , we set  $f_0 := f \circ \psi$ ,  $p_0 := p \circ \psi$ ,  $f_1 := f \circ \psi_1$ , and  $p_1 := p \circ \psi_1$ . If  $\Gamma \in \mathfrak{D}$ , then as was proved in [17, Lemma 1], the following relations hold:

$$\begin{aligned} f_1 &\in L^{p_1(\cdot)}(\mathbb{T}) \Leftrightarrow f \in L^{p(\cdot)}(\Gamma) \Leftrightarrow f_0 \in L^{p_0(\cdot)}(\mathbb{T}), \\ p_0 &\in \mathcal{P}(\mathbb{T}) \Leftrightarrow p \in \mathcal{P}(\Gamma) \Leftrightarrow p_1 \in \mathcal{P}(\mathbb{T}). \end{aligned}$$

For  $f \in L^{p(\cdot)}(\Gamma)$  we define the Cauchy type integrals

$$f_0^+(w) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D} \quad \text{and} \quad f_1^+(w) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D},$$

which are analytic in  $\mathbb{D}$ .

**Definition 3** For  $f \in L^{p(\cdot)}(\mathbb{T})$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{T})$ , and  $t > 0$  we set

$$\Delta_t^r f(w) := \sum_{s=0}^r (-1)^{r+s} \binom{r}{s} f(w e^{ist}), \quad r = 1, 2, 3, \dots$$

and define the *r*th modulus of smoothness by

$$\Omega_r(f, \delta)_{\mathbb{T}, p(\cdot)} := \sup_{0 < |h| \leq \delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f(w) dt \right\|_{L^{p(\cdot)}(\mathbb{T})}.$$

We also define the moduli of smoothness for  $f \in E^{p(\cdot)}(G)$  and  $f \in E^{p(\cdot)}(G^-)$  as follows:

$$\begin{aligned} \Omega_r(f, \delta)_{G, p(\cdot)} &:= \Omega_r(f_0^+, \delta)_{\mathbb{T}, p_0(\cdot)}, \\ \Omega_r(f, \delta)_{G^-, p(\cdot)} &:= \Omega_r(f_1^+, \delta)_{\mathbb{T}, p_1(\cdot)}, \quad \delta > 0. \end{aligned}$$

The approximation aggregates used by us are constructed via the Faber polynomials  $F_k(z)$  and  $\tilde{F}_k(1/z)$  with respect to  $z$  and  $1/z$ , respectively. These polynomials can be defined in particular by the following series representations [17] (see also [29]):

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \quad w \in \mathbb{D}^-, \quad z \in G, \tag{2}$$

$$\frac{\psi_1'(w)}{\psi_1(w) - z} = \sum_{k=1}^{\infty} -\frac{\tilde{F}_k(1/z)}{w^{k+1}}, \quad w \in \mathbb{D}^- \quad z \in G^-. \tag{3}$$

Using (2) and Cauchy’s integral formulae

$$f(z) = \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|w|=1} \frac{f_0(w)\psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G,$$

which hold for every  $f \in E^{p(\cdot)}(G) \subset E^1(G)$ , we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z), \quad z \in G, \tag{4}$$

where

$$a_k = a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \quad k = 0, 1, 2, \dots \tag{5}$$

Using (3) and the integral representation

$$f(z) = \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|w|=1} \frac{f_1(w)\psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G^-,$$

which holds for every  $f \in E^{p(\cdot)}(G^-) \subset E^1(G^-)$ , we also have

$$f(z) \sim \sum_{k=1}^{\infty} \tilde{a}_k \tilde{F}_k(1/z), \quad z \in G^-, \tag{6}$$

with the coefficients

$$\tilde{a}_k = \tilde{a}_k(f) := -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}} dw, \quad k = 1, 2, \dots \tag{7}$$

In this work, in terms of the higher order modulus of smoothness, the direct and inverse theorems of approximation theory in the variable exponent Smirnov classes  $E^{p(\cdot)}(G)$  and  $E^{p(\cdot)}(G^-)$  are proved. Moreover, the Marcinkiewicz and Littlewood–Paley type theorems are obtained. As a corollary some results on the constructive characterization problems in the generalized Lipschitz classes are represented.

By  $c(\cdot)$ ,  $c_1(\cdot)$ ,  $c_2(\cdot)$ , and  $c(\cdot, \cdot)$ ,  $c_1(\cdot, \cdot)$ ,  $c_2(\cdot, \cdot), \dots$  we denote the constants, depending in general only on the parameters, given in the corresponding brackets and independent of  $n$ .

Let  $\Pi_n$  be the class of algebraic polynomials of degree not exceeding  $n$  and let

$$E_n(f)_{G,p(\cdot)} := \inf \left\{ \|f - P_n\|_{L^{p(\cdot)}(\Gamma)} : P_n \in \Pi_n \right\}, \quad n = 1, 2, \dots$$

be the best approximation number of  $f \in E^{p(\cdot)}(G)$  in  $\Pi_n$ .

Our new results are presented as follows:

**Theorem 4** Let  $\Gamma \in \mathfrak{D}$ . If  $f \in E^{p(\cdot)}(G)$ ,  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ , Then there is a positive constant  $c(p, r)$  such that the inequality

$$E_n(f)_{G, p(\cdot)} \leq c(p, r) \Omega_r(f, 1/n)_{G, p(\cdot)}, \quad n = 1, 2, 3, \dots$$

holds.

**Theorem 5** Let  $\Gamma \in \mathfrak{D}$ . If  $f \in E^{p(\cdot)}(G)$ ,  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ , Then there is a positive constant  $c(p, r)$  such that the inequality

$$\Omega_r(f, 1/n)_{G, p(\cdot)} \leq \frac{c(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_k(f)_{G, p(\cdot)}, \quad n = 1, 2, 3, \dots$$

holds.

Theorems 4 and 5 in the case of  $r = 1$  were proved in [16] (see also [17]). In the variable exponent Lebesgue spaces  $L^{p(\cdot)}([0, 2\pi])$  these theorems in the case of  $r=1$  and  $p(\cdot) \in \mathcal{P}_0([0, 2\pi])$ , using some other modulus of smoothness, were proved in [1-3,10,20,21]. For a wider class of the exponents  $p(\cdot)$ , namely when  $p(\cdot) \in \mathcal{P}([0, 2\pi]) \supset \mathcal{P}_0([0, 2\pi])$ , the direct and inverse theorems in  $L^{p(\cdot)}([0, 2\pi])$  in term of the first modulus of smoothness were obtained in the papers [16, 25-28].

Note that in the classical Smirnov classes the direct and inverse problems of approximation theory were investigated adequately. Detailed information about these investigations can be found in [4,5,11-13,22,30] and the references given therein. In the variable exponent Smirnov classes some approximation problems were also investigated in [14, 16, 18].

**Corollary 6** If  $E_n(f)_{G, p(\cdot)} = \mathcal{O}(n^{-\alpha})$ ,  $\alpha > 0$ , Then under the conditions of Theorem 5,

$$\Omega_r(f, \delta)_{G, p(\cdot)} = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha \\ \mathcal{O}(\delta^r \log 1/\delta), & r = \alpha \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases}$$

If we define the generalized Lipschitz class  $Lip(G, p(\cdot), \alpha)$  with  $\alpha > 0$  and  $r := [\alpha] + 1$ , where  $[\alpha]$  is the integer part of  $\alpha$ , by

$$Lip(G, p(\cdot), \alpha) := \left\{ f \in E^{p(\cdot)}(G) : \Omega_r(f, \delta)_{G, p(\cdot)} = \mathcal{O}(\delta^\alpha), \delta > 0 \right\},$$

then from Corollary 6 we obtain:

**Corollary 7** If  $E_n(f)_{G, p(\cdot)} = \mathcal{O}(n^{-\alpha})$ ,  $\alpha > 0$ , Then under the conditions of Theorem 5 we have that  $f \in Lip(G, p(\cdot), \alpha)$ .

On the other hand, from Theorem 4 we have:

**Corollary 8** If  $f \in Lip(G, p(\cdot), \alpha)$ ,  $\alpha > 0$ , Then  $E_n(f)_{G, p(\cdot)} = \mathcal{O}(n^{-\alpha})$ .

Combining Corollaries 7 and 8 we obtain the following constructive characterization of the generalized Lipschitz class  $Lip(G, p(\cdot), \alpha)$ :

**Theorem 9** Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . Then for  $\alpha > 0$  the following statements are equivalent:

$$i) f \in Lip(G, p(\cdot), \alpha), \quad ii) E_n(f)_{G, p(\cdot)} = \mathcal{O}(n^{-\alpha}) \quad n = 1, 2, 3, \dots$$

Let  $\{\lambda_k\}_0^\infty$  be a sequence of complex numbers, satisfying for every natural numbers  $k$  and  $m$  the conditions

$$|\lambda_k| \leq c, \quad \sum_{k=2^{m-1}}^{2^m-1} |\lambda_k - \lambda_{k+1}| \leq c \quad (8)$$

with some positive constant  $c > 0$ .

**Theorem 10** Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . If  $f \in E^{p(\cdot)}(G)$  with the Faber series (4) and  $\{\lambda_k\}_0^\infty$  is a sequence of complex numbers satisfying the condition (8), then there exist a function  $F \in E^{p(\cdot)}(G)$  and a positive constant  $c(p)$  such that

$$F(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k(f) F_k(z), \quad z \in G$$

and  $\|F\|_{L^{p(\cdot)}(\Gamma)} \leq c \|f\|_{L^{p(\cdot)}(\Gamma)}$ .

Let

$$\Delta_k(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} a_j(f) F_j(z), \quad k = 1, 2, \dots$$

be the lacunary partial sums of Faber series of  $f \in E^{p(\cdot)}(G)$ . The following Littlewood–Paley type theorem holds:

**Theorem 11** Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . If  $f \in E^{p(\cdot)}(G)$ , Then there exist the constants  $c_1(p)$  and  $c_2(p)$  such that the inequalities

$$c_1(p) \|f\|_{L^{p(\cdot)}(\Gamma)} \leq \left\| \left( \sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\Gamma)} \leq c_2(p) \|f\|_{L^{p(\cdot)}(\Gamma)}$$

hold.

The results, similar to Theorems 10 and 11 in the classical Lebesgue spaces  $L^p([0, 2\pi])$ , were first proved by Littlewood and Paley in [24]. They play an important role in the various problems of approximation theory, especially for the improvements of the direct and inverse theorems. In the classical Smirnov classes, Theorems 10 and 11 were obtained in [9].

All of the results formulated above can be formulated also in the classes  $E^{p(\cdot)}(G^-)$ .

Let

$$E_n(f)_{G^-, p(\cdot)} := \inf \left\{ \|f - P_n^*\|_{L^{p(\cdot)}(\Gamma)} : P_n^* \in \Pi_n^* \right\}, \quad n = 1, 2, \dots$$

be the best approximation number of  $f \in E^{p(\cdot)}(G^-)$  in the class  $\Pi_n^*$ , of the algebraic polynomials with respect to  $1/z$ , of degree not exceeding  $n$ .

**Theorem 12** Let  $\Gamma \in \mathfrak{D}$ . If  $f \in E^{p(\cdot)}(G^-)$ ,  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ , Then there exists a positive constant  $c(p, r)$  such that the inequality

$$E_n(f)_{G^-, p(\cdot)} \leq c(p, r) \Omega_r(f, 1/n)_{G^-, p(\cdot)}, \quad n = 1, 2, 3, \dots$$

holds.

**Theorem 13** Let  $\Gamma \in \mathfrak{D}$ . If  $f \in E^{p(\cdot)}(G^-)$  with  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ , Then there exists a positive constant  $c(p, r)$  such that the inequality

$$\Omega_r(f, 1/n)_{G^-, p(\cdot)} \leq \frac{c(p, r)}{n^r} \sum_{k=1}^n (k+1)^{r-1} E_k(f)_{G^-, p(\cdot)}, \quad n = 1, 2, 3, \dots$$

holds.

**Corollary 14** If  $E_n(f)_{G^-, p(\cdot)} = \mathcal{O}(n^{-\alpha})$  for some  $\alpha > 0$ , Then under the conditions of Theorem 13

$$\Omega_r(f, \delta)_{G^-, p(\cdot)} = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha \\ \mathcal{O}(\delta^r \log(1/\delta)), & r = \alpha \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases}$$

Similarly, if we define the generalized Lipschitz class  $Lip(G^-, p(\cdot), \alpha)$  with  $\alpha > 0$  and  $r := [\alpha] + 1$  by

$$Lip(G^-, p(\cdot), \alpha) := \left\{ f \in E^{p(\cdot)}(G^-) : \Omega_r(f, \delta)_{G^-, p(\cdot)} = \mathcal{O}(\delta^\alpha), \delta > 0 \right\},$$

then we have:

**Corollary 15** If  $E_n(f)_{G^-, p(\cdot)} = \mathcal{O}(n^{-\alpha})$ ,  $\alpha > 0$ , Then under the conditions of Theorem 13 we have  $f \in Lip(G^-, p(\cdot), \alpha)$ .

On the other hand, Theorem 12 implies:

**Corollary 16** If  $f \in Lip(G^-, p(\cdot), \alpha)$  for some  $\alpha > 0$ , Then  $E_n(f)_{G^-, p(\cdot)} = \mathcal{O}(n^{-\alpha})$ .

Hence, by means of Corollaries 15 and 16, we can formulate the following theorem, which gives a constructive characterization of the generalized Lipschitz class  $Lip(G^-, p(\cdot), \alpha)$ :

**Theorem 17** Let  $\Gamma \in \mathfrak{D}$ ,  $p(\cdot) \in \mathcal{P}_0(\Gamma)$  and let  $\alpha > 0$ . The following statements are equivalent:

$$i) f \in Lip(G^-, p(\cdot), \alpha), \quad ii) E_n(f)_{G^-, p(\cdot)} = \mathcal{O}(n^{-\alpha}), \quad n = 1, 2, 3, \dots$$

**Theorem 18** Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . If  $f \in E^{p(\cdot)}(G^-)$  with the Faber series (6) and  $\{\lambda_k\}_0^\infty$  is a sequence of complex numbers, satisfying the condition (8), then there exists a function  $F \in E^{p(\cdot)}(G^-)$  such that

$$F(z) \sim \sum_{k=1}^\infty \lambda_k \tilde{a}_k(f) \tilde{F}_k(1/z), \quad z \in G^-$$

and  $\|F\|_{L^{p(\cdot)}(\Gamma)} \leq c \|f\|_{L^{p(\cdot)}(\Gamma)}$ .



Denoting the lacunary partial sums of Faber series of  $f \in E^{p(\cdot)}(G^-)$  by

$$\tilde{\Delta}_k(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} \tilde{a}_j(f) \tilde{F}_j(1/z),$$

we have the following Littlewood–Paley type theorem in the classes  $E^{p(\cdot)}(G^-)$ :

**Theorem 19** *Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . If  $f \in E^{p(\cdot)}(G^-)$ , Then there exist the positive constants  $c_3(p)$  and  $c_4(p)$  such that the inequalities*

$$c_3(p) \|f\|_{L^{p(\cdot)}(\Gamma)} \leq \left\| \left( \sum_{k=1}^{\infty} |\tilde{\Delta}_k(f)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\Gamma)} \leq c_4(p) \|f\|_{L^{p(\cdot)}(\Gamma)}$$

hold.

### 2. Auxiliary results

Let  $\mathcal{T}_n$  be the class of the trigonometric polynomials of degree not exceeding  $n$ . The *best approximation number* of  $f \in L^{p(\cdot)}(\mathbb{T})$  in  $\mathcal{T}_n$  is defined by

$$E_n(f)_{p(\cdot)} := \inf \left\{ \|f - T_n\|_{L^{p(\cdot)}(\mathbb{T})} : T_n \in \mathcal{T}_n \right\}, \quad n = 0, 1, 2, \dots .$$

We will use the following direct and inverse results proved in [19]:

**Theorem A** [19] *Let  $f \in L^{p(\cdot)}(\mathbb{T})$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{T})$ , and let  $r \in \mathbb{N}$ . Then there exists a positive constant  $c(p, r)$  such that for every  $n \in \mathbb{N}$  The inequality*

$$E_n(f)_{p(\cdot)} \leq c(p, r) \Omega_r(f, 1/n)_{\mathbb{T}, p(\cdot)}$$

holds.

**Theorem B** [19] *Let  $f \in L^{p(\cdot)}(\mathbb{T})$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{T})$ , and let  $r \in \mathbb{N}$ . Then there exists a positive constant  $c(p, r)$  such that for every  $n \in \mathbb{N}$  The inequality*

$$\Omega_r(f, 1/n)_{\mathbb{T}, p(\cdot)} \leq \frac{c(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_k(f)_{p(\cdot)}$$

holds.

Let

$$S_{\Gamma}(f)(z) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \{\zeta \in \Gamma: |\zeta - z| < \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

be the Cauchy singular integral of  $f \in L^{p(\cdot)}(\Gamma)$ . By Privalov’s theorem, the Cauchy type integrals

$$f^+(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{[\psi'(w)]}{\psi(w) - z} f_0(w) dw, \quad z \in G$$

$$f^-(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{[\psi'(w)]}{\psi(w) - z} f_1(w) dw, \quad z \in G^-$$

have the nontangential inside and outside limits  $f^+$  and  $f^-$ , respectively *a.e.* on  $\Gamma$ . Furthermore, the formulas

$$f^+(z) = S_{\Gamma}(f)(z) + \frac{1}{2}f(z) \quad \text{and} \quad f^-(z) = S_{\Gamma}(f)(z) - \frac{1}{2}f(z) \quad (9)$$

are valid *a.e.* on  $\Gamma$ , which implies that

$$f(z) = f^+(z) - f^-(z) \quad (10)$$

*a.e.* on  $\Gamma$ .

The following theorem is a special case of the more general result on the boundedness of Cauchy's singular operator  $S_{\Gamma}(f)$  in  $L^{p(\cdot)}(\Gamma)$ , proved in [23, p. 59, Theorem 2.45].

**Theorem C** *Let  $\Gamma \in \mathfrak{D}$  and  $p \in \mathcal{P}_0(\Gamma)$ . Then the Cauchy singular operator  $S_{\Gamma}(f)$  is bounded in  $L^{p(\cdot)}(\Gamma)$ .*

**Lemma 20** [16] *Let  $\Gamma \in \mathfrak{D}$ . If  $f \in L^{p(\cdot)}(\Gamma)$ ,  $p \in \mathcal{P}_0(\Gamma)$ , Then  $f^+ \in E^{p(\cdot)}(G)$  and  $f^- \in E^{p(\cdot)}(G^-)$ .*

Now we consider the operators

$$T(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w) \psi'(w)}{\psi(w) - z} dw, \quad f \in E^{p_0(\cdot)}(\mathbb{D}), \quad z \in G,$$

$$\tilde{T}(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w) \psi_1'(w)}{\psi_1(w) - z} dw, \quad f \in E^{p_1(\cdot)}(\mathbb{D}), \quad z \in G^-,$$

defined on the classes  $E^{p_0(\cdot)}(\mathbb{D})$  and  $E^{p_1(\cdot)}(\mathbb{D})$ , respectively.

The following lemma holds.

**Lemma 21** *Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . Then:*

- i) The operator  $T : E^{p_0(\cdot)}(\mathbb{D}) \rightarrow E^{p(\cdot)}(G)$  is linear, bounded, one-to-one, and onto. Moreover,  $T(f_0^+) = f$  for every  $f \in E^{p(\cdot)}(G)$ ,*
- ii) The operator  $\tilde{T} : E^{p_1(\cdot)}(\mathbb{D}) \rightarrow E^{p(\cdot)}(G^-)$  is linear, bounded, one-to-one, and onto. Moreover,  $\tilde{T}(f_1^+) = f$  for every  $f \in E^{p(\cdot)}(G^-)$ .*

Note that *assertion i)* was proved in [16]. *Assertion ii)* can be proved using the same techniques.

**Lemma 22** *Let  $\Gamma \in \mathfrak{D}$  and  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . Then there exist the positive constants  $c_i(p)$ ,  $i=5,6,7,8$ , such that the following assertions hold:*

- i) if  $f \in E^{p(\cdot)}(G)$ , Then  $E_n(f_0^+)_{p_0(\cdot)} \leq c_5(p)E_n(f)_{G,p(\cdot)} \leq c_6(p)E_n(f_0^+)_{p_0(\cdot)}$ ;*
- ii) if  $f \in E^{p(\cdot)}(G^-)$ , Then  $E_n(f_1^+)_{p_1(\cdot)} \leq c_7(p)E_n(f)_{G^-,p(\cdot)} \leq c_8(p)E_n(f_1^+)_{p_1(\cdot)}$ .*

**Proof** Assertion *i*) was proved in [16]. We will prove *ii*). Since the operator  $\tilde{T} : E^{p(\cdot)}(\mathbb{D}) \rightarrow E^{p(\cdot)}(G^-)$  is linear, bounded, one-to-one, and onto, it has the inverse operator  $\tilde{T}^{-1} : E^{p(\cdot)}(G^-) \rightarrow E^{p(\cdot)}(\mathbb{D})$ , which is also linear, bounded, one-to-one, and onto. If  $f \in E^{p(\cdot)}(G^-)$ , then  $\tilde{T}^{-1}(f) = f_1^+ \in E^{p(\cdot)}(\mathbb{D})$ . If  $P_n^*$  is the polynomial of the best approximation to  $f$  in  $E^{p(\cdot)}(G^-)$  with respect to  $1/z$  and degree not exceeding  $n$ , then  $\tilde{T}^{-1}(P_n^*)$  is a polynomial with respect to  $w$  and degree not exceeding  $n$ . Therefore, denoting  $c_7(p) := \|\tilde{T}^{-1}\|$ , we get

$$\begin{aligned} E_n(f_1^+)_{p(\cdot)} &\leq \|f_1^+ - \tilde{T}^{-1}(P_n^*)\|_{L^{p(\cdot)}(\mathbb{T})} \\ &\leq \|\tilde{T}^{-1}(f) - \tilde{T}^{-1}(P_n^*)\|_{L^{p(\cdot)}(\mathbb{T})} \\ &\leq \|\tilde{T}^{-1}\| \|f - P_n^*\|_{L^{p(\cdot)}(\Gamma)} = c_7(p) E_n(f)_{G^-, p(\cdot)}. \end{aligned}$$

On the other hand, by Lemma 21 we have that  $\tilde{T}(f_1^+) = f \in E^{p(\cdot)}(G^-)$  and then by boundedness of  $\tilde{T}$

$$\begin{aligned} E_n(f)_{G^-, p(\cdot)} &\leq \|f - \tilde{T}(P_n^*)\|_{L^{p(\cdot)}(\Gamma)} \\ &\leq \|\tilde{T}(f_1^+) - \tilde{T}(P_n^*)\|_{L^{p(\cdot)}(\Gamma)} \\ &\leq \|\tilde{T}\| \|f_1^+ - P_n^*\|_{L^{p(\cdot)}(\mathbb{T})} = c_8(p) E_n(f_1^+)_{p(\cdot)}. \end{aligned}$$

□

**Lemma 23** Let  $p(\cdot) \in \mathcal{P}_0(\Gamma)$  and let  $\{\lambda_k\}_0^\infty$  be a sequence of complex numbers satisfying the condition (8). If  $g \in E^{p(\cdot)}(\mathbb{D})$  has the Taylor series

$$g(w) = \sum_{k=0}^\infty \beta_k(g) w^k, \quad w \in \mathbb{D}$$

Then there exists a function  $g^* \in E^{p(\cdot)}(\mathbb{D})$  that has the Taylor series

$$g^*(w) = \sum_{k=0}^\infty \lambda_k \beta_k(g) w^k, \quad w \in \mathbb{D}$$

and  $\|g^*\|_{L^{p(\cdot)}(\mathbb{T})} \leq c(p) \|g\|_{L^{p(\cdot)}(\mathbb{T})}$ .

**Proof** Let  $g \in E^{p(\cdot)}(\mathbb{D})$  and  $c_k(g)$  ( $k = \dots, -1, 0, 1, \dots$ ) be the Fourier coefficients of the boundary function of  $g$ . Then (Theorem 3.4 in [7, p. 38]) we have

$$c_k(g) = \begin{cases} \beta_k(g), & k \geq 0 \\ 0, & k < 0. \end{cases}$$

By the Marcinkiewicz type theorem [23, p. 120, Theorem 2.103], there is a function  $h \in L^{p(\cdot)}(\mathbb{T})$  with the Fourier coefficients  $c_k(h) = \lambda_k c_k(g)$  such that  $\|h\|_{L^{p(\cdot)}(\mathbb{T})} \leq c(p) \|g\|_{L^{p(\cdot)}(\mathbb{T})}$ , for some positive constant  $c(p)$ .

Since  $g^* := h^+ \in E^{p(\cdot)}(\mathbb{D})$ , for the Taylor coefficients  $\beta_k(g^*)$ ,  $k = 0, 1, 2, \dots$ , of  $g^*$ , by (10) we have

$$\begin{aligned} \beta_k(g^*) &= \beta_k(h^+) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h^+(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(w)}{w^{k+1}} dw + \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h^-(w)}{w^{k+1}} dw \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(w)}{w^{k+1}} dw = c_k(h) = \lambda_k c_k(g) = \lambda_k \beta_k(g) \end{aligned}$$

and

$$\|g^*\|_{L^{p(\cdot)}(\mathbb{T})} \leq \|h^+\|_{L^{p(\cdot)}(\mathbb{T})} \leq c_9(p) \|h\|_{L^{p(\cdot)}(\mathbb{T})} \leq c_{10}(p) \|g\|_{L^{p(\cdot)}(\mathbb{T})}.$$

□

### 3. Proofs of main results

**Proof of Theorem 4** If  $f \in E^{p(\cdot)}(G)$ , then  $f_0^+ = T^{-1}(f) \in E^{p_0(\cdot)}(\mathbb{D})$ . Hence, applying the second inequality of assertion *i*) in Lemma 22 and Theorem A, we have

$$\begin{aligned} E_n(f)_{G,p(\cdot)} &\leq c_6(p) E_n(f_0^+)_{p_0(\cdot)} \\ &\leq c_6(p) c(p, r) \Omega_r(f_0^+, 1/n)_{\mathbb{T}, p_0(\cdot)} = c_1(p, r) \Omega_r(f, 1/n)_{G,p(\cdot)}. \end{aligned}$$

□

**Proof of Theorem 5** If  $f \in E^{p(\cdot)}(G)$ , then by Lemma 21 we have that  $f_0^+ \in E^{p_0(\cdot)}(\mathbb{D})$ . Applying Theorem B for the boundary values of  $f_0^+$  and the first inequality of assertion *i*) in Lemma 22, we obtain the desired inequality:

$$\begin{aligned} \Omega_r(f, 1/n)_{G,p(\cdot)} &= \Omega_r(f_0^+, 1/n)_{\mathbb{T}, p_0(\cdot)} \leq \frac{c(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_k(f_0^+)_{p_0(\cdot)} \\ &\leq \frac{c(p, r) c_5(p)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_n(f)_{G,p(\cdot)} \\ &\leq \frac{c_2(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_n(f)_{G,p(\cdot)} \end{aligned}$$

□

**Proof of Theorem 10** Let  $f \in E^{p(\cdot)}(G)$ . Then by (5) and (10) we have

$$\begin{aligned} a_k(f) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}} dw \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw = \beta_k(f_0^+), \quad k = 0, 1, 2, \dots \end{aligned}$$

This means that the Faber coefficients of  $f$  are also the Taylor coefficients of  $f_0^+$  at the origin; that is,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f) w^k, \quad w \in \mathbb{D}.$$

Since  $f \in E^{p(\cdot)}(G)$  we have  $f_0^+ \in E^{p_0(\cdot)}(\mathbb{D})$  and then by Lemma 23 there is a function  $F_0 \in E^{p_0(\cdot)}(\mathbb{D})$  with the Taylor coefficients  $\beta_k(F_0) = \lambda_k \beta_k(f_0^+) = \lambda_k a_k(f)$ ,  $k = 0, 1, 2, \dots$ , such that

$$\|F_0\|_{L^{p_0(\cdot)}(\mathbb{T})} \leq c(p) \|f_0^+\|_{L^{p_0(\cdot)}(\mathbb{T})}.$$

At the same time, by Lemma 21,  $F := T(F_0) \in E^{p(\cdot)}(G)$  and has the Faber coefficients  $\beta_k(F_0) = \lambda_k a_k(f)$ ,  $k = 0, 1, 2, \dots$ . Hence,

$$F(z) = T(F_0)(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k(f) F_k(z), \quad z \in G.$$

Now using the boundedness of  $T$ , (9), and Theorem C, we have

$$\begin{aligned} \|F\|_{L^{p(\cdot)}(\Gamma)} &= \|T(F_0)\|_{L^{p(\cdot)}(\Gamma)} \leq \|T\| \|F_0\|_{L^{p_0(\cdot)}(\mathbb{T})} \\ &\leq c(p) \|f_0^+\|_{L^{p_0(\cdot)}(\mathbb{T})} \leq c_{11}(p) \|f\|_{L^{p(\cdot)}(\Gamma)} \leq c_{12}(p) \|f\|_{L^{p(\cdot)}(\Gamma)}. \end{aligned}$$

□

Let  $\omega$  be a weight on  $\Gamma$ , i.e. an almost everywhere nonnegative integrable function on  $\Gamma$ , and let  $B_r(z) := \{t : |t - z| < r, z \in \Gamma\}$ ,  $r > 0$ .  $\omega$  is said to satisfy Muckenhoupt's  $A_p(\Gamma)$ ,  $1 < p < \infty$ , condition if

$$\sup_{z \in \Gamma} \sup_{r > 0} \left( \frac{1}{r} \int_{\Gamma \cap B_r(z)} \omega(z) |dz| \right) \left( \frac{1}{r} \int_{\Gamma \cap B_r(z)} \omega(z)^{1-p'} |dz| \right)^{p-1} < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let also  $\mathcal{M}f$  be the maximal function, defined as

$$\mathcal{M}f(z) := \sup_{\gamma \ni z} \left( \frac{1}{|\gamma|} \int_{\gamma} |f(z)| |dz| \right), \quad f \in L(\Gamma),$$

where the supremum is taken over all rectifiable arcs  $\gamma \subset \Gamma$  that contain  $z$  and  $|\gamma|$  is a Lebesgue measure of  $\gamma$ .

**Proof of Theorem 11** Let  $\omega \in A_p(\Gamma)$ ,  $\Gamma \in \mathfrak{D}$ , and  $1 < p < \infty$ . In [9, Theorem 3] it was proved that if  $f \in E^p(G, \omega)$ , then there exist the positive constants  $c_{13}(p)$  and  $c_{14}(p)$  such that

$$c_{13}(p) \|f\|_{L^p(\Gamma, \omega)} \leq \left\| \left( \sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_{L^p(\Gamma, \omega)} \leq c_{14}(p) \|f\|_{L^p(\Gamma, \omega)}. \tag{11}$$

Considering the operator  $\mathcal{A} : f \rightarrow \left( \sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2}$ , we see that by the second inequality of (11), it is bounded in  $L^p(\Gamma, \omega)$ . On the other hand, the maximal operator  $\mathcal{M}$  is bounded (see [23, p. 50, Theorem 2.29])

in  $L^{p(\cdot)}(\Gamma)$ ,  $p(\cdot) \in \mathcal{P}_0(\Gamma)$ . Hence, all of the conditions of Corollary 5.32 of extrapolation Theorem 5.28, proved in [6, pp. 209–211], are fulfilled. Then

$$\left\| \left( \sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\Gamma)} \leq c_{15}(p) \|f\|_{L^{p(\cdot)}(\Gamma)},$$

and therefore, the second inequality of Theorem 11 is proved. The proof of the first inequality goes similarly.  $\square$

**Proof of Theorem 12** If  $f \in E^{p(\cdot)}(G^-)$ , then by assertion *ii*) of Lemma 21, we get  $f_1^+ \in E^{p_1(\cdot)}(\mathbb{D})$ . Applying the second inequality of assertion *ii*) in Lemma 22 and Theorem A for  $f_1^+$  we have that

$$\begin{aligned} E_n(f)_{G^-, p(\cdot)} &\leq c(p) E_n(f_1^+)_{p_1(\cdot)} \\ &\leq c(p, r) \Omega_r(f_1^+, 1/n)_{\mathbb{T}, p_1(\cdot)} = c(p, r) \Omega_r(f, 1/n)_{G^-, p(\cdot)}. \end{aligned}$$

$\square$

**Proof of Theorem 13** If  $f \in E^{p(\cdot)}(G^-)$ , then by assertion *ii*) of Lemma 21 we have  $f_1^+ \in E^{p_1(\cdot)}(\mathbb{D})$ . Hence, applying Theorem B for the boundary values of  $f_1^+$  and the first inequality of assertion *ii*) in Lemma 22, we get

$$\begin{aligned} \Omega_r(f, 1/n)_{G^-, p(\cdot)} &= \Omega_r(f_1^+, 1/n)_{\mathbb{T}, p_1(\cdot)} \leq \frac{c(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_k(f_1^+)_{p_1(\cdot)} \\ &\leq \frac{c(p, r) c_7(p)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_n(f)_{G^-, p(\cdot)} \\ &\leq \frac{c_3(p, r)}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_n(f)_{G^-, p(\cdot)}. \end{aligned}$$

$\square$

**Proof of Theorem 18** Let  $f \in E^{p(\cdot)}(G^-)$ . By (7) and (10)

$$\begin{aligned} \tilde{a}_k(f) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1^-(w)}{w^{k+1}} dw \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}} dw = \beta_k(f_1^+), \end{aligned}$$

where  $\beta_k(f_1^+)$ ,  $k = 0, 1, 2, \dots$ , are the Taylor coefficients of  $f_1^+ \in E^{p_1(\cdot)}(\mathbb{D})$ . This means that the Faber coefficients  $\tilde{a}_k(f)$  of  $f$  are the Taylor coefficients of  $f_1^+$  at the origin; that is,

$$f_1^+(w) = \sum_{k=0}^{\infty} \tilde{a}_k(f) w^k, \quad w \in \mathbb{D}.$$

Since  $f \in E^{p(\cdot)}(G^-)$  by assertion *ii*) of Lemma 21, we have  $f_1^+ \in E^{p_1(\cdot)}(\mathbb{D})$  and then by Lemma 23 there is a function  $F_1 \in E^{p_1(\cdot)}(\mathbb{D})$  with the Taylor coefficients  $\beta_k(F_1) = \lambda_k \beta_k(f_1^+) = \lambda_k \tilde{a}_k(f)$ ,  $k = 0, 1, 2, \dots$ , such that

$$\|F_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c(p) \|f_1^+\|_{L^{p_1(\cdot)}(\mathbb{T})}.$$

Since  $\tilde{T}(F_1) \in E^{p(\cdot)}(G^-)$ , its Faber coefficients are  $\beta_k(F_1) = \lambda_k \tilde{a}_k(f)$  and hence

$$\tilde{T}(F_1)(z) \sim \sum_{k=0}^{\infty} \lambda_k \tilde{a}_k(f) \tilde{F}_k(1/z), \quad z \in G^-.$$

Now denoting  $F := \tilde{T}(F_1)$ , using the boundedness of  $\tilde{T}$  in  $E^{p_1(\cdot)}(\mathbb{D})$  and the relation (9), and applying Theorem C, we have

$$\begin{aligned} \|F\|_{L^{p(\cdot)}(\Gamma)} &= \left\| \tilde{T}(F_1) \right\|_{L^{p(\cdot)}(\Gamma)} \leq \left\| \tilde{T} \right\| \|F_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \\ &\leq c_{16}(p) \|f_1^+\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c_{17}(p) \|f_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c(p) \|f\|_{L^{p(\cdot)}(\Gamma)}. \end{aligned}$$

□

**Proof of Theorem 19** The proof can be realized similarly to the proof of Theorem 11. We just need to apply the relation

$$c \|f\|_{L^p(\Gamma, \omega)} \leq \left\| \left( \sum_{k=1}^{\infty} |\tilde{\Delta}_k(f)|^2 \right)^{1/2} \right\|_{L^p(\Gamma, \omega)} \leq c \|f\|_{L^p(\Gamma, \omega)},$$

proved in [9, Theorem 4], instead of (11). □

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