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A new class of generalized polynomials

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Abstract: Motivated by their importance and potential for applications in a variety of research fields, recently, various polynomials and their extensions have been introduced and investigated. In this sequel, we modify the known generating functions of polynomials, due to both Milne-Thomson and Dere and Simsek, to introduce a new class of generalized polynomials and present some of their involved properties. As obvious special cases of the newly introduced polynomials, we also introduce so-called power sum-Laguerre–Hermite polynomials and generalized Laguerre and Euler polynomials and we present some of their involved identities and formulas. The results presented here, being very general, are pointed out to be specialized to yield a number of known and new identities involving relatively simple and familiar polynomials.

Key words: Milne-Thomson polynomials, Dere–Simsek polynomials, Laguerre polynomials, Hermite polynomials, Euler polynomials, generalized Laguerre–Euler polynomials, sum of integer powers, summation formulae, symmetric identities

1. Introduction and preliminaries

Throughout this paper, let \mathbb{C} , \mathbb{R} , \mathbb{R}^+ , \mathbb{Z} , and \mathbb{N} be the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The two variable Laguerre polynomials $L_n(x, y)$ are generated by (see [7])

$$\frac{1}{1-yt} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x, y) t^n \quad (|yt| < 1). \quad (1)$$

Also, equivalently, the polynomials $L_n(x, y)$ are given by (see [8])

$$e^{yt} C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}, \quad (2)$$

where $C_0(x)$ denotes the 0th order Tricomi function. The n th order Tricomi functions $C_n(x)$ are generated by

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=0}^{\infty} C_n(x) t^n \quad (t \in \mathbb{C} \setminus \{0\}, x \in \mathbb{C}). \quad (3)$$

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We have

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \quad (n \in \mathbb{N}_0). \tag{4}$$

The Tricomi functions $C_n(x)$ are connected with the Bessel function of the first kind $J_n(x)$ (see [6]):

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}). \tag{5}$$

From (2) and (4), we find

$$L_n(x, y) = n! \sum_{s=0}^n \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!} = y^n L_n(x/y), \tag{6}$$

where $L_n(x)$ are the ordinary Laguerre polynomials (see, e.g., [1, 24]). We have

$$L_n(x, 0) = \frac{(-1)^n x^n}{n!}, \quad L_n(0, y) = y^n, \quad L_n(x, 1) = L_n(x). \tag{7}$$

Milne-Thomson [20] defined polynomials $\Phi_n^{(\alpha)}(x)$ of degree n and order α by the following generating function:

$$f(t, \alpha) e^{xt+g(t)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x) \frac{t^n}{n!}, \tag{8}$$

where $f(t, \alpha)$ is a function of t and $\alpha \in \mathbb{Z}$ and $g(t)$ is a function of t . Then, by choosing some explicit functions of $f(t, \alpha)$ and $g(t)$, he [20] presented several interesting properties for certain polynomials such as Bernoulli polynomials and Hermite polynomials.

Dere and Simsek [9] made a slight modification of the Milne-Thomson polynomials $\Phi_n^{(\alpha)}(x)$ to give polynomials $\Phi_n^{(\alpha)}(x, \nu)$ of degree n and order α by means of the following generating function:

$$G(t, x; \alpha, \nu) := f(t, \alpha) e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, \nu) \frac{t^n}{n!}, \tag{9}$$

where $f(t, \alpha)$ and $h(t, \nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t = 0$.

Observe that $\Phi_n^{(\alpha)}(x, 0) = \Phi_n^{(\alpha)}(x)$ (see, for details, [20]). In particular, choosing $f(t, \alpha) = \left(\frac{2}{e^t+1}\right)^\alpha$ in (9), we obtain

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt+h(t,\nu)} := \sum_{n=0}^{\infty} E_n^{(\alpha)}(x, \nu) \frac{t^n}{n!}. \tag{10}$$

Note that the polynomials $E_n^{(\alpha)}(x, \nu)$ are related to both Euler polynomials and the Hermite polynomials. For example, if $h(t, 0) = 0$ in (10), we have

$$E_n^{(\alpha)}(x, 0) = E_n^{(\alpha)}(x),$$

where $E_n^{(\alpha)}(x)$ denotes the Euler polynomials of higher order defined by means of the following generating function (see, e.g., [25, p. 88]):

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}. \tag{11}$$

We find

$$\left(\frac{2}{e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)} \frac{t^n}{n!}, \tag{12}$$

where $E_n^{(\alpha)}$ are generalized Euler numbers. For more information about Euler numbers and Euler polynomials, we refer the reader, for example, to [3, 18, 19, 25].

The 2-variable Hermite–Kampé de Fériet polynomials $H_n(x, y)$ (see [2, 5]) are generated by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{13}$$

Note that

$$H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{14}$$

and $H_n(2x, -1) = H_n(x)$ are the ordinary Hermite polynomials (see, e.g., [2]; see also [24, Chapter 11]). Dere and Simsek [9] generalized the polynomials $H_n(x, y)$ in (13) to define two variable Hermite polynomials $H_n^{(\ell)}(x, y)$ by the following generating function:

$$e^{xt+yt^\ell} = \sum_{n=0}^{\infty} H_n^{(\ell)}(x, y) \frac{t^n}{n!} \quad (\ell \in \mathbb{N} \setminus \{1\}). \tag{15}$$

Taking $h(t, y) = yt^2$ in (10), we get the generalized Hermite–Euler polynomials of two variables ${}_H E_n^{(\alpha)}(x, y)$ introduced by Pathan [21]:

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y) \frac{t^n}{n!}. \tag{16}$$

Note that the polynomials ${}_H E_n^{(\alpha)}(x, y)$ generalize Euler numbers, Euler polynomials, Hermite polynomials, and Hermite-Euler polynomials ${}_H E_n(x, y)$ introduced by Dattoli et al. [5, p. 386, Eq. (1.6)]:

$$\frac{2}{e^t + 1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n(x, y) \frac{t^n}{n!}. \tag{17}$$

Luo et al. [18, 19] introduced the generalized Euler numbers $E_n(a, b)$, which are generated by

$$\frac{2}{a^t + b^t} = \sum_{n=0}^{\infty} E_n(a, b) \frac{t^n}{n!} \tag{18}$$

$$(|t| < 2\pi; n \in \mathbb{N}_0; a, b \in \mathbb{R}^+ \text{ with } a \neq b).$$

Also, Luo et al. [19] introduced the generalized Euler polynomials $E_n(x; a, b, \mathbf{e})$, which are generated by

$$\frac{2e^{xt}}{a^t + b^t} = \sum_{n=0}^{\infty} E_n(x; a, b, \mathbf{e}) \frac{t^n}{n!} \tag{19}$$

$$(|t| < 2\pi; n \in \mathbb{N}_0; a, b \in \mathbb{R}^+ \text{ with } a \neq b).$$

The sum of integer power (simply, power sum)

$$S_k(\mathbf{n}) := \sum_{j=0}^{\mathbf{n}} j^k \quad (k \in \mathbb{N}_0; \mathbf{n} \in \mathbb{N})$$

is generated by

$$\sum_{k=0}^{\infty} S_k(\mathbf{n}) \frac{t^k}{k!} = 1 + e^t + e^{2t} + \dots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1}. \tag{20}$$

Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other fields of applied mathematics, a number of certain numbers and polynomials, and their generalizations, have recently been extensively investigated (see, e.g., [1–28]). Here, we also make a slight modification of Milne-Thomson polynomials $\Phi_n^{(\alpha)}(x)$ in (8) and Dere and Simsek polynomials $\Phi_n^{(\alpha)}(x, \nu)$ in (9) to define polynomials $\Phi_{n,\ell}^{(\alpha)}(x, y, \nu)$ by the following generating function:

$$H(t, x, y; \alpha, \nu) := f(t, \alpha) e^{xt+yt^\ell+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha,\ell)}(x, y, \nu) \frac{t^n}{n!} \tag{21}$$

$$(x, y \in \mathbb{C}; \ell \in \mathbb{N} \setminus \{1\}),$$

where $f(t, \alpha)$ and $h(t, \nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t = 0$. Obviously, $\Phi_n^{(\alpha,\ell)}(x, 0, \nu) = \Phi_n^{(\alpha)}(x, \nu)$. Then we establish various identities involving the polynomials $\Phi_n^{(\alpha,\ell)}(x, y, \nu)$. Also, as special cases of the generalized generating function in (21), we introduce two new polynomials, power sum-Laguerre–Hermite polynomials and generalized Laguerre–Euler polynomials, and we investigate some of their involved properties.

Some of the results presented here will include certain known identities and formulas involving relatively simple and familiar numbers and polynomials as particular cases, which will be easy for the interested reader to check (see, e.g., [7, 11–16, 18, 21, 22, 27, 28]).

2. Some formulas involving the polynomials $\Phi_{n,\ell}^{(\alpha)}(x, y, \nu)$

Here we present certain formulas associated with the polynomials $\Phi_{n,\ell}^{(\alpha)}(x, y, \nu)$. For easy reference, we begin by recalling some formal manipulations of double series in the following lemma (see, e.g., [4], [16], [24, pp. 56-57], and [26, p. 52]).

Lemma 2.1 *The following identities hold:*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/p \rfloor} A_{k,n-pk} \quad (p \in \mathbb{N}); \tag{22}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/p \rfloor} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n+pk} \quad (p \in \mathbb{N}); \tag{23}$$

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n) \frac{x^n y^m}{n! m!}. \tag{24}$$

Here, $A_{k,n}$ and $f(N)$ ($k, n, N \in \mathbb{N}_0$) are real or complex valued functions indexed by k, n , and N , respectively, and x and y are real or complex numbers. Also, for possible rearrangements of the involved double series, all the associated series should be absolutely convergent.

Theorem 2.2 *Let $\alpha \in \mathbb{Z}$, $\nu \in \mathbb{N}_0$, and $\ell \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned} \Phi_n^{(\alpha,\ell)}(x_1 + x_2, y, \nu) &= \sum_{k=0}^n \binom{n}{k} x_1^k \Phi_{n-k}^{(\alpha,\ell)}(x_2, y, \nu) \\ &= \sum_{k=0}^n \binom{n}{k} x_2^k \Phi_{n-k}^{(\alpha,\ell)}(x_1, y, \nu) \quad (n \in \mathbb{N}_0, x_1, x_2, y \in \mathbb{C}); \end{aligned} \tag{25}$$

$$\begin{aligned} \Phi_n^{(\alpha,\ell)}(x, y_1 + y_2, \nu) &= \sum_{k=0}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! y_1^k}{(n - \ell k)! k!} \Phi_{n-\ell k}^{(\alpha,\ell)}(x, y_2, \nu) \\ &= \sum_{k=0}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! y_2^k}{(n - \ell k)! k!} \Phi_{n-\ell k}^{(\alpha,\ell)}(x, y_1, \nu) \\ &(n \in \mathbb{N}_0, x, y_1, y_2 \in \mathbb{C}); \end{aligned} \tag{26}$$

$$\Phi_n^{(\alpha,\ell)}(x, y, \nu) = \sum_{k=0}^n \binom{n}{k} x^k \Phi_{n-k}^{(\alpha,\ell)}(0, y, \nu); \quad (n \in \mathbb{N}_0, x, y \in \mathbb{C}); \tag{27}$$

$$\begin{aligned} \Phi_n^{(\alpha,\ell)}(x, y, \nu) &= \sum_{k=0}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! y^k}{(n - \ell k)! k!} \Phi_{n-\ell k}^{(\alpha,\ell)}(x, 0, \nu) \\ &(n \in \mathbb{N}_0, x, y \in \mathbb{C}); \end{aligned} \tag{28}$$

$$\frac{\partial}{\partial x} \Phi_n^{(\alpha,\ell)}(x, y, \nu) = n \Phi_{n-1}^{(\alpha,\ell)}(x, y, \nu) \quad (n \in \mathbb{N}, x, y \in \mathbb{C}); \tag{29}$$

$$\frac{\partial^r}{\partial x^r} \Phi_n^{(\alpha, \ell)}(x, y, \nu) = \frac{n!}{(n-r)!} \Phi_{n-r}^{(\alpha, \ell)}(x, y, \nu) \tag{30}$$

$(n, r \in \mathbb{N}$ with $1 \leq r \leq n; x, y \in \mathbb{C})$;

$$\frac{\partial}{\partial y} \Phi_n^{(\alpha, \ell)}(x, y, \nu) = \frac{n!}{(n-\ell)!} \Phi_{n-\ell}^{(\alpha, \ell)}(x, y, \nu) \tag{31}$$

$(n, \ell \in \mathbb{N}$ with $1 \leq \ell \leq n; x, y \in \mathbb{C})$;

$$\int_a^x \Phi_n^{(\alpha, \ell)}(u, y, \nu) du = \frac{\Phi_{n+1}^{(\alpha, \ell)}(x, y, \nu) - \Phi_{n+1}^{(\alpha, \ell)}(a, y, \nu)}{n+1} \tag{32}$$

$(n \in \mathbb{N}_0, a, x \in \mathbb{R}, y \in \mathbb{C})$.

$$\int_a^y \Phi_n^{(\alpha, \ell)}(x, u, \nu) du = \frac{n!}{(n+\ell)!} \left\{ \Phi_{n+\ell}^{(\alpha, \ell)}(x, y, \nu) - \Phi_{n+\ell}^{(\alpha, \ell)}(x, a, \nu) \right\} \tag{33}$$

$(n \in \mathbb{N}_0, x \in \mathbb{C}, a, y \in \mathbb{R})$.

Proof From (21), we write

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha, \ell)}(x_1 + x_2, y, \nu) \frac{t^n}{n!} = e^{x_1 t} \cdot f(t, \alpha) e^{x_2 t + y t^\ell + h(t, \nu)}.$$

Expanding $e^{x_1 t}$ as the Maclaurin series and using (21) to expand the second factor, with the aid of (22) with $p = 1$, we find

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha, \ell)}(x_1 + x_2, y, \nu) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_1^k}{(n-k)!k!} \Phi_{n-k}^{(\alpha, \ell)}(x_2, y, \nu) t^n,$$

which, upon equating the coefficients of t^n , yields the first equality of (25). For the second equality of (25), we just change the role of x_1 and x_2 in the above proof.

Similarly as in the proof of (25), with the aid of (22) with $p = \ell$, we prove (26).

Setting $x_1 = x$ and $x_2 = 0$ in the first equality in (25), we obtain (27). Similarly, setting $y_1 = y$ and $y_2 = 0$ in the first equality in (26), we get (28).

Differentiating both sides of (27) with respect to the variable x , we have

$$\begin{aligned} \frac{\partial}{\partial x} \Phi_n^{(\alpha, \ell)}(x, y, \nu) &= \sum_{k=1}^n k \binom{n}{k} x^{k-1} \Phi_{n-k}^{(\alpha, \ell)}(0, y, \nu) \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} x^k \Phi_{n-1-k}^{(\alpha, \ell)}(0, y, \nu) \\ &= n \Phi_{n-1}^{(\alpha, \ell)}(x, y, \nu), \end{aligned} \tag{34}$$

where the identity (27) is used for the last equality. This proves (29).

Then, differentiating both sides of (29) with respect to the variable x by using the identity (29) on the right side of each resulting identity, consecutively, $r - 1$ times, we obtain (30).

Differentiating both sides of (28) with respect to the variable y , we have

$$\frac{\partial}{\partial y} \Phi_n^{(\alpha, \ell)}(x, y, \nu) = \sum_{k=1}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! y^{k-1}}{(n - \ell k)! (k - 1)!} \Phi_{n - \ell k}^{(\alpha, \ell)}(x, 0, \nu). \tag{35}$$

Taking $k - 1 = k'$ on the right side of (35) and considering

$$\lfloor \frac{n}{\ell} \rfloor - 1 = \lfloor \frac{n}{\ell} - 1 \rfloor = \lfloor \frac{n - \ell}{\ell} \rfloor,$$

we get

$$\frac{\partial}{\partial y} \Phi_n^{(\alpha, \ell)}(x, y, \nu) = \frac{n!}{(n - \ell)!} \sum_{k=0}^{\lfloor \frac{n - \ell}{\ell} \rfloor} \frac{(n - \ell)! y^k}{(n - \ell - \ell k)! k!} \Phi_{n - \ell - \ell k}^{(\alpha, \ell)}(x, 0, \nu),$$

which, upon using (28), proves (31).

Replacing x by u in (29) and integrating both sides of the resulting identity with respect to the variable u from a to x by using the fundamental theorem of calculus, and substituting $n + 1$ for n in the last resulting identity, we obtain (32).

Similarly as in getting (32), using (31), we get (33). □

3. Power sum-Laguerre–Hermite polynomials

Here, replacing x by y and ν by z in (9) and setting $h(t, z) = z t^2$ and

$$f(x; t, \mathbf{n}) = \frac{e^{(n+1)t} - 1}{e^t - 1} C_0(xt),$$

we introduce a new class of power sum-Laguerre–Hermite polynomials ${}^S_H L_n(x, y, z; \mathbf{n})$ by the following generating function:

$$\frac{e^{(n+1)t} - 1}{e^t - 1} e^{yt + zt^2} C_0(xt) = \sum_{n=0}^{\infty} {}^S_H L_n(x, y, z; \mathbf{n}) \frac{t^n}{n!} \quad (|t| < 2\pi). \tag{36}$$

Now we present various implicit summation formulae for the power sum-Laguerre–Hermite polynomials.

Theorem 3.1 *The following implicit summation formulas for the power sum-Laguerre–Hermite polynomials hold.*

$${}^S_H L_n(x, y, 0; \mathbf{n}) = \sum_{k=0}^n \binom{n}{k} L_{n-k}(x, y) S_k(\mathbf{n}) \quad (n \in \mathbb{N}_0; \mathbf{n} \in \mathbb{N}); \tag{37}$$

$${}^S_H L_n(x, y, z; \mathbf{n}) = n! \sum_{r=0}^n \sum_{k=0}^{n-r} \frac{(-1)^r x^r H_{n-k-r}(y, z) S_k(\mathbf{n})}{(r!)^2 k! (n - k - r)!} \quad (n \in \mathbb{N}_0; \mathbf{n} \in \mathbb{N}); \tag{38}$$

$${}_H^S L_n(x, u + v, z; \mathbf{n}) = \sum_{k=0}^n \binom{n}{k} u^k {}_H^S L_{n-k}(x, v, z; \mathbf{n}) \quad (n \in \mathbb{N}_0; \mathbf{n} \in \mathbb{N}); \tag{39}$$

$${}_H^S L_n(x, y, a + b; \mathbf{n}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} b^k {}_H^S L_{n-2k}(x, y, a; \mathbf{n}) \quad (n \in \mathbb{N}_0; \mathbf{n} \in \mathbb{N}). \tag{40}$$

Proof Setting $z = 0$ in (36) and using (2) and (20) with the aid of (22) with $p = 1$, we obtain

$$\sum_{n=0}^{\infty} {}_H^S L_n(x, y, z; \mathbf{n}) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n L_{n-k}(x, y) S_k(\mathbf{n}) \frac{t^n}{(n-k)!k!},$$

which, upon equating the coefficients of t^n , yields the desired result (37).

The other identities can be proved as in the proof of Theorem 2.2. We omit the details. □

4. Generalized Laguerre–Euler polynomials

Here, replacing x by y and ν by z in (9) and $f(x; t, \alpha) = \left(\frac{2}{a^t + b^t}\right)^\alpha C_0(xt)$, we introduce a new class of the generalized Laguerre–Euler polynomials.

Let $\alpha \in \mathbb{R}$ or \mathbb{C} be a parameter. Also, let $a, b \in \mathbb{R}^+$ with $a \neq b$. The generalized Euler polynomials $E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e})$ are defined by the following generating function:

$$\left(\frac{2}{a^t + b^t}\right)^\alpha e^{yt+h(t,z)} C_0(xt) = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{t^n}{n!} \tag{41}$$

$$\left(x \in \mathbb{R}; |t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

In particular, setting $h(t, z) = zt^2$ in (41), we get the following.

Let $\alpha \in \mathbb{R}$ or \mathbb{C} be a parameter. Also, let $a, b \in \mathbb{R}^+$ with $a \neq b$. The generalized Laguerre–Euler polynomials ${}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e})$ are defined by

$$\left(\frac{2}{a^t + b^t}\right)^\alpha e^{yt+zt^2} C_0(xt) = \sum_{n=0}^{\infty} {}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{t^n}{n!} \tag{42}$$

$$\left(x \in \mathbb{R}; |t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

We have

$${}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) = \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{E_{n-m}^{(\alpha)} L_{m-2k}(x, y) z^k n!}{(m-2k)!k!(n-m)!}. \tag{43}$$

Remark 1 Consider some special cases of (42).

(i) The case $x = 0$ of (42) reduces to the known generalized Hermite–Bernoulli polynomials defined by (see [22])

$$\left(\frac{2}{a^t + b^t}\right)^\alpha e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(y, z; a, b, \mathbf{e}) \frac{t^n}{n!} \quad (44)$$

$$\left(|t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

(ii) The case $x = z = 0$ of (42) reduces to the known generalized Euler polynomials defined by (see [19])

$$\left(\frac{2}{a^t + b^t}\right)^\alpha e^{yt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(y; a, b, \mathbf{e}) \frac{t^n}{n!} \quad (45)$$

$$\left(|t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

(iii) The case $x = y = z = 0$ of (42) reduces to the generalized Euler number $E_n^{(\alpha)}(a, b)$ defined by

$$\left(\frac{2}{a^t + b^t}\right)^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)}(a, b) \frac{t^n}{n!} \quad (46)$$

$$\left(|t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

We find that $E_n^{(1)}(a, b) = E_n(a, b)$ in (18) and

$$E_n^{(\alpha+\beta)}(a, b) = \sum_{k=0}^n \binom{n}{k} E_k^{(\alpha)}(a, b) E_{n-k}^{(\beta)}(a, b) \quad (n \in \mathbb{N}_0). \quad (47)$$

Here we present various implicit summation formulae for the generalized Laguerre–Euler polynomials.

Theorem 4.1 *Let $\alpha, \beta \in \mathbb{R}$ or \mathbb{C} be parameters. Also, let $a, b \in \mathbb{R}^+$ with $a \neq b$. Further, let $u, v, w, x, y, z \in \mathbb{R}$, and $n \in \mathbb{N}_0$. Then the following implicit summation formulas for the generalized Laguerre–Euler polynomials in (42) hold:*

$${}_L E_{m+n}^{(\alpha)}(x, w, z; a, b, \mathbf{e}) = \sum_{s=0}^m \sum_{k=0}^n \binom{m}{s} \binom{n}{k} (w - y)^{s+k} {}_L E_{m+n-s-k}^{(\alpha)}(x, y, z; a, b, \mathbf{e}); \quad (48)$$

$${}_L E_n^{(\alpha)}(x, y + \alpha, z; a, b, \mathbf{e}) = n! \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} \frac{(-1)^k x^k z^j E_{n-2j-k}^{(\alpha)}(y; \frac{a}{\mathbf{e}}, \frac{b}{\mathbf{e}}, \mathbf{e})}{(n - 2j - k)! j! (k!)^2}; \quad (49)$$

$${}_L E_n^{(\alpha+\beta)}(x, y + v, z; a, b, \mathbf{e}) = \sum_{k=0}^n \binom{n}{k} {}_L E_{n-k}^{(\alpha)}(x, y, z; a, b, \mathbf{e}) E_k^{(\beta)}(v; a, b, \mathbf{e}); \quad (50)$$

$${}_L E_n^{(\alpha+\beta)}(x, y + z, v + u; a, b, \mathbf{e}) = \sum_{k=0}^n \binom{n}{k} E_{n-k}^{(\alpha)}(x, z, v; a, b, \mathbf{e}) {}_H E_k^{(\beta)}(y, u; a, b, \mathbf{e}); \tag{51}$$

$${}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) = n! \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} \frac{E_k^{(\alpha)}(a, b, \mathbf{e}) L_{n-k-2j}(x, y) z^j}{k! j! (n - k - 2j)!}. \tag{52}$$

Proof For (48), replacing t by $t + u$ in (42) and using the binomial theorem, we have

$$\begin{aligned} & \left(\frac{2}{a^{t+u} + b^{t+u}} \right)^\alpha e^{y(t+u)+z(t+u)^2} C_0(x(t+u)) \\ &= \sum_{n=0}^{\infty} {}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{(t+u)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n {}_L E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{t^{n-m} u^m}{(n-m)! m!}. \end{aligned} \tag{53}$$

Using (23) with $p = 1$ in the last double summation in (53), we obtain

$$\left(\frac{2}{a^{t+u} + b^{t+u}} \right)^\alpha e^{z(t+u)^2} C_0(x(t+u)) = e^{-y(t+u)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_L E_{n+m}^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{t^n u^m}{n! m!}. \tag{54}$$

Since the left side of (54) is independent of the variable y , we introduce another variable w for the variable y in the right side of (54) and equate the two resulting identities to find

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_L E_{n+m}^{(\alpha)}(x, w, z; a, b, \mathbf{e}) \frac{t^n u^m}{n! m!} = e^{(w-y)(t+u)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_L E_{n+m}^{(\alpha)}(x, y, z; a, b, \mathbf{e}) \frac{t^n u^m}{n! m!}. \tag{55}$$

We use (24) to find

$$e^{(w-y)(t+u)} = \sum_{N=0}^{\infty} (w-y)^N \frac{(t+u)^N}{N!} = \sum_{k,s=0}^{\infty} (w-y)^{k+s} \frac{t^k u^s}{k! s!}. \tag{56}$$

Using (56) in the right side of (55) and applying (22) with $p = 1$ in the resulting four-ple series two times, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_L E_{n+m}^{(\alpha)}(x, w, z; a, b, \mathbf{e}) \frac{t^n u^m}{n! m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^n \sum_{s=0}^m {}_L E_{n+m-s-k}^{(\alpha)}(x, y, z; a, b, \mathbf{e}) (w-y)^{k+s} \frac{t^n u^m}{(n-k)! k! (m-s)! s!}. \end{aligned} \tag{57}$$

Finally, equating the coefficients of t^n and u^m in both sides of (57), consecutively, we obtain the identity (48).

For (49), we find from (42) that

$$\sum_{n=0}^{\infty} {}_L E_n^{(\alpha)}(x, y + \alpha, z; a, b, \mathbf{e}) \frac{t^n}{n!} = \left(\frac{2}{\left(\frac{a}{\mathbf{e}}\right)^t + \left(\frac{b}{\mathbf{e}}\right)^t} \right)^\alpha e^{yt} \cdot e^{zt^2} \cdot C_0(xt). \tag{58}$$

By using (45) and (22) with $p = 2$, we have

$$\begin{aligned} \left(\frac{2}{\left(\frac{a}{e}\right)^t + \left(\frac{b}{e}\right)^t}\right)^\alpha e^{yt} \cdot e^{zt^2} &= \sum_{n=0}^\infty E_n^{(\alpha)}\left(y; \frac{a}{e}, \frac{b}{e}, e\right) \frac{t^n}{n!} \cdot \sum_{j=0}^\infty \frac{z^j t^{2j}}{j!} \\ &= \sum_{n=0}^\infty \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} E_{n-2j}^{(\alpha)}\left(y; \frac{a}{e}, \frac{b}{e}, e\right) z^j \frac{t^n}{(n-2j)! j!}. \end{aligned} \tag{59}$$

Setting the result (59) in (58) and using (4) with $n = 0$, with the help of (22) with $p = 1$, we obtain

$$\sum_{n=0}^\infty {}_L E_n^{(\alpha)}(x, y + \alpha, z; a, b, e) \frac{t^n}{n!} = \sum_{n=0}^\infty \left\{ \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} E_{n-2j-k}^{(\alpha)}\left(y; \frac{a}{e}, \frac{b}{e}, e\right) \frac{z^j x^k (-1)^k}{(n-2j-k)! j! (k!)^2} \right\} t^n. \tag{60}$$

Finally, equating the coefficients of t^n on both sides of (60), we get the identity (49).

Similarly as above, we can prove the other identities. We omit the details. □

5. Symmetry identities for the generalized Laguerre–Euler polynomials

A number of interesting symmetry identities for various polynomials were presented (see, e.g., [11–16, 27, 28]).

Here we give symmetry identities for the generalized Laguerre–Euler polynomials ${}_L E_n^{(\alpha)}(x, y, z; a, b, e)$ in (42). To do this, we consider the following function:

$$\begin{aligned} g(t) &:= \left\{ \frac{4}{(cat + dat)(c^bt + d^bt)} \right\}^\alpha \left\{ \frac{4}{(cat + dat)(c^bt + d^bt)} \right\}^\beta \\ &\quad \times e^{(a+b)(y_1+y_2)t + (a^2+b^2)(z_1+z_2)t^2} \\ &\quad \times C_0(x_1at) C_0(x_1bt) C_0(x_2at) C_0(x_2bt). \end{aligned} \tag{61}$$

We see that the function $g(t)$ in (61) is symmetric with respect to α and β , a and b , c and d , x_1 and x_2 , y_1 and y_2 , and z_1 and z_2 , respectively. To make the generalized Laguerre–Euler polynomials in (42), we have 16 combinations. Then we will get 15 symmetry identities for the generalized Laguerre–Euler polynomials in (42), two of which will be asserted in the following theorem and the other 13 of which are left to the interested reader.

Theorem 5.1 *Let $\alpha, \beta \in \mathbb{R}$ or \mathbb{C} be parameters. Also, let $c, d \in \mathbb{R}^+$ with $c \neq d$. Further, let $a, b, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &\sum_{r=0}^n \sum_{m=0}^{n-r} \sum_{s=0}^r {}_L E_{n-m-r}^{(\alpha)}(x_1, y_1, z_1; c, d, e) {}_L E_m^{(\alpha)}(x_1, y_1, z_1; c, d, e) \\ &\quad \times {}_L E_{r-s}^{(\beta)}(x_2, y_2, z_2; c, d, e) {}_L E_s^{(\beta)}(x_2, y_2, z_2; c, d, e) \frac{a^{n-m-s} b^{m+s}}{(n-m-r)! m! (r-s)! s!} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{r=0}^n \sum_{m=0}^{n-r} \sum_{s=0}^r {}_L E_{n-m-r}^{(\alpha)}(x_2, y_2, z_2; c, d, \mathbf{e}) {}_L E_m^{(\alpha)}(x_2, y_2, z_2; c, d, \mathbf{e}) \\
 &\quad \times {}_L E_{r-s}^{(\beta)}(x_1, y_1, z_1; c, d, \mathbf{e}) {}_L E_s^{(\beta)}(x_1, y_1, z_1; c, d, \mathbf{e}) \frac{a^{n-m-s} b^{m+s}}{(n-m-r)! m! (r-s)! s!}
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 &= \sum_{r=0}^n \sum_{m=0}^{n-r} \sum_{s=0}^r E_{n-m-r}^{(\beta)}(x_2, y_1, z_1; c, d, \mathbf{e}) {}_L E_m^{(\beta)}(x_2, y_1, z_1; c, d, \mathbf{e}) \\
 &\quad \times {}_L E_{r-s}^{(\alpha)}(x_1, y_2, z_2; c, d, \mathbf{e}) {}_L E_s^{(\alpha)}(x_1, y_2, z_2; c, d, \mathbf{e}) \frac{b^{n-m-s} a^{m+s}}{(n-m-r)! m! (r-s)! s!}.
 \end{aligned} \tag{63}$$

Proof We try to combine $g(t)$ as follows:

$$\begin{aligned}
 g(t) &= \left\{ \frac{2}{c^{at} + d^{at}} \right\}^{\alpha} e^{ay_1 t + a^2 z_1 t} C_0(x_1 at) \\
 &\quad \times \left\{ \frac{2}{c^{bt} + d^{bt}} \right\}^{\alpha} e^{by_1 t + b^2 z_1 t} C_0(x_1 bt) \\
 &\quad \times \left\{ \frac{2}{c^{at} + d^{at}} \right\}^{\beta} e^{ay_2 t + a^2 z_2 t} C_0(x_2 at) \\
 &\quad \times \left\{ \frac{2}{c^{bt} + d^{bt}} \right\}^{\beta} e^{by_2 t + b^2 z_2 t} C_0(x_2 bt),
 \end{aligned} \tag{64}$$

which, upon using (42), gives

$$\begin{aligned}
 g(t) &= \sum_{n=0}^{\infty} {}_L E_n^{(\alpha)}(x_1, y_1, z_1; c, d, \mathbf{e}) \frac{(at)^n}{n!} \\
 &\quad \times \sum_{m=0}^{\infty} {}_L E_m^{(\alpha)}(x_1, y_1, z_1; c, d, \mathbf{e}) \frac{(bt)^m}{m!} \\
 &\quad \times \sum_{r=0}^{\infty} {}_L E_r^{(\beta)}(x_2, y_2, z_2; c, d, \mathbf{e}) \frac{(at)^r}{r!} \\
 &\quad \times \sum_{s=0}^{\infty} {}_L E_s^{(\beta)}(x_2, y_2, z_2; c, d, \mathbf{e}) \frac{(bt)^s}{s!}.
 \end{aligned} \tag{65}$$

Now we apply (22) with $p = 1$ to combine the first and second series into a single series and the third and fourth series into another single series. Then we use (22) with $p = 1$ to combine the two resulting single series into one series to find

$$\begin{aligned}
 g(t) &= \sum_{n=0}^{\infty} \left\{ \sum_{r=0}^n \sum_{m=0}^{n-r} \sum_{s=0}^r {}_L E_{n-m-r}^{(\alpha)}(x_1, y_1, z_1; c, d, \mathbf{e}) {}_L E_m^{(\alpha)}(x_1, y_1, z_1; c, d, \mathbf{e}) \right. \\
 &\quad \times \left. {}_L E_{r-s}^{(\beta)}(x_2, y_2, z_2; c, d, \mathbf{e}) {}_L E_s^{(\beta)}(x_2, y_2, z_2; c, d, \mathbf{e}) \frac{a^{n-m-s} b^{m+s}}{(n-m-r)! m! (r-s)! s!} \right\} t^n.
 \end{aligned} \tag{66}$$

Considering another combination of $g(t)$ as in (64), similarly as above, we can get another single series of $g(t)$ as in (66). Then, equating the coefficients of t^n in both sides of the two single series, we can find 15 identities, two of which are recorded. \square

6. Concluding remarks

The results presented here, being very general, can be specialized to yield a number of known and new identities involving relatively simple and familiar polynomials. For example, setting $x = 0$ in (48), we have

$${}_H E_{m+n}^{(\alpha)}(w, z; a, b, \mathbf{e}) = \sum_{s=0}^m \sum_{k=0}^n \binom{m}{s} \binom{n}{k} (w - y)^{s+k} {}_H E_{m+n-s-k}^{(\alpha)}(y, z; a, b, \mathbf{e}).$$

The power sum-Laguerre-Hermite polynomials ${}_H^S L_n(x, y, z; \mathbf{n})$ in (3) and the generalized Laguerre-Euler polynomials $E_n^{(\alpha)}(x, y, z; a, b, \mathbf{e})$ in (42) can be further extended and have their differential and integral formulas as in Theorem 2.2.

Obviously, if we replace e^{yt+zt^2} in our newly introduced polynomials (36) and (42), we can provide more generalized polynomials associated with the two variable Hermite polynomials $H_n^{(\ell)}(x, y)$ in (15).

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