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A study on (strong) order-congruences in ordered semihypergroups

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Abstract: In this paper, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup S , and obtain the relationship between strong order-congruences and pseudoorders on S . Furthermore, we characterize the (strong) order-congruences by the ρ -chains, where ρ is a (strong) congruence on S . Moreover, we give a method of constructing order-congruences, and prove that every hyperideal I of an ordered semihypergroup S is congruence class of one order-congruence on S if and only if I is convex. Finally, we define and study the strong order-congruence generated by a strong congruence. As an application of the results of this paper, we solve an open problem on ordered semihypergroups given by Davvaz et al.

Key words: Ordered semihypergroup, pseudoorder, (strong) order-congruence, ρ -chain

1. Introduction

In mathematics, an ordered semigroup (S, \cdot, \leq) is a semigroup (S, \cdot) with an order relation “ \leq ” such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. As we know, congruences on ordered semigroups play an important role in studying the structures of ordered semigroups, for example, see [13–15, 22–24]. For any congruence ρ on an ordered semigroup S , in general, we do not know whether the quotient semigroup S/ρ is also an ordered semigroup. Even if S/ρ is an ordered semigroup, the order on S/ρ is not necessarily relative to the order on the original ordered semigroup S . As to the above-mentioned questions, Kehayopulu and Tsingelis [13, 14] introduced the concept of pseudoorder on an ordered semigroup S and proved that if σ is a pseudoorder on S , then there exists a congruence $\bar{\sigma}$ on S such that $S/\bar{\sigma}$ is an ordered semigroup. In the same papers a necessary and sufficient condition such that S is a subdirect product of some ordered semigroups was given and two isomorphism theorems of S were established. Since then, Xie [23] introduced the concept of regular congruences on an ordered semigroup S , and proved that ρ is a regular congruence on S if and only if there exists a pseudoorder σ on S such that $\rho = \sigma \cap \sigma^{-1}$.

On the other hand, algebraic hyperstructures, particularly hypergroups, were introduced by Marty [17] in 1934. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a set. Thus algebraic hyperstructures are a suitable

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generalization of classical algebraic structures. Surveys of hyperstructure theory can be found in the books by Corsini [4], Corsini and Leoreanu [5], Davvaz and Leoreanu-Fotea [8], and Vougiouklis [20]. In the hyperstructure theory, semihypergroups are the simplest algebraic hyperstructures that are a generalization of the concept of semigroups. At present, many researchers have studied different aspects of semihypergroups. For more details, the reader is referred to [1, 3, 6, 9, 10, 12, 16, 18, 25]. Especially, regular and strong regular relations on semihypergroups have been introduced and investigated in [4].

A theory of hyperstructures on ordered semigroups has been recently developed. In [11], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Later on, a lot of papers on ordered semihypergroups have been written; for instance, see [2, 7, 19, 26]. It is worth pointing out that Davvaz et al. [7] introduced the concept of a pseudoorder on an ordered semihypergroup, and extended some results in [13] on ordered semigroups to ordered semihypergroups. In particular, they posed an open problem about ordered semihypergroups: Is there a regular relation ρ on an ordered semihypergroup $(S, *, \leq)$ for which S/ρ is an ordered semihypergroup? As a further study, in this paper we define and study the order-congruences and strong order-congruences on an ordered semihypergroup, and extend some results in ordered semigroups to ordered semihypergroups. The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory. In Section 3, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup S , and establish the relationship between strong order-congruences and pseudoorders on S . Moreover, we described the least pseudoorder containing a strong order-congruence on an ordered semihypergroup, and give out a homomorphism theorem of ordered semihypergroups by pseudoorders. In Section 4, we characterize the strong order-congruences (resp. order-congruences) by the ρ -chains, where ρ is a strong congruence (resp. congruence). Furthermore, we provide a method of constructing order-congruences, and prove that every hyperideal I of an ordered semihypergroup S is congruence class of one order-congruence on S if and only if I is convex. By this constructing method of order-congruences, we answer to the open problem given by Davvaz et al. in [7]. Finally, we define and discuss the strong order-congruence generated by a strong congruence.

2. Preliminaries and some notations

Recall that a *hypergroupoid* $(S, *)$ is a nonempty set S together with a hyperoperation, that is a map $* : S \times S \rightarrow P^*(S)$, where $P^*(S)$ denotes the set of all the nonempty subsets of S . The image of the pair (x, y) is denoted by $x * y$. If $x \in S$ and A, B are nonempty subsets of S , then $A * B$ is defined by $A * B = \bigcup_{a \in A, b \in B} a * b$.

Also $A * x$ is used for $A * \{x\}$ and $x * A$ for $\{x\} * A$. Generally, the singleton $\{x\}$ is identified by its element x .

We say that a hypergroupoid $(S, *)$ is a *semihypergroup* if the hyperoperation “ $*$ ” is associative, that is, $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$ (see [4]).

We now recall the notion of ordered semihypergroups from [11].

Definition 2.1 *An algebraic hyperstructure $(S, *, \leq)$ is called an ordered semihypergroup (also called po-semihypergroup in [11]) if $(S, *)$ is a semihypergroup and (S, \leq) is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a * x \leq a * y$ and $x * a \leq y * a$. Here, if $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.*

Clearly, every ordered semigroup is an ordered semihypergroup.

Let (S, \leq) be a partially ordered set (or briefly poset). For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

A nonempty subset A of a poset S is called *convex* if $a \leq b \leq c$ implies $b \in A$ for all $a, c \in A, b \in S$. A nonempty subset B of a poset S is called *strongly convex* if $B = (B]$ (equivalently, $a \in S, b \in B$ and $a \leq b$ imply $a \in B$). Any strongly convex subset of S is clearly a convex subset; however, the converse does not hold in general.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A * A \subseteq A$. A nonempty subset A of a semihypergroup $(S, *)$ is called a *left* (resp. *right*) *hyperideal* of S if $S * A \subseteq A$ (resp. $A * S \subseteq A$). If A is both a left and a right hyperideal of S , then it is called a *hyperideal* of S . A nonempty subset A of an ordered semihypergroup $(S, *, \leq)$ is called an *ordered hyperideal* of S if A is a strongly convex hyperideal of S .

Let ρ be an equivalence relation on a semihypergroup $(S, *)$ or an ordered semigroup $(S, *, \leq)$. If A and B are nonempty subsets of S , then we write $A\bar{\rho}B$ to denote that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b \in B$ there exists $a \in A$ such that $a\rho b$. We write $A\bar{\rho}B$ if for every $a \in A$ and for every $b \in B$ we have $a\rho b$. The equivalence relation ρ is called *congruence* (also called *regular relation* in [4, 7]) if for every $(x, y) \in S \times S$ the implication $x\rho y \Rightarrow a * x \bar{\rho} a * y$ and $x * a \bar{\rho} y * a$, for all $a \in S$, is valid. ρ is called *strong congruence* (also called *strongly regular relation* in [4, 7]) if for every $(x, y) \in S \times S$, from $x\rho y$, it follows that $a * x \bar{\rho} a * y$ and $x * a \bar{\rho} y * a$ for all $a \in S$.

Lemma 2.2 ([4]) *Let $(S, *)$ be a semihypergroup and ρ an equivalence relation on S . Then*

(i) *If ρ is a congruence, then $(S/\rho, \otimes)$ is a semihypergroup with respect to the following hyperoperation:*

$$(a)_\rho \otimes (b)_\rho = \bigcup_{c \in a * b} (c)_\rho, \text{ and it is called a factor semihypergroup.}$$

(ii) *If ρ is a strong congruence, then $(S/\rho, \otimes)$ is a semigroup with respect to the following operation: $(a)_\rho \otimes$*

$$(b)_\rho = (c)_\rho \text{ for all } c \in a * b, \text{ and it is called a factor semigroup.}$$

Let I be a hyperideal of a semihypergroup $(S, *)$. The relation ρ_I on S is defined as follows:

$$\rho_I := \{(x, y) \in S \setminus I \times S \setminus I \mid x = y\} \cup (I \times I).$$

Clearly, ρ_I is an equivalence relation on S . Moreover, we have the following lemma.

Lemma 2.3 *Let $(S, *)$ be a semihypergroup and I a hyperideal of S . Then ρ_I is a congruence on S and it is called Rees congruence induced by I .*

Proof Let $x, y \in S$ and $x\rho_I y$. Then $x = y \in S \setminus I$ or $x, y \in I$. We consider the following cases:

Case 1. If $x = y \in S \setminus I$, then, for any $z \in S$, $x * z = y * z$. Hence $x * z \bar{\rho}_I y * z$.

Case 2. Let $x, y \in I$. Since I is a hyperideal of S , we have $x * z \subseteq I, y * z \subseteq I$ for any $z \in S$. Thus, for any $a \in x * z, b \in y * z$, we have $(a, b) \in I \times I \subseteq \rho_I$. Therefore, $x * z \bar{\rho}_I y * z$. \square

Similarly, we can show that $z * x \bar{\rho}_I z * y$ for any $z \in S$. We have thus shown that ρ_I is a congruence on S .

Remark 2.4 (1) $S/\rho_I = \{\{x\} \mid x \in S \setminus I\} \cup \{I\}$.

(2) By Lemmas 2.2 and 2.3, $(S/\rho_I, \otimes_I)$ forms a factor semihypergroup, which is called Rees factor semihypergroup (also called Rees quotient semihypergroup). Here the hyperoperation \otimes_I on S/ρ_I is defined by

$$(a)_{\rho_I} \otimes_I (b)_{\rho_I} = \bigcup_{c \in a * b} (c)_{\rho_I}.$$

A relation ρ on an ordered semihypergroup $(S, *, \leq)$ is called pseudoorder if it satisfies the following conditions: (1) $\leq \subseteq \rho$, (2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e. $\rho \circ \rho \subseteq \rho$ and (3) $a\rho b$ implies $a * c \bar{\rho} b * c$ and $c * a \bar{\rho} c * b$, for all $c \in S$ (see [7]).

Lemma 2.5 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a pseudoorder on S . Then $(S/\rho^*, \otimes, \preceq_\rho)$ is an ordered semigroup, where $\rho^* (= \rho \cap \rho^{-1})$ is a strong congruence on S , and the order relation \preceq_ρ is defined as follows:

$$\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.$$

Let $(S, *, \leq)$ and (T, \diamond, \preceq) be two ordered semihypergroups, $f : S \rightarrow T$ a mapping from S to T . f is called isotone if $x \leq y$ implies $f(x) \preceq f(y)$, for all $x, y \in S$. f is called reverse isotone if $x, y \in S$, $f(x) \preceq f(y)$ implies $x \leq y$. f is called homomorphism (resp. strong homomorphism) if it is isotone and satisfies $f(x) \diamond f(y) = \bigcup_{z \in x * y} f(z)$ (resp. $f(x) \diamond f(y) = f(z)$, $\forall z \in x * y$), for all $x, y \in S$. f is called isomorphism (resp. strong isomorphism) if it is homomorphism (resp. strong homomorphism), onto, and reverse isotone. The ordered semihypergroups S and T are called strongly isomorphic, in symbol $S \cong T$, if there exists a strong isomorphism between them.

Remark 2.6 Let S and T be two ordered semihypergroups. Then

- (1) If f is a strong homomorphism and reverse isotone mapping from S to T , then $S \cong \text{Im}(f)$.
- (2) In particular, if S and T are both ordered semigroups, then, in this case, the concepts of strong isomorphisms of ordered semihypergroups and isomorphisms of ordered semigroups coincide.

The reader is referred to [5, 21] for notation and terminology not defined in this paper.

3. Strong order-congruences and order-congruences on ordered semihypergroups

As we know, pseudoorders on ordered semihypergroups play an important role in studying the structures of ordered semihypergroups (see [7]). To investigate the properties of pseudoorders on ordered semihypergroups in detail, in this section we shall introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup, and study the relationship between strong order-congruences and pseudoorders.

Definition 3.1 Let $(S, *, \leq)$ be an ordered semihypergroup. A congruence (resp. strong congruence) ρ is called an order-congruence (resp. a strong order-congruence) if there exists an order relation “ \preceq ” on $(S/\rho, \otimes)$ such that:

- (1) $(S/\rho, \otimes, \preceq)$ is an ordered semihypergroup (resp. ordered semigroup), where the hyperoperation “ \otimes ” is defined as one in Lemma 2.2.

(2) The canonical epimorphism $\varphi : S \rightarrow S/\rho, x \mapsto (x)_\rho$ is isotone, that is, φ is a homomorphism (resp. strong homomorphism) from S onto S/ρ .

It is clear that the equality relation 1_S and the universal relation $S \times S$ on S are both order-congruences. In general, an example of order-congruence is given as follows:

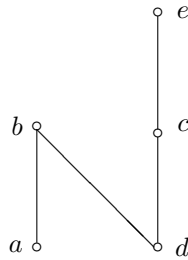
Example 3.2 We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation “ $*$ ” and the order “ \leq ”:

$*$	a	b	c	d	e
a	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, e), (d, b), (d, c), (d, e), (d, d), (e, e)\}.$$

We give the covering relation “ \prec ” and the figure of S as follows:

$$\prec = \{(a, b), (d, b), (d, c), (c, e)\}.$$



Then $(S, *, \leq)$ is an ordered semihypergroup. Let ρ_1, ρ_2 be congruences on S defined as follows:

$$\rho_1 := \{(a, a), (b, b), (c, c), (d, d), (e, e), (d, e), (e, d)\},$$

$$\rho_2 := \{(a, a), (b, b), (c, c), (d, d), (e, e), (c, e), (e, c)\}.$$

Then $S/\rho_1 = \{\{a\}, \{b\}, \{c\}, \{d, e\}\}, S/\rho_2 = \{\{a\}, \{b\}, \{c, e\}, \{d\}\}$. Moreover, we have

(1) ρ_1 is not an order-congruence on S . In fact, if ρ_1 is an order-congruence on S , then there exists an order “ \preceq_1 ” on S/ρ_1 such that $(S/\rho_1, \otimes_1, \preceq_1)$ is an ordered semihypergroup and the mapping $\varphi_1 : S \rightarrow S/\rho_1, x \mapsto (x)_{\rho_1}$ is isotone. Since $d \leq c$, we have $(d)_{\rho_1} \preceq_1 (c)_{\rho_1}$. Also, since $c \leq e$, we have $(c)_{\rho_1} \preceq_1 (e)_{\rho_1} = (d)_{\rho_1}$. Then $(d)_{\rho_1} = (c)_{\rho_1}$. Impossible.

(2) ρ_2 is an order-congruence on S . In fact, let $S/\rho_2 = \{x, y, z, w\}$, where $x = \{a\}, y = \{b\}, z = \{c, e\}, w = \{d\}$.

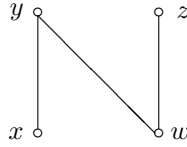
The hyperoperation “ \otimes_2 ” and the order “ \preceq_2 ” on S/ρ_2 are as follows:

\otimes_2	x	y	z	w
x	$\{y, w\}$	$\{y, w\}$	$\{w\}$	$\{w\}$
y	$\{y, w\}$	$\{y, w\}$	$\{w\}$	$\{w\}$
z	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

$$\preceq_2 := \{(x, x), (y, y), (z, z), (w, w), (x, y), (w, y), (w, z)\}.$$

We give the covering relation “ \prec ” and the figure of S/ρ_2 as follows:

$$\prec_2 = \{(x, y), (w, y), (w, z)\}.$$



Then $(S/\rho_2, \otimes_2, \preceq_2)$ is an ordered semihypergroup and the mapping $\varphi_2 : S \rightarrow S/\rho_2, x \mapsto (x)_{\rho_2}$ is isotone. Hence ρ_2 is an order-congruence on S .

Proposition 3.3 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a pseudoorder on S . Then ρ^* is a strong order-congruence on S , where $\rho^* = \rho \cap \rho^{-1}$.

Proof By Lemma 2.5, $(S/\rho^*, \otimes, \preceq_\rho)$ is an ordered semigroup, where the order relation \preceq_ρ is defined as follows:

$$\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.$$

Also, let $x, y \in S$ and $x \leq y$. Then, since ρ is a pseudoorder on S , $(x, y) \in \rho \subseteq \rho^*$, and thus $((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_\rho$, i.e. $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$. Therefore, ρ^* is a strong order-congruence on S . \square

In order to establish the relationship between strong order-congruences and pseudoorders on an ordered semihypergroup, the following lemma is essential.

Lemma 3.4 Let $(S, *, \leq)$ be an ordered semihypergroup and σ a relation on S . Then the following statements are equivalent:

- (1) σ is a pseudoorder on S .
- (2) There exist an ordered semihypergroup (T, \diamond, \preceq) and a strong homomorphism $\varphi : S \rightarrow T$ such that

$$\overrightarrow{\ker \varphi} := \{(a, b) \in S \times S \mid \varphi(a) \preceq \varphi(b)\} = \sigma,$$

where $\overrightarrow{\ker \varphi}$ is called the directed kernel of φ .

Proof (1) \Rightarrow (2). Let σ be a pseudoorder on S . We denote by σ^* the strong congruence on S defined by

$$\sigma^* := \{(a, b) \in S \times S \mid (a, b) \in \sigma, (b, a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Lemma 2.5, the set $S/\sigma^* := \{(a)_{\sigma^*} \mid a \in S\}$ with the operation $(a)_{\sigma^*} \otimes (b)_{\sigma^*} = (c)_{\sigma^*}, \forall c \in a * b$, for all $a, b \in S$ and the order

$$\preceq_\sigma := \{((x)_{\sigma^*}, (y)_{\sigma^*}) \in S/\sigma^* \times S/\sigma^* \mid (x, y) \in \sigma\}$$

is an ordered semigroup. Let $T = (S/\sigma^*, \otimes, \preceq_\sigma)$ and φ be the mapping of S onto S/σ^* defined by $\varphi : S \rightarrow S/\sigma^* \mid a \mapsto (a)_{\sigma^*}$. Then, by Proposition 3.3, φ is a strong homomorphism from S onto S/σ^* and clearly, $\overrightarrow{\ker \varphi} = \sigma$.

(2) \Rightarrow (1). Let $(S, *, \leq)$ be an ordered semihypergroup. If there exist an ordered semihypergroup (T, \diamond, \preceq) and a strong homomorphism $\varphi : S \rightarrow T$ such that $\overrightarrow{\ker\varphi} = \sigma$, then σ is a pseudoorder on S . Indeed, let $(a, b) \in \leq$. Then, by hypothesis, $\varphi(a) \preceq \varphi(b)$. Thus $(a, b) \in \overrightarrow{\ker\varphi} = \sigma$, and we have $\leq \subseteq \sigma$. Moreover, let $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Then $\varphi(a) \preceq \varphi(b) \preceq \varphi(c)$. Hence $\varphi(a) \preceq \varphi(c)$, i.e. $(a, c) \in \overrightarrow{\ker\varphi} = \sigma$. Also, if $(a, b) \in \sigma$, then $\varphi(a) \preceq \varphi(b)$. Since (T, \diamond, \preceq) is an ordered semihypergroup, for any $c \in S$ we have $\varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c)$. Since φ is a strong homomorphism from S to T , for every $x \in a * c$ and $y \in b * c$, we have

$$\varphi(x) = \varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c) = \varphi(y).$$

Then $(x, y) \in \overrightarrow{\ker\varphi} = \sigma$, and thus $a * c \bar{\sigma} b * c$. In the same way, it can be shown that $c * a \bar{\sigma} c * b$. □

Theorem 3.5 *Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a strong congruence on S . Then the following statements are equivalent:*

- (1) ρ is a strong order-congruence on S .
- (2) There exists a pseudoorder σ on S such that $\rho = \sigma \cap \sigma^{-1}$.
- (3) There exist an ordered semihypergroup T and a strong homomorphism $\varphi : S \rightarrow T$ such that $\rho = \ker(\varphi)$, where $\ker\varphi = \{(a, b) \in S \times S \mid \varphi(a) = \varphi(b)\}$ is the kernel of φ .

Proof (1) \Rightarrow (2). Let ρ be a strong order-congruence on S . Then there exist an order relation “ \preceq ” on the factor semigroup $(S/\rho, \otimes)$ such that $(S/\rho, \otimes, \preceq)$ is an ordered semigroup, and $\varphi : S \rightarrow S/\rho$ is a strong homomorphism. Let $\sigma = \overrightarrow{\ker\varphi}$. By Lemma 3.4, σ is a pseudoorder on S and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

(2) \Rightarrow (3). For a pseudoorder σ on S , by Lemma 3.4, there exist an ordered semihypergroup T and a strong homomorphism $\varphi : S \rightarrow T$ such that $\sigma = \overrightarrow{\ker\varphi}$. Then we have

$$\ker\varphi = \overrightarrow{\ker\varphi} \cap (\overrightarrow{\ker\varphi})^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3) \Rightarrow (1). By hypothesis and Lemma 3.4, $\overrightarrow{\ker\varphi}$ is a pseudoorder on S . Then, by Lemma 2.5, $\rho = \overrightarrow{\ker\varphi} \cap (\overrightarrow{\ker\varphi})^{-1}$ is a strong congruence on S . Thus, by the proof of Lemma 3.4, ρ is a strong order-congruence on S . □

Lemma 3.6 (1) *For a strong order-congruence ρ on S , since the order “ \preceq ” such that $(S/\rho, \otimes, \preceq)$ is an ordered semigroup is not unique in general, we have the pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique.*

(2) *If σ is a pseudoorder on an ordered semihypergroup S , then $\rho = \sigma \cap \sigma^{-1}$ is the greatest strong order-congruence on S contained in σ . In fact, if ρ_1 is a strong order-congruence on S contained in σ , then $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$.*

Theorem 3.7 *Let ρ be a strong order-congruence on an ordered semihypergroup $(S, *, \leq)$. Then the least pseudoorder σ containing ρ is the transitive closure of relations $\leq \circ \rho$ (resp. $\rho \circ \leq$), that is,*

$$\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n = \bigcup_{n=1}^{\infty} (\rho \circ \leq)^n.$$

Proof

- (1) Let $\sigma_1 = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n$. Clearly, $\rho \subseteq \leq \subseteq \circ \rho \subseteq \sigma_1$. Similarly, since $\leq \subseteq \leq \circ \rho$, we have $\leq \subseteq \sigma_1$.
- (2) If $(a, b) \in \sigma_1$, $(b, c) \in \sigma_1$, then there exist $m, n \in \mathbb{Z}^+$ such that $(a, b) \in (\leq \circ \rho)^m$ and $(b, c) \in (\leq \circ \rho)^n$, where \mathbb{Z}^+ denotes the set of positive integers. Thus $(a, c) \in (\leq \circ \rho)^{m+n} \subseteq \sigma_1$, i.e. σ_1 is transitive.
- (3) Let $(a, b) \in \sigma_1$ and $c \in S$. Then there exists $n \in \mathbb{Z}^+$ such that $(a, b) \in (\leq \circ \rho)^n$, that is, there exist $a_1, b_1, a_2, b_2, \dots, a_n \in S$ such that

$$a \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \dots \leq a_n \rho b.$$

Since $(S, *, \leq)$ is an ordered semihypergroup and ρ is a strong congruence on S , we have

$$a * c \leq a_1 * c \bar{\rho} b_1 * c \leq a_2 * c \bar{\rho} b_2 * c \leq \dots \leq a_n * c \bar{\rho} b * c.$$

Then, for any $x \in a * c, y \in b * c$, there exist $x_i \in a_i * c (i = 1, 2, \dots, n), y_j \in b_j * c (j = 1, 2, \dots, n - 1)$ such that

$$x \leq x_1 \rho y_1 \leq x_2 \rho y_2 \leq \dots \leq x_n \rho y.$$

It thus implies that $(x, y) \in (\leq \circ \rho)^n \subseteq \sigma_1$, and we obtain that $a * c \bar{\sigma}_1 b * c$. Similar to the above way, it can be shown that $c * a \bar{\sigma}_1 c * b$. Thus $\bigcup_{n=1}^{\infty} (\leq \circ \rho)^n$ is a pseudoorder on S containing ρ .

Furthermore, since σ is transitive, and $\rho \subseteq \sigma, \leq \subseteq \sigma$, we have $\bigcup_{n=1}^{\infty} (\leq \circ \rho)^n \subseteq \sigma$. Thus, by hypothesis, $\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n$. In the same way, we can conclude that $\sigma = \bigcup_{n=1}^{\infty} (\rho \circ \leq)^n$. □

Let σ be a pseudoorder on an ordered semihypergroup $(S, *, \leq)$. Then, by Theorem 3.5, $\rho = \sigma \cap \sigma^{-1}$ is a strong order-congruence on S . We denote by ρ^\sharp the canonical epimorphism from S onto S/ρ , i.e. $\rho^\sharp : S \rightarrow S/\rho \mid x \mapsto (x)_\rho$, which is a strong homomorphism. In the following, we give out a homomorphism theorem of ordered semihypergroups by pseudoorders, which is a generalization of Theorem 1 in [14]. In fact, in Theorem 3.8, if our ordered semihypergroup is an ordered semigroup, i.e. the hyperoperation is an ordinary binary operation, we shall obtain Theorem 1 in [14].

Theorem 3.8 *Let $(S, *, \leq)$ and (T, \diamond, \preceq) be two ordered semihypergroups, $\varphi : S \rightarrow T$ a strong homomorphism. Then: If σ is a pseudoorder on S such that $\sigma \subseteq \overrightarrow{\ker \varphi}$, then there exists the unique strong homomorphism $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \rho^\sharp \downarrow & \nearrow f & \\ S/\rho & & \end{array}$$

commutes, where $\rho = \sigma \cap \sigma^{-1}$. Moreover, $Im(\varphi) = Im(f)$. Conversely, if σ is a pseudoorder on S for which there exists a strong homomorphism $f : (S/\rho, \otimes, \preceq_\sigma) \rightarrow (T, \diamond, \preceq)$ ($\rho = \sigma \cap \sigma^{-1}$) such that the above diagram commutes, then $\sigma \subseteq \overrightarrow{\ker \varphi}$.

Proof Let σ be a pseudoorder on S such that $\sigma \subseteq \overrightarrow{\ker\varphi}$, $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$. Then

- (1) f is well defined. Indeed, if $(a)_\rho = (b)_\rho$, then $(a, b) \in \rho \subseteq \sigma$. Since $\sigma \subseteq \overrightarrow{\ker\varphi}$, we have $(\varphi(a), \varphi(b)) \in \preceq$. Furthermore, since $(b, a) \in \sigma \subseteq \overrightarrow{\ker\varphi}$, we have $(\varphi(b), \varphi(a)) \in \preceq$. Therefore, $\varphi(a) = \varphi(b)$.
- (2) f is a strong homomorphism and $\varphi = f \circ \rho^\sharp$. In fact: By Lemma 3.4, there exist an order relation “ \preceq_σ ” on the factor semigroup $(S/\rho, \otimes)$ such that $(S/\rho, \otimes, \preceq_\sigma)$ is an ordered semigroup and the canonical epimorphism ρ^\sharp is a strong homomorphism. Moreover, we have

$$\begin{aligned} (a)_\rho \preceq_\sigma (b)_\rho &\Rightarrow (a, b) \in \sigma \subseteq \overrightarrow{\ker\varphi} \\ &\Rightarrow \varphi(a) \preceq \varphi(b) \\ &\Rightarrow f((a)_\rho) \preceq f((b)_\rho). \end{aligned}$$

Also, let $(a)_\rho, (b)_\rho \in S/\rho$. For any $(c)_\rho \in (a)_\rho \otimes (b)_\rho$, we have $c \in a * b$. Since φ is a strong homomorphism from S to T , we have

$$f((a)_\rho) \diamond f((b)_\rho) = \varphi(a) \diamond \varphi(b) = \varphi(c) = f((c)_\rho).$$

Furthermore, for any $a \in S$, $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$, and thus $\varphi = f \circ \rho^\sharp$.

We claim that f is a unique strong homomorphism from S/ρ to T . To prove our claim, let g be a strong homomorphism from S/ρ to T such that $\varphi = g \circ \rho^\sharp$. Then

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover, $Im(f) = \{f((a)_\rho) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = Im(\varphi)$.

Conversely, let σ be a pseudoorder on S , $f : S/\rho \rightarrow T$ is a strong homomorphism, and $\varphi = f \circ \rho^\sharp$. Then $\sigma \subseteq \overrightarrow{\ker\varphi}$. Indeed, by hypothesis, we have

$$\begin{aligned} (a, b) \in \sigma &\Leftrightarrow (a)_\rho \preceq_\sigma (b)_\rho \Rightarrow f((a)_\rho) \preceq f((b)_\rho) \\ &\Rightarrow (f \circ \rho^\sharp)(a) \preceq (f \circ \rho^\sharp)(b) \\ &\Rightarrow \varphi(a) \preceq \varphi(b) \Rightarrow (a, b) \in \overrightarrow{\ker\varphi}, \end{aligned}$$

where the order \preceq_σ on S/ρ is defined as in the proof of Lemma 3.4, that is

$$\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

□

Corollary 3.9 Let $(S, *, \leq)$ and (T, \diamond, \preceq) be two ordered semihypergroups and $\varphi : S \rightarrow T$ a strong homomorphism. Then $S/\ker\varphi \cong Im(\varphi)$, where $\ker\varphi$ is the kernel of φ .

Proof Let $\sigma = \overrightarrow{\ker\varphi}$ and $\rho = \overrightarrow{\ker\varphi} \cap (\overrightarrow{\ker\varphi})^{-1}$. Then, by Theorems 3.5 and 3.8, ρ is a strong order-congruence on S and $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$ is a strong homomorphism. Moreover, f is inverse isotone. In fact, let $(a)_\rho, (b)_\rho$ be two elements of S/ρ such that $f((a)_\rho) \preceq f((b)_\rho)$. Then $\varphi(a) \preceq \varphi(b)$, and we have $(a, b) \in \overrightarrow{\ker\varphi}$. Thus, by Lemma 3.4, $((a)_\rho, (b)_\rho) \in \preceq_\sigma$, i.e. $(a)_\rho \preceq_\sigma (b)_\rho$. Clearly, $\rho = \ker\varphi$. By Remark 2.6(1), $S/\ker\varphi \cong Im(f)$. Also, by Theorem 3.8, $Im(f) = Im(\varphi)$. Therefore, $S/\ker\varphi \cong Im(\varphi)$. □

Note that if S and T are both ordered semigroups, then Corollary 3.9 coincides with Corollary in [14].

4. Characterizations of (strong) order-congruences on ordered semihypergroups

In the above section, we have characterized the strong order-congruences by the properties of pseudoorders on an ordered semihypergroup. In the current section, we shall give out other characterizations of (strong) order-congruences on ordered semihypergroups. In order to prove the main results in this section, we first introduce the following concept.

Definition 4.1 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ an equivalence relation on S . A finite sequence of the form $(x, a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1}, a_n, y)$ of elements in S is called a ρ -chain if

- (1) $(a_1, b_1) \in \rho, (a_2, b_2) \in \rho, \dots, (a_{n-1}, b_{n-1}) \in \rho, (a_n, y) \in \rho$;
- (2) $x \leq a_1, b_1 \leq a_2, b_2 \leq a_3, \dots, b_{n-2} \leq a_{n-1}, b_{n-1} \leq a_n$.

Briefly we write

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \dots \leq a_n \rho y.$$

The number n is called the length, x and y initial and terminal elements, respectively, of the ρ -chain. A ρ -chain is called close if its initial and terminal elements are equal, i.e. $x = y$.

We denote by $\rho^{C_{xy}}$ the set of all ρ -chains with x as the initial and y as the terminal elements in the sequel.

Lemma 4.2 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a congruence on S . Then the following statements are true:

- (1) $(x, y) \in (\leq \circ \rho)^n$ if and only if there exists a ρ -chain with length n in $\rho^{C_{xy}}$, i.e. $\rho^{C_{xy}} \neq \emptyset$.
- (2) For any $z \in S$, if $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in S$, then for every $u \in x * z$, there exists $v \in y * z$ such that $\rho^{C_{uv}} \neq \emptyset$.
- (3) For any $z \in S$, if $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in S$, then for every $u' \in z * x$, there exists $v' \in z * y$ such that $\rho^{C_{u'v'}} \neq \emptyset$.

Proof

(1) The proof is straightforward by Definition 4.1 and we omit it.

(2) Let $(x, a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1}, a_n, y) \in \rho^{C_{xy}}$ and $z \in S$. Then

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \dots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho b.$$

Since $(S, *, \leq)$ is an ordered semihypergroup and ρ is a congruence on S , we have

$$x * z \leq a_1 * z \bar{\rho} b_1 * z \leq a_2 * z \bar{\rho} b_2 * z \leq \dots \leq a_{n-1} * z \bar{\rho} b_{n-1} * z \leq a_n * z \bar{\rho} y * z.$$

Then, for any $u \in x * z$, there exist $x_i \in a_i * z (i = 1, 2, \dots, n), y_j \in b_j * z (j = 1, 2, \dots, n - 1), v \in y * z$ such that

$$u \leq x_1 \rho y_1 \leq x_2 \rho y_2 \leq \dots \leq x_{n-1} \rho y_{n-1} \leq x_n \rho v.$$

It thus implies that $(u, x_1, y_1, x_2, y_2, \dots, x_{n-1}, y_{n-1}, x_n, v) \in \rho^{C_{uv}}$, i.e. $\rho^{C_{uv}} \neq \emptyset$.

(3) It is similar to that of (2) and we omit it. □

Lemma 4.3 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a strong congruence on S . If $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in S$, then, for any $z \in S$, we have $\rho^{C_{uv}} \neq \emptyset$ and $\rho^{C_{u'v'}} \neq \emptyset$ for every $u \in x * z, v \in y * z, u' \in z * x, v' \in z * y$.

Proof The proof is similar to that of Lemma 4.2 with a slight modification. □

Lemma 4.4 Let S be an ordered semihypergroup and ρ a (strong) congruence on S . If $(x, y) \in \rho, (k, z) \in \rho$, then $\rho^{C_{xk}} \neq \emptyset$ if and only if $\rho^{C_{yz}} \neq \emptyset$.

Proof (\implies). If $\rho^{C_{xk}} \neq \emptyset$, by Lemma 4.2(1), there exists $n \in \mathbb{Z}^+$ such that $(x, k) \in (\leq \circ \rho)^n$. Since $(x, y) \in \rho, (z, k) \in \rho$, we have

$$y \leq y\rho x(\leq \circ \rho)^n k \leq k\rho z,$$

which implies that $(y, z) \in (\leq \circ \rho)^{n+2}$. By Lemma 4.2(1), we have $\rho^{C_{yz}} \neq \emptyset$. □

(\impliedby). Similar to the proof of necessity and we omit it.

Now we shall give a characterization of order-congruences on an ordered semihypergroup.

Theorem 4.5 Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a congruence on S . Then ρ is an order-congruence on S if and only if every close ρ -chain is contained in a single equivalent class of ρ .

Proof Let ρ be an order-congruence on S . Then there exists an order \preceq on the factor semihypergroup $(S/\rho, \otimes)$ such that $(S/\rho, \otimes, \preceq)$ is an ordered semihypergroup and $\varphi : S \rightarrow S/\rho$ is a homomorphism. For any $x \in S$, and every close ρ -chain $(x, a_1, b_1, \dots, a_n, x)$ in $\rho^{C_{xx}}$, we have

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \dots \leq a_n\rho x.$$

Then,

$$\varphi(x) \preceq \varphi(a_1) = \varphi(b_1) \preceq \varphi(a_2) = \varphi(b_2) \preceq \dots \preceq \varphi(a_n) = \varphi(x).$$

It implies that $\varphi(x) = \varphi(a_1) = \varphi(b_1) = \varphi(a_2) = \varphi(b_2) = \dots = \varphi(a_n)$. Consequently, $(x, a_1, b_1, \dots, a_n, x)$ is contained in a single ρ -class.

Conversely, since ρ is a congruence on S , by Lemma 2.2, $(S/\rho, \otimes)$ is a semihypergroup. We define a relation “ \preceq ” on the factor semihypergroup $(S/\rho, \otimes)$ as follows:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid \rho^{C_{xy}} \neq \emptyset\}.$$

(1) \preceq is well defined. In fact, let $x_1, y_1 \in S$ be such that $(x)_\rho = (x_1)_\rho, (y)_\rho = (y_1)_\rho$. If $(x)_\rho \preceq (y)_\rho$, then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.4, we have $\rho^{C_{x_1y_1}} \neq \emptyset$, and $(x_1)_\rho \preceq (y_1)_\rho$.

(2) \preceq is an ordered relation on S/ρ .

(α) \preceq is reflexive. In fact, since for any $x \in S, x \leq x\rho x$, and we have $\rho^{C_{xx}} \neq \emptyset$, i.e. $((x)_\rho, (x)_\rho) \in \preceq$.

(β) \preceq is transitive. Indeed, let $((x)_\rho, (y)_\rho) \in \preceq, ((y)_\rho, (z)_\rho) \in \preceq$. Then we have $\rho^{C_{xy}} \neq \emptyset, \rho^{C_{yz}} \neq \emptyset$. By Lemma 4.2(1), there exist $m, n \in \mathbb{Z}^+$ such that $(x, y) \in (\leq \circ \rho)^m, (y, z) \in (\leq \circ \rho)^n$. Then we have

$$(x, z) \in (\leq \circ \rho)^m \circ (\leq \circ \rho)^n = (\leq \circ \rho)^{m+n},$$

i.e. $\rho^{C_{xz}} \neq \emptyset$. Thus $((x)_\rho, (z)_\rho) \in \preceq$.

(γ) \preceq is anti-symmetric. In fact, if $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (x)_\rho) \in \preceq$, then $\rho^{C_{xy}} \neq \emptyset$, $\rho^{C_{yx}} \neq \emptyset$. Similar to the above proof, it can be obtained that $\rho^{C_{xx}} \neq \emptyset$, i.e. there exists a close ρ -chain in ${}_\rho C_{xx}$ containing x and y . By hypothesis, $(x)_\rho = (y)_\rho$.

(3) $(S/\rho, \otimes, \preceq)$ is an ordered semihypergroup. Indeed, let $(x)_\rho \preceq (y)_\rho$ and $(z)_\rho \in S/\rho$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.2(2), for every $u \in x * z$, there exists $v \in y * z$ such that $\rho^{C_{uv}} \neq \emptyset$, i.e. $(u)_\rho \preceq (v)_\rho$. Thus

$$(x)_\rho \otimes (z)_\rho = \bigcup_{u \in x * z} (u)_\rho \preceq \bigcup_{v \in y * z} (v)_\rho = (y)_\rho \otimes (z)_\rho.$$

Similarly, it can be shown that $(z)_\rho \otimes (x)_\rho \preceq (z)_\rho \otimes (y)_\rho$.

(4) The mapping $\varphi : S \rightarrow S/\rho \mid x \mapsto (x)_\rho$ is isotone. In fact, let $x, y \in S$ be such that $x \leq y$. Then $(x, y) \in \leq \circ \rho$, we have $\rho^{C_{xy}} \neq \emptyset$, i.e. $(x)_\rho \preceq (y)_\rho$.

Therefore, ρ is an order-congruence on S . □

Similarly, strong order-congruences on an ordered semihypergroup can be characterized as follows:

Theorem 4.6 *Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a strong congruence on S . Then ρ is a strong order-congruence on S if and only if every close ρ -chain is contained in a single equivalent class of ρ .*

Proof The proof is similar to that of Theorem 4.5 with suitable modification by using Lemma 4.3. □

By Theorem 4.5, we immediately obtain the following corollary:

Corollary 4.7 *If ρ is an order-congruence on an ordered semihypergroup S , then every ρ -class in S is convex.*

Proof Let ρ be an order-congruence on S and I a congruence class of ρ . If $x \leq y \leq z$ and $x, z \in I$, then $(x)_\rho = (z)_\rho$. Thus we have $x \leq y\rho y \leq z\rho x$. Hence (x, y, y, z, x) is a close ρ -chain and by Theorem 4.5 we have $(x)_\rho = (y)_\rho = (z)_\rho$. It thus follows that $y \in I$, and I is convex. □

Furthermore, we have the following theorem.

Theorem 4.8 *Let $(S, *, \leq)$ be an ordered semihypergroup and I a hyperideal of S . Then I is a congruence class of one order-congruence on S if and only if I is convex.*

Proof The proof is straightforward by Corollary 4.7.

Conversely, let ρ_I be the Rees congruence induced by I on S . By Remark 2.4(1), I is a congruence class of ρ_I . Now we define a relation “ \preceq_I ” on the factor semihypergroup $(S/\rho_I, \otimes)$ as follows:

$$(x)_{\rho_I} \preceq_I (y)_{\rho_I} \Leftrightarrow (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).$$

We claim that ρ_I is an order-congruence on S . To prove our claim, we first show that \preceq_I is order relation on S/ρ_I , i.e. \preceq_I is reflexive, anti-symmetric, and transitive.

(1) Let $(x)_{\rho_I}$ be any element of S/ρ_I . Then, since $x \leq x$, we have $(x)_{\rho_I} \preceq_I (x)_{\rho_I}$.

(2) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (x)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq x$ or $y \leq b, b' \leq x$ for some $b, b' \in I$. We consider the following four cases:

Case 1. If $x \leq y$ and $y \leq x$, then $x = y$, and thus $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq x$ for some $b, b' \in I$, then $b' \leq x \leq y \leq b$. Since I is convex and $b, b' \in I$, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I} = I$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq x$. Similar to the proof of Case 2, we have $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq x$ for some $b, b' \in I$. Then $b' \leq x \leq a$ and $a' \leq y \leq b$. Since I is convex, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I}$.

(3) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (z)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq z$ or $y \leq b, b' \leq z$ for some $b, b' \in I$. There are four cases to be considered:

Case 1. If $x \leq y$ and $y \leq z$, then $x \leq z$, and thus $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq z$ for some $b, b' \in I$, then $x \leq y \leq b$ and $b' \leq z$. By the definition of \preceq_I , $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq z$. Analogous to the proof of Case 2, we have $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq z$ for some $b, b' \in I$. Then $x \leq a$ and $b' \leq z$. Hence $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Now we show that $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semihypergroup. Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $z \in S$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$. We consider the following two cases:

Case 1. If $x \leq y$, then $x * z \leq y * z$. Thus for every $u \in x * z$, there exists $v \in y * z$ such that $u \leq v$, and we have $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$. Thus

$$(x)_{\rho_I} \otimes_I (z)_{\rho_I} = \bigcup_{u \in x * z} (u)_{\rho_I} \preceq_I \bigcup_{v \in y * z} (v)_{\rho_I} = (y)_{\rho_I} \otimes_I (z)_{\rho_I}.$$

Case 2. Let $x \leq a, a' \leq y$ for some $a, a' \in I$. Then $x * z \leq a * z, a' * z \leq y * z$. Thus for every $u \in x * z$, there exists $b \in a * z$ such that $u \leq b$, and for some $b' \in a' * z$ there exists $v \in y * z$ such that $b' \leq v$. Since I is a hyperideal of S and $a, a' \in I$, we have $b \in a * z \subseteq I, b' \in a' * z \subseteq I$. On the other hand, $u \leq b, b' \leq v$ for some $b, b' \in I$. Hence $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$, and thus $(x)_{\rho_I} \otimes_I (z)_{\rho_I} \preceq_I (y)_{\rho_I} \otimes_I (z)_{\rho_I}$.

Similar to the above way, we can show that $(z)_{\rho_I} \otimes_I (x)_{\rho_I} \preceq_I (z)_{\rho_I} \otimes_I (y)_{\rho_I}$. Therefore, $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semihypergroup.

Furthermore, by the definition of \preceq_I , it can be obtained that the canonical epimorphism $\varphi : S \rightarrow S/\rho_I, x \mapsto (x)_{\rho_I}$ is isotone. Thus ρ_I is an order-congruence on S . The proof is completed. \square

By the proof of above theorem, we immediately obtain the following corollary:

Corollary 4.9 *Let $(S, *, \leq)$ be an ordered semihypergroup and I an ordered hyperideal of S . Then $(S/\rho_I, \otimes_I, \preceq_I)$ forms an ordered semihypergroup and the Rees congruence ρ_I induced by I on S is an order-congruence, where the order relation “ \preceq_I ” on S/ρ_I is defined as follows:*

$$(x)_{\rho_I} \preceq_I (y)_{\rho_I} \Leftrightarrow (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).$$

As an application of Corollary 4.9, we can give an answer to the open problem given by Davvaz et al. in [7].

Open Problem Is there a congruence relation (also called regular relation in [7]) ρ on an ordered semihypergroup $(S, *, \leq)$ for which S/ρ is an ordered semihypergroup?

To solve the above problem, we need only show that ρ_I is not a strong congruence on S in general. We illustrate it by the following example.

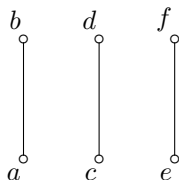
Example 4.9 We consider a set $S := \{a, b, c, d, e, f\}$ with the following hyperoperation “ $*$ ” and the order “ \leq ”:

$*$	a	b	c	d	e	f
a	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e, f\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (f, f)\}.$$

We give the covering relation “ \prec ” and the figure of S as follows:

$$\prec = \{(a, b), (c, d), (e, f)\}.$$



Then (S, \circ, \leq) is an ordered semihypergroup. Let $I = \{c, d\}$. One can easily verify that I is an ordered hyperideal of S . Then $\rho_I = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (c, d), (d, c)\}$. By Lemma 2.3, ρ_I is a congruence on S . However, we claim that ρ_I is not a strong congruence on S . In fact, since $(e, e) \in \rho_I$, while $e * b \overline{\rho_I} e * b$ does not hold.

As a generalization of Proposition 2.7 in [23], we have the following theorem. The following theorem can be proved using similar techniques as in the proof of Theorem 4.8.

Theorem 4.10 Let $(S, *, \leq)$ be an ordered semihypergroup and I an ordered hyperideal of S . We define a relation “ \preceq_1 ” on $S/\rho_I (= \{\{x\} \mid x \in S \setminus I\} \cup \{I\})$ as follows:

$$\preceq_1 := \{(I, \{x\}) \mid x \in S \setminus I\} \cup \{(\{x\}, \{y\}) \mid x, y \in S \setminus I, x \leq y\} \cup \{(I, I)\}.$$

Then $(S/\rho_I, \otimes_I, \preceq_1)$ is an ordered semihypergroup, and ρ_I is an order-congruence on S .

Proposition 4.11 Let $(S, *, \leq)$ be an ordered semihypergroup and I an ordered hyperideal of S . Then the order relations in Corollary 4.9 and Theorem 4.11 are different. Moreover, $\preceq_I \subseteq \preceq_1$.

Proof Let $(x)_{\rho_I}, (y)_{\rho_I} \in S/\rho_I$ and $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$. Since $(I] = I$, we have $x \leq y$ or $x \in I$ and $a' \leq y$ for some $a' \in I$. The first case implies $(x)_{\rho_I} \preceq_1 (y)_{\rho_I}$, and the second case implies $(x)_{\rho_I} \preceq_1 (a')_{\rho_I} \preceq_1 (y)_{\rho_I}$, i.e. $(x)_{\rho_I} \preceq_1 (y)_{\rho_I}$. Thus $\preceq_I \subseteq \preceq_1$. \square

The following example shows that $\preceq_I \not\subseteq \preceq_1$ in general.

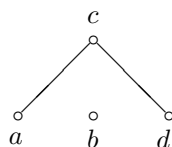
Example 4.12 We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation “ $*$ ” and the order “ \leq ”:

$*$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$\leq := \{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}.$$

We give the covering relation “ \prec ” and the figure of S as follows:

$$\prec = \{(a, c), (d, c)\}.$$



Then $(S, *, \leq)$ is an ordered semihypergroup. It is easy to check that $I = \{a, d\}$ is an ordered hyperideal of S . Since $a \not\leq b$ and there does not exist $x \in I$ such that $x \leq b$, we have $(a)_{\rho_I} \not\preceq_I (b)_{\rho_I}$. However, by the definition of \preceq_1 , we have $(a)_{\rho_I} \preceq_1 (b)_{\rho_I}$.

In the following we shall consider the strong order-congruence generated by a strong congruence on an ordered semihypergroup.

Definition 4.13 Let ρ be a strong congruence on an ordered semihypergroup S . A strong order-congruence σ is called the strong order-congruence generated by ρ on S , if σ satisfies the following conditions:

- (1) $\rho \subseteq \sigma$.
- (2) If there is a strong order-congruence η on S such that $\rho \subseteq \eta$, then $\sigma \subseteq \eta$.

Theorem 4.14 Let ρ be a strong congruence on an ordered semihypergroup $(S, *, \leq)$. Then

- (1) If we define a relation $\underline{\rho}$ on S as follows:

$$(x, y \in S) (x, y) \in \underline{\rho} \text{ if and only if } \rho^{C_{xy}} \neq \emptyset,$$

then $\underline{\rho}$ is a pseudoorder on S .

- (2) R_ρ is a relation on S defined as follows:

$$(x, y \in S) (x, y) \in R_\rho \iff (x, y) \in \underline{\rho} \text{ and } (y, x) \in \underline{\rho}.$$

Then R_ρ is the strong order-congruence generated by ρ on S .

Proof

(1) Let $x, y \in S$ such that $x \leq y$. Then there is a ρ -chain from x to y : (x, y, y) , i.e. $\rho^{C_{xy}} \neq \emptyset$. Thus $x \leq y$ implies $x\rho y$, and we have $\leq \subseteq \underline{\rho}$. Suppose that $(x, y) \in \underline{\rho}$ and $(y, z) \in \underline{\rho}$. Then there exist $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{n-1}, c_1, c_2, \dots, c_m, d_1, d_2, \dots, d_m \in S$ such that

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \dots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y,$$

$$y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \dots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho z.$$

Thus, $x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \dots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \dots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho z$, which is a ρ -chain from x to z . Hence $(x, z) \in \underline{\rho}$ and $\underline{\rho}$ is transitive. Furthermore, let $(x, y) \in \underline{\rho}$ and $z \in S$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.3, for every $u \in x * z, v \in y * z, u' \in z * x, v' \in z * y$, we have $\rho^{C_{uv}} \neq \emptyset$ and $\rho^{C_{u'v'}} \neq \emptyset$, which imply that $(u, v) \in \underline{\rho}$ and $(u', v') \in \underline{\rho}$. It thus follows that $x * z \bar{\rho} y * z$ and $z * x \bar{\rho} z * y$. Therefore, $\underline{\rho}$ is a pseudoorder on S .

(2) By (1), $\underline{\rho}$ is a pseudoorder on S . Since $R_\rho = \underline{\rho} \cap \underline{\rho}^{-1}$, by Proposition 3.3 R_ρ is a strong order-congruence on S . We claim that R_ρ is the strong order-congruence generated by ρ on S . To prove our claim, let $(x, y) \in \rho$. Since ρ is a strong congruence on S , we have $(y, x) \in \rho$. Consequently, $(x, y) \in R_\rho$. Hence $\rho \subseteq R_\rho$. Furthermore, suppose that η is a strong order-congruence on S and $\rho \subseteq \eta$. Then $R_\rho \subseteq \eta$. Indeed, let $(x, y) \in R_\rho$. Then $(x, y) \in \underline{\rho}$ and $(y, x) \in \underline{\rho}$. By definition of $\underline{\rho}$, there exist $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{n-1}, c_1, c_2, \dots, c_m, d_1, d_2, \dots, d_{m-1} \in S$ such that

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \dots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y,$$

$$y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \dots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho x.$$

Thus, by $\rho \subseteq \eta$, we have $x \leq a_1\eta b_1 \leq a_2\eta b_2 \leq \dots \leq a_{n-1}\eta b_{n-1} \leq a_n\eta y \leq c_1\eta d_1 \leq c_2\eta d_2 \leq \dots \leq c_{m-1}\eta d_{m-1} \leq d_m\eta x$. Since η is a strong order-congruence on S , by Theorem 4.6 we can conclude that the closed η -chain $(x, a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1}, a_n, y, c_1, d_1, c_2, d_2, \dots, c_{m-1}, d_{m-1}, d_m, x)$ is contained in a single equivalence class of η . In particular, we have $(x, y) \in \eta$. Therefore, R_ρ is the strong order-congruence generated by ρ on S .

□

By Theorem 4.15, we immediately obtain the following corollary:

Corollary 4.15 *Every strong congruence of an ordered semihypergroup S is contained in a strong order-congruence of S .*

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