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Iterative roots of some functions

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Abstract: The iterative equation $f^q(x) = g(x)$, $x \in X$ for a given function g and a positive integer q is solved in the following two main cases:

(i) $X = \mathbb{Z}$, $g(x) = ax + b$, $(a, b \in \mathbb{Z};\ a \neq 0, 1)$;

(ii) $X = \mathbb{N} \cup \{0\}$, g is increasing with no fixed point.

Key words: Iterative functional equations, monotone functions, cycles

1. Introduction

For a function f and a positive integer q, define $f^q = f \circ f \circ \cdots \circ f$ (q times). Given a function g, the polynomial-like iterative functional equation of the form

$$a_1 f(x) + a_2 f^2(x) + \cdots + a_q f^q(x) = g(x) \qquad (x \in X) \tag{1.1}$$

has been studied in many different settings of $a_1, a_2, \ldots, a_n, X$ and $g(x)$.

In a recent paper [6], using Schauder's fixed point theorem and the Banach contraction principle, sufficient conditions for the existence, uniqueness, and stability of the periodic and continuous solutions of (1.1) were given when $X = \mathbb{R}$. In particular, the solution of $2f(x) + \lambda f^2(x) = \sin x$, $\lambda \in [-1, 1]$, was established, as well as a similar result when the right-hand expression is a cosine function. In [4, 5], the equation of the form $a_1 f(x) + a_2 f^2(x) = bx + c$ was solved for $a_1, a_2, b, c \in \mathbb{R}$, $a_2 \neq 0$. When $g(x)$ is continuous and strictly monotonic, a comprehensive work dealing with continuous solutions of $f^q(x) = g(x)$ can be found in [2, Chapter XV].

In 2008, Sarkaria [9] (see also his unpublished paper at http://kssarkaria.org/docs/RootsFunctions.pdf) found all functions $f : X \to X$, where $X = \mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$, satisfying the iterative functional solution $f^q(n) = n+k$

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for given q and $k \in \mathbb{N}$. This equation arose from one of the problems posed at the International Mathematical Olympiad in 1987: prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\left(f(n)\right) = n + 1987.$$

More precisely, Sarkaria proved that:

(i) for $q \geq 1, k \geq 1$, there exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f^q(n) = n + k$ if and only if q divides k;

(ii) there are exactly $k!/(k/q)!$ such functions and their shapes can be explicitly determined;

(iii) there are infinitely many functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f^q(n) = n + k$ if and only if q divides k;

(iv) there exist continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f^q(n) = n + k$, and explicit forms of such functions can be determined.

In another direction, as mentioned in [1], Mallows observed that there is a unique increasing sequence $(a(n))_{n\geq 0}$ of nonnegative integers such that $a\left(a(n)\right) = 2n$ for $n \neq 1$. In 1979, Propp [7, 8] introduced the sequence $(s(n))_{n\geq 0}$, defined to be the unique increasing sequence such that $s\left(s(n)\right) = 3n$. In 2005, Allouche et al. [1] showed that there are uncountably many increasing sequences $(a(n))_{n\geq 0}$ such that $a\left(a(n)\right) = dn$ for all $d \geq 4$, while for $d = 2$ and $d = 3$ there is a unique increasing sequence satisfying $a\left(a(n)\right) = dn$. Recently, in 2014, the results of Propp [7, 8] and Allouche et al. [1] were generalized in [3], where it was proved that for $q \geq 2, D \geq 2$, if $D - 1$ divides q, then there exists a unique increasing function $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying $f^q(n) = Dn$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; otherwise, there are uncountably many increasing functions satisfying this iterative functional equation.

Here, we consider the iterative functional equation

$$f^q(x) = g(x) \qquad (x \in X), \tag{1.2}$$

for two different domains of $X = \mathbb{Z}$ and $X = \mathbb{N}$.

Since the domains considered in this work are discrete, the results in the continuous case as given in [2] and [4–6] are independent from ours here. In the next section, we find all solutions f for the case $X = \mathbb{Z}$ and $g(x) = ax + b$, where $a, b \in \mathbb{Z}$ with $a \neq 0, 1$. Our approach is to look closely at the sequence of iterative values $\{f^n(\alpha)\}_{n\in\mathbb{Z}}$ for fixed α. This sequence is periodic with somewhat arbitrarily given values in each period, and the main task is to systematize the values in each period to take care of all possibilities. It turns out that the analysis is most involved when $a = -1$. When $a \notin \{-1, 0, 1\}$, the situation is simpler in the sense that the sequence of iterative values has a so-called initial element, called a starter, which enables us to systematize the values in each period more easily and cleanly.

In Section 3, we solve (1.2) when $X = \mathbb{N}_0$ and g is an increasing function having no fixed point. The approach in this part is similar to that of the case $a \notin \{-1, 0, 1\}$ in the previous section.

2. Over the set of integers

In this section, we solve the iterative functional equation

$$f^q(n) = an + b \quad (n \in \mathbb{Z}) \tag{2.1}$$

where $q \geq 2$, $a \neq 0, 1$, and b are integers. For brevity, put

$$g(n) := an + b, \quad p := \frac{b}{1-a}.$$

The simple proof of the next lemma is left to the reader.

Lemma 2.1 *If f is a solution function of (2.1), then:*

(i) *f is one-to-one;*

(ii) *f partitions $\mathbb{Z}$ into equivalence classes via the relation $x \sim y \Longleftrightarrow y = f^{sq}(x)$ for some $s \in \mathbb{Z}$;*

(iii) *f has a unique fixed point at $x = p$ provided that it is an integer.*

For convenience, we will define some terminologies about equivalence classes. Let $A_1, A_2, \ldots$ be equivalence classes on the set $\mathbb{Z}$ or $\mathbb{N}_0$ with respect to some equivalence relation. If A_i for $i = 1, 2, \ldots$ has the smallest absolute element, then this element is called the **starter** of class A_i and an element of A_i not being a starter is called a **nonstarter** of A_i.

We start with the case $a = -1$.

Theorem 2.2 *Let $q \geq 2$ and b be integers. If $f : \mathbb{Z} \to \mathbb{Z}$ satisfies the iterative functional equation*

$$f^q(n) = -n + b, \tag{2.2}$$

then f must be of the form f_π for some π defined below.

Proof Let $f : \mathbb{Z} \to \mathbb{Z}$ be a solution of $f^q(n) = -n + b$. Let $\alpha \in \mathbb{Z} \backslash \left\{\frac{b}{2}\right\}$. Consider the set

$$\left\{\alpha, f(\alpha), f^2(\alpha), \ldots, f^q(\alpha) = -\propto + b, \ldots, f^{2q}(\alpha) = \alpha\right\},$$

which has at most $2q$ elements. Let m be the smallest natural number such that $f^m(\alpha) = \alpha$. Since f^{-1} exists and $f^{2q}(\alpha) = \alpha$, we get $m \mid 2q$. If m is odd, then $m \mid q$, say $m = qk$, so $f^q(\alpha) = f^{mk}(\alpha) = \alpha$, a contradiction. Hence, m is even, say

$$m = 2r. \tag{2.3}$$

If $\alpha, f(\alpha), f^2(\alpha), \ldots, f^{2r-1}(\alpha)$ are not distinct, then $f^i(\alpha) = f^j(\alpha)$ for some i, j with $0 \leq i < j \leq 2r - 1$ implying that $f^{j-i}(\alpha) = \alpha$, which is a contradiction. Then $\alpha, f(\alpha), f^2(\alpha), \ldots, f^{2r-1}(\alpha)$ are distinct. Note also that $f^l(\alpha) = \alpha$ for $l \in 2r\mathbb{Z}$. Since $r \mid q$, if q/r is even, say $q = (2t)r$, then $f^q(\alpha) = f^{(2r)t}(\alpha) = \alpha$, a contradiction. Then q/r must be an odd number, say $q = (2t+1)r$, so

$$f^r(\alpha) = f^r(f^{2r(t)}(\alpha)) = f^{(2t+1)r}(\alpha) = f^q(\alpha) = -\alpha + b.$$

Hence,

$$\left\{\alpha, f(\alpha), f^2(\alpha)\ , \ldots, f^{2r-1}(\alpha)\right\} = \left\{\alpha, f(\alpha), f^2(\alpha), \ldots, f^{r-1}(\alpha), -\alpha + b, -f(\alpha) + b, \ldots, -f^{r-1}(\alpha) + b\right\}.$$

For $h = 0, 1, \ldots, r-1$, let $D_h^\alpha := \left\{ f^h(\alpha), f^{h+r}(\alpha) \right\} = \left\{ f^h(\alpha), -f^h(\alpha) + b \right\}$. Observe that $D_h^\alpha \cap D_k^\alpha = \emptyset$ for all $h, k \in \{0, 1, \ldots, r-1\}$ and $h \neq k$. Denote the set $D_0^\alpha \cup \cdots \cup D_{r-1}^\alpha$ by $\mathscr{D}(r, \alpha)$ and observe that

$$f : D_h^\alpha = \left\{ f^h(\alpha), -f^h(\alpha) + b \right\} \to D_{h+1}^\alpha = \left\{ f^{h+1}(\alpha), -f^{h+1}(\alpha) + b \right\}$$

satisfies

$$f(f^h(\alpha)) = f^{h+1}(\alpha), \qquad f(-f^h(\alpha) + b) = f^{h+1+r}(\alpha) = -f^{h+1}(\alpha) + b.$$

Taking $\alpha_2 \in \mathbb{Z} \backslash \left(\mathscr{D}(r, \alpha) \cup \{b/2\} \right)$ and repeating the above procedure, we obtain

$$\mathscr{D}(r_2, \alpha_2) = D_0^{\alpha_2} \cup \cdots \cup D_{r_2-1}^{\alpha_2},$$

where $r_2 \mid q$ and q/r_2 is odd. This process can be continued until we eventually exhaust the set $\mathbb{Z}$ and so

$$\mathbb{Z} \backslash \left\{ \frac{b}{2} \right\} = \mathscr{D}(r, \alpha) \cup \mathscr{D}(r_2, \alpha_2) \cup \cdots .$$

We turn now to obtain an explicit form of the solution function f. Let

$$E_i = \{i, b-i\}, \quad i \in \left\{ m \in \mathbb{Z} \mid m \geq \frac{b}{2} \right\}.$$

Observe that $\mathbb{Z} = \cup_{i \in \mathbb{Z}} E_i$. Let π be a partition of the set $\{E_i\}_{i \neq b/2}$ into subsets, each of which contains s of the sets E_i, where s is a divisor of q with q/s odd as obtained in (2.3). Since q/s is an odd number, let $q = (2l+1)s$. Consider the set

$$\mathscr{E}_{s(\pi(i))} = E_{i_0} \cup E_{i_1} \cup \cdots \cup E_{i_{s-1}},$$

where $\pi(i) = \{i_0, i_1, \ldots, i_{s-1}\}$ is the ith component of the partition π, and

$$E_{i_j} = \left\{ \alpha_{i_j}, \beta_{i_j} \right\}, \quad \alpha_{i_j} \in \{i_j, b - i_j\}, \beta_{i_j} = b - \alpha_{i_j}$$

for $j = 0, 1, 2, \ldots, s-1$.

Define the function $f_\pi : \mathbb{Z} \to \mathbb{Z}$ by assigning its values on each $\mathscr{E}_{s(\pi(i))}$ as

$$f_\pi : E_{i_j} \to E_{i_{j+1}} \qquad (j = 0, 1, 2, \ldots, s-2)$$

by

$$f_\pi(\alpha_{i_j}) = \alpha_{i_{j+1}}, \qquad f_\pi(\beta_{i_j}) = \beta_{i_{j+1}}$$

and

$$f_\pi : E_{i_{s-1}} \to E_{i_0}$$

by

$$f_\pi(\alpha_{i_{s-1}}) = \beta_{i_0}, \qquad f_\pi(\beta_{i_{s-1}}) = \alpha_{i_0}.$$

The mapping of function f_π on each $\mathscr{E}_{s(\pi(i))}$ is illustrated by

$$\alpha_{i_0} \xrightarrow{f_\pi} \alpha_{i_1} \xrightarrow{f_\pi} \alpha_{i_2} \xrightarrow{f_\pi} \cdots \xrightarrow{f_\pi} \alpha_{i_{s-1}} \xrightarrow{f_\pi} \beta_{i_0} \xrightarrow{f_\pi} \beta_{i_1} \xrightarrow{f_{pi}} \beta_{i_2} \xrightarrow{f_\pi} \cdots \xrightarrow{f_\pi} \beta_{i_{s-1}} \xrightarrow{f_\pi} \alpha_{i_0}.$$

If $b/2 \in \mathbb{Z}$, define $f_\pi(b/2) = b/2$ so that $f_\pi^q(b/2) = b/2 = -(b/2) + b$.

We now show that f_π satisfies (2.2). For $j = 0, 1, \ldots, s-2$, since

$$
\begin{aligned}
f_\pi(\alpha_{i_j}) &= \alpha_{i_{j+1}}, & f_\pi^2(\alpha_{i_j}) &= f_\pi(\alpha_{i_{j+1}}) = \alpha_{i_{j+2}}, \\
&\vdots & &\vdots \\
f_\pi^{s-j-1}(\alpha_{i_j}) &= \alpha_{i_{j+(s-j-1)}} = \alpha_{i_{s-1}}, & f_\pi^{s-j}(\alpha_{i_j}) &= f_\pi(\alpha_{i_{s-1}}) = \beta_{i_0}, \\
&\vdots & &\vdots \\
f_\pi^{s}(\alpha_{i_j}) &= f_\pi^{j}(\beta_{i_0}) = \beta_{i_j}, & -\alpha_{i_j} + b &= \beta_{i_j},
\end{aligned}
$$

we get

$$f_\pi^s(\alpha_{i_j}) = -\alpha_{i_j} + b \quad \text{and} \quad f_\pi^{2s}(\alpha_{i_j}) = \alpha_{i_j}$$

so

$$f_\pi^q(\alpha_{i_j}) = f_\pi^{2ls+s}(\alpha_{i_j}) = f_\pi^s(\alpha_{i_j}) = -\alpha_{i_j} + b.$$

Similarly, we have $f_\pi^q(\beta_{i_j}) = -\beta_{i_j} + b$ for $j = 0, 1, \ldots, s-2$. Finally, on the set $E_{i_{s-1}}$, we have

$$
\begin{aligned}
f_\pi(\alpha_{i_{s-1}}) &= \beta_{i_0}, \\
f_\pi^2(\alpha_{i_{s-1}}) &= f_\pi(\beta_{i_0}) = \beta_{i_1}, \\
&\vdots \\
f_\pi^s(\alpha_{i_{s-1}}) &= \beta_{i_{s-1}} = -\alpha_{i_{s-1}} + b, \\
f_\pi^{2s}(\alpha_{i_{s-1}}) &= \alpha_{i_{s-1}},
\end{aligned}
$$

so

$$f_\pi^q(\alpha_{i_{s-1}}) = f_\pi^{2ls+s}(\alpha_{i_{s-1}}) = f_\pi^s(\alpha_{i_{s-1}}) = -\alpha_{i_{s-1}} + b,$$

and similarly we also have

$$f_\pi^q(\beta_{i_{s-1}}) = -\beta_{i_{s-1}} + b.$$

Since $\mathbb{Z} = \cup_{i\in\mathbb{Z}} E_i$, it follows that $f_\pi^q(n) = -n + b$.

From the above construction, we deduce that the totality of the $\mathscr{E}_{s(\pi(i))}$s is identical with that of the $\mathscr{D}(\alpha, s)$s and so each solution function f of (2.2) must be of the form f_π for some π (with s being some divisor r of q). □

The last part of the proof in Theorem 2.2 gives the following collorary.

Corollary 2.3 *Let $q, b \in \mathbb{Z}$, $q \geq 2$ and let $C_i = \{i, b-i\}$, $i \in \mathbb{Z}$. For each divisor s of q such that q/s is odd, let $I_s = \{i_0, i_1, \ldots, i_{s-1}\} \subset \mathbb{Z}$ be a set of s indices $i_0 < i_1 < \cdots < i_{s-1}$ such that $C_{i_j} \cap C_{i_k} = \emptyset$ whenever $i_j \notin \{i_k, b-i_k\}$ and let $\mathscr{C}_s^{(I_s)} = C_{i_0} \cup \cdots \cup C_{i_{s-1}}$.*

For each decomposition of $\mathbb{Z}\setminus\{b/2\}$ into a countable union of pointwise disjoint $\mathscr{C}_s^{(I_s)}$ $\left(i.e.\ ,\mathbb{Z}\setminus\{b/2\} = \bigcup_{I_s} \mathscr{C}_s^{(I_s)}\right)$, define $f_s : \mathbb{Z} \to \mathbb{Z}$ by

$$f_s\left(\frac{b}{2}\right) = \frac{b}{2} \quad \text{if} \quad \frac{b}{2} \in \mathbb{Z},$$

from each $\mathscr{C}_s^{(I_s)}$ *into itself*

$$f_s : C_{i_j} := \left\{\alpha_{i_j}, \beta_{i_j}\right\} \to C_{i_{j+1}} \qquad (j = 0, 1, \ldots, s-2)$$

where $\alpha_{i_j} \in \{i_j, b - i_j\}$, $\beta_{i_j} = b - \alpha_{i_j}$, *by*

$$f_s(\alpha_{i_j}) = \alpha_{i_{j+1}}, \qquad f_s(\beta_{i_j}) = \beta_{i_{j+1}},$$

and

$$f_s : C_{i_{s-1}} := \left\{\alpha_{i_{s-1}}, \beta_{i_{s-1}}\right\} \to C_{i_0}$$

by

$$f_s(\alpha_{i_{s-1}}) = \beta_{i_0}, \qquad f_s(\beta_{i_{s-1}}) = \alpha_{i_0}.$$

Then f_s *satisfies* $f_s^q(n) = -n + b$.

Example 2.4 *Let* $f : \mathbb{Z} \to \mathbb{Z}$ *satisfy the iterative functional equation*

$$f^6(n) = g(n) := -n + 5. \tag{2.4}$$

Then $g(n)$ *has no fixed points in* $\mathbb{Z}$. *Since* $q = 6$, *there are two possible values of* s *for which* q/s *is odd, i.e.* $s = 2, 6$. *Let*

$$C_i = \{i, 5-i\}, \quad i \in \left\{m \in \mathbb{Z} \mid m \geq \frac{5}{2}\right\}.$$

Case $s = 2$***.*** *To illustrate how to obtain a solution function, as an example of a possible partition* π, *we take*

$$\pi = \{C_3, C_4\} \cup \{C_5, C_6\} \cup \{C_7, C_8\} \cup \{C_9, C_{10}\} \cup \cdots.$$

The solution function f_π *is shown via the following diagram of element maps:*

$$\begin{matrix} f_\pi : C_i \to C_{i+1} \\ f_\pi : C_{i+1} \to C_i \end{matrix} \qquad i = 3, 5, 7, \ldots$$

$$\begin{aligned}
i = 3 &: 3 \overset{f_\pi}{\to} 4 \overset{f_\pi}{\to} 2 \overset{f_\pi}{\to} 1 \overset{f_\pi}{\to} 3 \overset{f_\pi}{\to} 4 \overset{f_\pi}{\to} 2 \overset{f_\pi}{\to} 1 \overset{f_\pi}{\to} \cdots \\
i = 5 &: 5 \to 6 \to 0 \to -1 \to 5 \to 6 \to 0 \to -1 \to \cdots \\
i = 7 &: 7 \to 8 \to -2 \to -3 \to 7 \to 8 \to -2 \to -3 \to \cdots \\
&\vdots
\end{aligned}$$

As another example of possible partition π, *take*

$$\pi = \{C_3, C_5\} \cup \{C_4, C_6\} \cup \{C_7, C_9\} \cup \{C_8, C_{10}\} \cup \cdots,$$

which gives another solution function f_π:

$$\begin{matrix} f_\pi : C_i \to C_{i+2} \\ f_\pi : C_{i+2} \to C_i \end{matrix} \qquad i = 3, 4, 7, 8, 11, 12, \ldots$$

$$\begin{aligned}
i=3 &\ : \ 3 \xrightarrow{f_\pi} 5 \xrightarrow{f_\pi} 2 \xrightarrow{f_\pi} 0 \xrightarrow{f_\pi} 3 \xrightarrow{f_\pi} 5 \xrightarrow{f_\pi} 2 \xrightarrow{f_\pi} 0 \xrightarrow{f_\pi} \cdots \\
i=4 &\ : \ 4 \to 6 \to 1 \to -1 \to 4 \to 6 \to 1 \to -1 \to \cdots \\
i=7 &\ : \ 7 \to 9 \to -2 \to -4 \to 7 \to 9 \to -2 \to -4 \to \cdots \\
i=8 &\ : \ 8 \to 10 \to -3 \to -5 \to 8 \to 10 \to -3 \to -5 \to \cdots \\
&\ \ \vdots
\end{aligned}$$

Since there are infinite many partitions of the set $\{C_i\}_{i\geq 3}$ *into subsets, each of which contains two elements of the sets* C_i*, the equation* (2.4) *has infinitely many solutions.*

Case $s=6$***.*** *As an example of the partition* π*, take*

$$\pi = \{C_3, C_4, C_5, C_6, C_7, C_8\} \cup \{C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}\} \cup \cdots .$$

Then the solution f_π *is given by*

$$\begin{aligned}
&f_\pi : C_i \to C_{i+1} \text{ for } i \geq 3 \text{ and } i \neq 8, 14, 20, 26, \ldots \\
&f_\pi : C_i \to C_{i-5} \text{ for } i = 8, 14, 20, 26, \ldots
\end{aligned}$$

$$\begin{aligned}
&3 \xrightarrow{f_\pi} 4 \xrightarrow{f_\pi} 5 \xrightarrow{f_\pi} 6 \xrightarrow{f_\pi} 7 \xrightarrow{f_\pi} 8 \xrightarrow{f_\pi} 2 \xrightarrow{f_\pi} 1 \xrightarrow{f_\pi} 0 \xrightarrow{f_\pi} -1 \xrightarrow{f_\pi} -2 \xrightarrow{f_\pi} -3 \xrightarrow{f_\pi} 3 \xrightarrow{f_\pi} \cdots \\
&9 \to 10 \to 11 \to 12 \to 13 \to 14 \to -4 \to -5 \to -6 \to -7 \to -8 \to -9 \to 9 \cdots \\
&\vdots
\end{aligned}$$

Since there are infinitely many partitions of the set $\{C_i\}_{i\geq 3}$ *into subsets of six elements, there are uncountably many solutions of* (2.4)*.*

Throughout the rest of this section, we assume that q, a, and b are integers such that $q \geq 2$ and $a \neq -1, 0, 1$.

Lemma 2.5 *Let* $\beta \in \mathbb{Z}$*.*

(i) *If* $g^{-i}(\beta) \notin \mathbb{Z}$ *for some* $i \in \mathbb{N}$*, then* $g^{-(i+1)}(\beta) \notin \mathbb{Z}$*.*

(ii) *If* $\beta = p$*, then* $g^{-j}(\beta) \in \mathbb{Z}$ *for all* $j \geq 1$*.*

(iii) *If* $\beta \neq p$*, then there exists a positive integer* J *such that* $g^{-j}(\beta) \in \mathbb{Z}$ *for all* $j \leq J$ *and* $g^{-j}(\beta) \notin \mathbb{Z}$ *for all* $j > J$*.*

Proof

(i) Assume that $g^{-(i+1)}(\beta) := n \in \mathbb{Z}$. Then $g^{-i}(\beta) = g(n) \in \mathbb{Z}$, a contradiction.

(ii) We see that

$$g(p) = a\left(\frac{b}{1-a}\right) + b = \frac{b}{1-a} = p,$$

and so $p = g^{-j}(p)$ for all $j \geq 1$.

(iii) Suppose $\beta \neq p$. If $p \in \mathbb{Z}$, then let J be the largest nonnegative integer such that a^J divides $\beta - p$. For $i = 0, 1, \ldots, J$, we have $\frac{\beta-p}{a^i} + p \in \mathbb{Z}$ and

$$\begin{aligned}\frac{\beta - p}{a^i} + p &= \frac{\beta}{a^i} - \frac{b}{a-1}\left(\frac{a^i - 1}{a^i}\right) = \frac{\beta - b}{a^i} - \frac{b}{a^{i-1}} - \cdots - \frac{b}{a} \\ &= \frac{1}{a}\left\{\left(\frac{\beta - b}{a^{i-1}} - \frac{b}{a^{i-2}} - \cdots - \frac{b}{a}\right) - b\right\} = g^{-1}\left(\frac{\beta - b}{a^{i-1}} - \frac{b}{a^{i-2}} - \cdots - \frac{b}{a}\right) \\ &= \cdots = g^{-i}(\beta).\end{aligned}$$

For $j > J$, we easily see that $g^{-j}(\beta) = \frac{\beta-p}{a^j} + p \notin \mathbb{Z}$.

If $p \notin \mathbb{Z}$, then consider the numbers of the form $\frac{\beta-p}{a^k} + p$ $(k \in \mathbb{N}_0)$. For $k = 0$, note that $\frac{\beta-p}{a^k} + p \in \mathbb{Z}$. Let $K \in \mathbb{N}$ be large enough so that $\left|\frac{\beta-p}{a^K}\right| < |p| - \lfloor |p| \rfloor$. Thus, $\frac{\beta-p}{a^K} + p \notin \mathbb{Z}$. Let J be the largest positive integer such that $\frac{\beta-p}{a^j} + p \in \mathbb{Z}$ for $j = 0, 1, \ldots, J$. For $j > J$, we then have $g^{-j}(\beta) = \frac{\beta-p}{a^j} + p \notin \mathbb{Z}$. □

By Lemma 2.5, each equivalence class constructed via the equivalence relation in Lemma 2.1 contains a unique element called a starter (α is a starter if there is no integer n such that $g(n) = \alpha$). Let S denote the set of all starters in $\mathbb{Z}\backslash\{p\}$ and let N denote the set of all nonstarters in $\mathbb{Z}$ together with p if $p \in \mathbb{Z}$. We see that S is an infinite set since $\{ma + b - 1 \mid m \in \mathbb{N}_0\} \subset S$.

By Lemma 2.5, the equivalence classes constructed via the equivalence relation in Lemma 2.1 are either of the form

$$C_\alpha = \{g^m(\alpha) \mid m \in \mathbb{N}_0\} \text{ where } \alpha \in S$$

or of the form

$$C_p = \begin{cases} \{p\}, & \text{if } p \in \mathbb{Z}, \\ \emptyset, & \text{if } p \notin \mathbb{Z}, \end{cases}$$

which yields

$$\mathbb{Z} = \left(\bigcup_{\alpha \in S} C_\alpha\right) \bigcup C_p.$$

For convenience, let $\mathcal{C} = \{C_\alpha \mid \alpha \in S\}$. Useful properties of these classes are gathered below.

Lemma 2.6 *If r and l are in S but $r \neq l$, then r and l are in different classes.*

Proof If r and l are in the same equivalence class, then there is $t \in \mathbb{Z}\backslash\{0\}$ such that $r = f^{tq}(l)$. Interchanging r, l, we may assume $t > 0$. Thus, $r = f^q\left(f^{(t-1)q}(l)\right) = g(f^{(t-1)q}(l))$, i.e. $r \notin S$, a contradiction. □

Lemma 2.7 (i) *For each $\alpha \in S$, f sends the class C_α into a unique class in $\mathcal{C}$ that contains $f(\alpha)$.*

(ii) *We have $f^q(C_\alpha) \subset C_\alpha$ for all $\alpha \in S$.*

(iii) *The function f is not onto.*

Proof

(i) Let $x \in C_\alpha$, i.e. there exists a nonnegative integer m such that $x = g^m(\alpha)$. We see that

$$f(x) = f(g^m(\alpha)) = f^{qm}(f(\alpha)) = g^m(f(\alpha)),$$

i.e. $f(x)$ and $f(\alpha)$ are in the same class, say, $C_{\overline{\alpha}}$ for some $\overline{\alpha} \in S$. If $C_{\overline{\alpha}} \cap C_p \neq \emptyset$, then there is a nonnegative integer m such that $p = g^m(f(\alpha)) = f(g^m(\alpha))$. Since p is the fixed point of f and f is one-to-one, we get $g^m(\alpha) = p$, and so $\alpha = p$, which is a contradiction.

(ii) Let $\alpha \in S$ and $x \in C_\alpha$, i.e. $x = g^m(\alpha)$ for some $m \in \mathbb{N}_0$. Then $f^q(x) = g^{m+1}(\alpha) \in C_\alpha$, which shows that $f^q(C_\alpha) \subset C_\alpha$ for all $\alpha \in S$.

(iii) Suppose that f is onto. Let $\alpha \in S$. Then there exists $\alpha_1 \in \mathbb{Z}$ such that $\alpha = f(\alpha_1)$. Similarly, there exists $\alpha_2 \in \mathbb{Z}$ such that $\alpha_1 = f(\alpha_2)$, i.e. $\alpha = f^2(\alpha_2)$. Continuing in this way, we get $\alpha = f^q(\alpha_q)$ for some $\alpha_q \in \mathbb{Z}$, i.e. $\alpha = g(\alpha_q)$, a contradiction.

□

Lemma 2.7(iii) tells us that $R_f \neq \mathbb{Z}$. Since $N \subset R_f$, it follows that the set $S \backslash R_f$ is nonempty and we explore it further.

Lemma 2.8 *If $k \in S \backslash R_f$, then $f^{q-1}(k), f^{q-2}(k), \ldots, f(k), k$ are the starters of q distinct classes.*

Proof For $i = 0, 1, \ldots, q-1$, suppose $f^i(k) \in C_{\alpha_i}$ where the α_is are not necessarily distinct. We now show that $f^i(k)$ is the starter of the class C_{α_i}. Starting from class $C_{\alpha_{q-1}}$, we have $f^{q-1}(k) = g^t(\alpha_{q-1})$ for some $t \in \mathbb{N}_0$. Since both α_{q-1} and $f^{q-1}(k)$ are in class $C_{\alpha_{q-1}}$, Lemma 2.7(ii) implies that both $f(\alpha_{q-1})$ and $f^q(k)$ are in the same class. By Lemma 2.7(iii), we have $f^q(k) \in C_{\alpha_0}$, and so $f(\alpha_{q-1}) \in C_{\alpha_0}$, i.e. $f(\alpha_{q-1}) = g^s(k)$ for some $s \in \mathbb{N}$. Then

$$g(k) = f^q(k) = f(f^{q-1}(k)) = f(g^t(\alpha_{q-1})) = g^t(f(\alpha_{q-1})) = g^{t+s}(k).$$

Since g is one-to-one, we have $k = g^{t+s-1}(k) = a^{t+s-1}(k-p)+p$. Since k is not a fixed point, we have $t+s = 1$, so $t = 0$ and $s = 1$. Hence, $f^{q-1}(k) = \alpha_{q-1}$. The same reasoning also shows that $f^{q-2}(k), \ldots, f(k), k$ are the starters of the distinct classes $C_{\alpha_{q-2}}, \ldots, C_{\alpha_1}, C_{\alpha_0}$, respectively. □

We can now determine our solution function.

Theorem 2.9 *If $f : \mathbb{Z} \to \mathbb{Z}$ satisfies the iterative functional equation $f^q(n) = an + b$, then f must be of the form f_π for some π defined below.*

Proof Let $f : \mathbb{Z} \to \mathbb{Z}$ be a solution of $f^q(n) = an + b$. Let $\alpha_0 \in S \backslash R_f$ so that $\alpha_0 \neq p$. From Lemma 2.8, we see that $\alpha_0, f(\alpha_0) = \alpha_1, \ldots, f^{q-1}(\alpha_0) = \alpha_{q-1}$ are starters of q distinct classes. Since f is 1-1, we have $\alpha_i \neq p$ for $i = 0, 1, \ldots, q-1$. Indeed, $\alpha_i = f^i(\alpha_0)$ is the starter of C_{α_i}.

Let $\mathscr{C}(\alpha_0) = C_{\alpha_0} \cup \cdots \cup C_{\alpha_{q-1}}$ and observe that for $h = 0, 1, \ldots, q-2$,

$$f : C_{\alpha_h} = \{g^m(\alpha_h) \mid m \in \mathbb{N}_0\} \to C_{\alpha_{h+1}} = \{g^m(\alpha_{h+1}) \mid m \in \mathbb{N}_0\}$$

via

$$\begin{aligned}
f(g^0(\alpha_h)) &= f(\alpha_h) = \alpha_{h+1}, \\
f(g(\alpha_h)) &= f(f^q(\alpha_h)) = f^q(f(\alpha_h)) = af(\alpha_h) + b = a\alpha_{h+1} + b = a\alpha_{h+1} - (a-1)p \\
&\vdots \\
f(g^m(\alpha_h)) &= g^m(\alpha_{h+1}) = a^m(\alpha_{h+1}) - (a^m - 1)p
\end{aligned}$$

and

$$f : C_{\alpha_{q-1}} = \{g^m(\alpha_{q-1}) \mid m \in \mathbb{N}_0\} \to C_{\alpha_0} = \{g^m(\alpha_0) \mid m \in \mathbb{N}_0\}$$

via

$$\begin{aligned}
f(g^0(\alpha_{q-1})) &= f(\alpha_{q-1}) = f^q(\alpha_0) = a\alpha_0 + b = a\alpha_0 - (a-1)p, \\
f(g(\alpha_{q-1})) &= f(f^q(\alpha_{q-1})) = f^{q+1}(f^{q-1}(\alpha_0)) = f^{2q}(\alpha_0) = af^q(\alpha_0) + b = a^2\alpha_0 - (a^2-1)p \\
&\vdots \\
f(g^m(\alpha_{q-1})) &= g^m(a\alpha_0 + b) = a^{m+1}(\alpha_0) - (a^{m+1} - 1)p.
\end{aligned}$$

Take $\alpha_q \in (S \backslash R_f) \cap (\mathbb{Z} \backslash (\mathscr{C}(\alpha_0) \cup \{p\}))$ and repeat the above procedure to obtain

$$\mathscr{C}(\alpha_q) = C_{\alpha_q} \cup \cdots \cup C_{\alpha_{2q-1}}.$$

Continuing in the same manner, at stage $i+1$ choose

$$\alpha_{iq} \in (S \backslash R_f) \cap \left(\mathbb{Z} \backslash \left(\mathscr{C}(\alpha_0) \cup \mathscr{C}(\alpha_q) \cup \cdots \cup \mathscr{C}(\alpha_{(i-1)q}) \cup \{p\}\right)\right)$$

and construct

$$\mathscr{C}(\alpha_{iq}) = C_{\alpha_{iq}} \cup \cdots \cup C_{\alpha_{(i+1)q-1}}.$$

We finally arrive at

$$\mathbb{Z} \backslash \{p\} = \mathscr{C}(\alpha_0) \cup \mathscr{C}(\alpha_q) \cup \mathscr{C}(\alpha_{2q}) \cup \cdots .$$

We turn now to obtain an explicit form of the solution function f. Define $f_\pi : \mathbb{Z} \to \mathbb{Z}$ on each $\mathscr{C}(\alpha_h)$, $h \in q\mathbb{N}_0$, by

$$f_\pi : C_{\alpha_{h+j}} \to C_{\alpha_{h+j+1}} \qquad (j = 0, 1, \ldots, q-2)$$

via

$$f_\pi(g^m(\alpha_{h+j})) = g^m(\alpha_{h+j+1})$$

and

$$f_\pi : C_{\alpha_{h+q-1}} \to C_{\alpha_h}$$

via

$$f_\pi(g^m(\alpha_{h+q-1})) = g^{m+1}(\alpha_h)$$

and if $p \in \mathbb{Z}$, define $f_\pi(p) = p$, so that $f_\pi^q(p) = ap + b$.

For $j = 0, 1, \ldots, q-2$, we have

$$\begin{aligned}
f_\pi(g^m(\alpha_{h+j})) &= g^m(\alpha_{h+j+1}),\\
f_\pi^2(g^m(\alpha_{h+j})) &= f_\pi(g^m(\alpha_{h+j+1})) = g^m(\alpha_{h+j+2}),\\
&\vdots\\
f_\pi^{q-1-j}(g^m(\alpha_{h+j})) &= g^m(\alpha_{h+j+(q-1-j)}) = g^m(\alpha_{h+q-1}),\\
f_\pi^{(q-1-j)+1}(g^m(\alpha_{h+j})) &= f_\pi(g^m(\alpha_{h+q-1})) = g^{m+1}(\alpha_h),\\
&\vdots\\
f_\pi^q(g^m(\alpha_{h+j})) &= f_\pi^{(q-1-j)+(1+j)}(g^m(\alpha_{h+j})) = g^{m+1}(\alpha_{h+j}).
\end{aligned}$$

Since $g^{m+1}(\alpha_{h+j}) = g(g^m(\alpha_{h+j})) = ag^m(\alpha_{h+j}) + b$, it follows that

$$f_\pi^q(g^m(\alpha_{h+j})) = ag^m(\alpha_{h+j}) + b.$$

On the set $C_{\alpha_{h+q-1}}$, we have

$$\begin{aligned}
f_\pi(g^m(\alpha_{h+q-1})) &= g^{m+1}(\alpha_h),\\
f_\pi^2(g^m(\alpha_{h+q-1})) &= f_\pi(g^{m+1}(\alpha_h)) = g^{m+1}(\alpha_{h+1}),\\
&\vdots\\
f_\pi^{q-1}(g^m(\alpha_{h+q-1})) &= g^{m+1}(\alpha_{h+q-2}),\\
f_\pi^q(g^m(\alpha_{h+q-1})) &= f_\pi(g^{m+1}(\alpha_{h+q-2})) = g^{m+1}(\alpha_{h+q-1}).
\end{aligned}$$

Since $g^{m+1}(\alpha_{h+q-1}) = g(g^m(\alpha_{h+q-1})) = ag^m(\alpha_{h+q-1}) + b$, it follows that

$$f_\pi^q(g^m(\alpha_{h+q-1})) = ag^m(\alpha_{h+q-1}) + b.$$

Hence, f is of the form f_π . $\square$

Corollary 2.10 *Let $\alpha \in \mathbb{Z}\setminus\{p\}$ and $C_\alpha = \{g^m(\alpha) \mid m \in \mathbb{N}_0\}$. Let*

$$I_j = \left\{\alpha_{jq}, \alpha_{jq+1}, \ldots, \alpha_{(j+1)q-1}\right\} \subset \mathbb{Z}\setminus\{p\} \quad (j \in \mathbb{N}_0)$$

be a set of q indices $\alpha_{jq} < \alpha_{jq+1} < \cdots < \alpha_{(j+1)q-1}$ such that $C_{\alpha_l} \cap C_{\alpha_t} = \emptyset$ whenever $\alpha_l \neq \alpha_t$ and let $\mathscr{C}_j^{(I_j)} = C_{\alpha_{jq}} \cup C_{\alpha_{jq+1}} \cup \cdots \cup C_{\alpha_{(j+1)q-1}}$.

For each decomposition of $\mathbb{Z}\setminus\{p\}$ into a countable union of pointwise disjoint $\mathscr{C}_j^{(I_j)}$ $\left(\text{i.e. } \mathbb{Z}\setminus\{p\} = \bigcup_{j\in\mathbb{N}_0} \mathscr{C}_j^{(I_j)}\right)$, denote this partition (I_j) by π.

Define $F_\pi : \mathbb{Z} \to \mathbb{Z}$ by $F_\pi(p) = p$ if $p \in \mathbb{Z}$, from each class $\mathscr{C}_j^{(I_j)}$ into itself by

$$F_\pi : C_{\alpha_{jq+l}} := \{g^m(\alpha_{jq+l}) \mid m \in \mathbb{N}_0\} \to C_{\alpha_{jq+l+1}} \qquad (l = 0, 1, \ldots, q-2)$$

via

$$F_\pi(g^m(\alpha_{jq+l})) = g^m(\alpha_{jq+l+1}),$$

and

$$F_\pi : C_{\alpha_{(j+1)q-1}} := \left\{g^m(\alpha_{(j+1)q-1}) \mid m \in \mathbb{N}_0\right\} \to C_{\alpha_{jq}}$$

via

$$F_\pi(g^m(\alpha_{(j+1)q-1})) = g^{m+1}(\alpha_{jq}).$$

Then F_π *satisfies* $f^q(n) = an + b$.

Proof Choose $\alpha_0 \in \mathbb{Z}\setminus\{p\}$ and construct $C_{\alpha_0} = \{g^m(\alpha_0) \mid m \in \mathbb{N}_0\}$. Choose $\alpha_1 \in \mathbb{Z}\setminus(C_{\alpha_0} \cup \{p\})$ and construct $C_{\alpha_1} = \{g^m(\alpha_1) \mid m \in \mathbb{N}_0\}$. Choose $\alpha_2 \in \mathbb{Z}\setminus(C_{\alpha_0} \cup C_{\alpha_1} \cup \{p\})$ and construct $C_{\alpha_2} = \{g^m(\alpha_2) \mid m \in \mathbb{N}_0\}$. Continue in the same manner until we choose $\alpha_{q-1} \in \mathbb{Z}\setminus\left(C_{\alpha_0} \cup \cdots \cup C_{\alpha_{q-2}} \cup \{p\}\right)$ and construct $C_{\alpha_{q-1}} = \{g^m(\alpha_{q-1}) \mid m \in \mathbb{N}_0\}$. Let $I_0 = \{\alpha_0, \dots, \alpha_{q-1}\}$.

Let $\mathscr{C}_0^{(I_0)} = \bigcup_{i=0}^{q-1} C_{\alpha_i}$. Let $\alpha_q \in \mathbb{Z}\setminus(\mathscr{C}(\alpha_0) \cup \{p\})$. We repeat the above procedure to obtain $I_1 = \{\alpha_q, \dots, \alpha_{2q-1}\}$ and

$$\mathscr{C}_1^{(I_1)} = \bigcup_{i=q}^{2q-1} C_{\alpha_i}.$$

Continuing in the same manner, choose

$$\alpha_{jq} \in \mathbb{Z}\setminus\left(\mathscr{C}(\alpha_0) \cup \mathscr{C}(\alpha_q) \cup \cdots \cup \mathscr{C}(\alpha_{(j-1)q}) \cup \{p\}\right)$$

and construct $I_j = \left\{\alpha_{jq}, \dots, \alpha_{(j+1)q-1}\right\}$

$$\mathscr{C}_j^{(I_j)} = \bigcup_{i=jq}^{(j+1)q-1} C_{\alpha_i}.$$

Repeat the process until we finally exhaust $\mathbb{Z}\setminus\{p\}$, and so

$$\mathbb{Z}\setminus\{p\} = \mathscr{C}_0^{(I_0)} \cup \mathscr{C}_1^{(I_1)} \cup \mathscr{C}_2^{(I_2)} \cup \cdots; \qquad \mathscr{C}_j^{(I_j)} \cap \mathscr{C}_k^{(I_k)} = \emptyset \quad (j \neq k).$$

Denote this partition of $\mathbb{Z}\setminus\{p\}$ by π.

Define $F_\pi : \mathbb{Z} \to \mathbb{Z}$ by $F_\pi(p) = p$ if $p \in \mathbb{Z}$ and on each class $\mathscr{C}_j^{(I_j)}$ $(j \geq 0)$ (of the partition π) onto itself by

$$\begin{aligned} F_\pi : C_{\alpha_i}\left(\in \mathscr{C}_j^{(I_j)}\right) &\to C_{\alpha_{i+1}} \qquad (i = (j-1)q, (j-1)q+1, \dots, jq-2),\\ g^m(\alpha_i) &\longmapsto g^m(\alpha_{i+1}),\\ F_\pi : C_{\alpha_{jq-1}} &\to C_{\alpha_{(j-1)q}},\\ g^m(\alpha_{jq-1}) &\longmapsto g^{m+1}(\alpha_{(j-1)q}). \end{aligned}$$

We see that $F_\pi^q(p) = p$ if $p \in \mathbb{Z}$ and for $l \in \{0, 1, \dots, q-2\}$,

$$F_\pi\left(g^m(\alpha_{(j-1)q+l})\right) = g^m(\alpha_{(j-1)q+l+1}),$$

$$
\begin{aligned}
F_\pi^2\left(g^m(\alpha_{(j-1)q+l})\right) &= F_\pi\left(g^m(\alpha_{(j-1)q+l+1})\right) = g^m(\alpha_{(j-1)q+l+2}),\\
&\vdots\\
F_\pi^{q-l-1}\left(g^m(\alpha_{(j-1)q+l})\right) &= g^m(\alpha_{(j-1)q+l+(q-l-1)}) = g^m(\alpha_{jq-1}),\\
F_\pi^{(q-l-1)+1}\left(g^m(\alpha_{(j-1)q+l})\right) &= F_\pi\left(g^m(\alpha_{jq-1})\right) = g^{m+1}(\alpha_{(j-1)q}),\\
F_\pi^{q}\left(g^m(\alpha_{(j-1)q+l})\right) &= F_\pi^{(q-l)+l}\left(g^m(\alpha_{(j-1)q+l})\right) = F_\pi^{l}\left(g^{m+1}(\alpha_{(j-1)q})\right)\\
&= g^{m+1}(\alpha_{(j-1)q+1}) = g\left(g^m(\alpha_{(j-1)q+1})\right) = ag^m(\alpha_{(j-1)q+1}) + b,
\end{aligned}
$$

and

$$
\begin{aligned}
F_\pi\left(g^m(\alpha_{jq-1})\right) &= g^{m+1}(\alpha_{(j-1)q}),\\
F_\pi^2\left(g^m(\alpha_{jq-1})\right) &= g^{m+1}(\alpha_{(j-1)q+1}),\\
&\vdots\\
F_\pi^{q-1}\left(g^m(\alpha_{jq-1})\right) &= g^{m+1}(\alpha_{(j-1)q+(q-2)}) = g^{m+1}(\alpha_{jq-2}),\\
F_\pi^{q}\left(g^m(\alpha_{jq-1})\right) &= F_\pi^{(q-1)+1}\left(g^m(\alpha_{jq-1})\right) = F_\pi\left(g^{m+1}(\alpha_{jq-2})\right)\\
&= g^{m+1}(\alpha_{jq-1}) = g\left(g^m(\alpha_{jq-1})\right) = ag^m(\alpha_{jq-1}) + b.
\end{aligned}
$$

Hence, F_π satisfies $f^q(n) = an + b$. $\square$

Example 2.11 *Let $f : \mathbb{Z} \to \mathbb{Z}$ satisfy the iterative functional equation*

$$f^3(n) = g(n) := 3n + 1. \tag{2.5}$$

Then $g(n)$ has no fixed points in $\mathbb{Z}$. The existence of a solution function is guaranteed by Theorem 2.9. The set of all starters is infinite of the form

$$S = \mathbb{Z}\setminus(3\mathbb{Z}+1) = 3\mathbb{Z}\cup(3\mathbb{Z}+2) = \{\ldots,-4,-3,-1,0,2,3,5,\ldots\}.$$

Then we have

$$
\begin{gathered}
\ldots,\ C_{-4} = \{-4,-11,-32,\ldots\},\ C_{-3} = \{-3,-8,-23,\ldots\},\ C_{-1} = \{-1,-2,-5,\ldots\},\\
C_0 = \{0,1,4,\ldots\},\ C_2 = \{2,7,22,\ldots\},\ C_3 = \{3,10,31,\ldots\},\ \ldots.
\end{gathered}
$$

As an example of a partition π of $\mathbb{Z}$, choose

$$I_0 = \{0,2,3\},\ I_1 = \{-4,-3,-1\},\ I_2 = \{5,6,8\},\ \ldots$$

and let

$$\mathscr{C}_0^{(I_0)} = C_0\cup C_2\cup C_3,\ \mathscr{C}_1^{(I_1)} = C_{-4}\cup C_{-3}\cup C_{-1},\ \mathscr{C}_2^{(I_2)} = C_5\cup C_6\cup C_8,\ \ldots.$$

Then the set

$$\pi = \left\{\mathscr{C}_0^{(I_0)}, \mathscr{C}_1^{(I_1)}, \mathscr{C}_2^{(I_2)}, \ldots\right\}$$

forms a partition of $\mathbb{Z}$*. The following diagram shows element maps of one such function* F_π*.*

$$\begin{array}{llll} F_\pi : C_0 \to C_2 & F_\pi : C_{-4} \to C_{-3} & F_\pi : C_5 \to C_6 & \\ F_\pi : C_2 \to C_3 & F_\pi : C_{-3} \to C_{-1} & F_\pi : C_6 \to C_8 & \cdots \\ F_\pi : C_3 \to C_0 & F_\pi : C_{-1} \to C_{-4} & F_\pi : C_8 \to C_5 & \end{array}$$

$$\begin{aligned} &0 \overset{F_\pi}{\to} 2 \overset{F_\pi}{\to} 3 \overset{F_\pi}{\to} 1 \overset{F_\pi}{\to} 7 \overset{F_\pi}{\to} 10 \overset{F_\pi}{\to} 4 \overset{F_\pi}{\to} 22 \overset{F_\pi}{\to} 31 \overset{F_\pi}{\to} \cdots \\ &-4 \to -3 \to -1 \to -11 \to -8 \to -2 \to -32 \to -23 \to -5 \to \cdots \\ &5 \to 6 \to 8 \to 16 \to 19 \to 25 \to 49 \to 58 \to 76 \to \cdots \\ &9 \to 11 \to 12 \to 28 \to 34 \to 37 \to 85 \to 103 \to 112 \to \cdots \\ &\cdots \end{aligned}$$

As there are many ways to construct such a partition π*, there are infinitely many solutions of* (2.5)*.*

3. Over the set of natural numbers

In this section, we prove the following theorem.

Theorem 3.1 *Let* $q \in \mathbb{N},\ q \geq 2$ *and* $g : \mathbb{N}_0 \to \mathbb{N}_0$*. Assume that* g *is an increasing function having no fixed point. If* $f : \mathbb{N}_0 \to \mathbb{N}_0$ *satisfies the iterative functional equation*

$$f^q(n) = g(n), \tag{3.1}$$

then f *must be of the form* $f_\mathcal{K}$ *for some* $\mathcal{K}$ *defined in the proof of this theorem.*

Adopting the notation in Sections 1 and 2, let R_g and R_f denote the range of g and f, respectively. We begin with some preliminary observations.

(i) Since $g = f^q$ is increasing, it is injective.

(ii) That g has no fixed points is equivalent to the condition that $g(1) \neq 1$.

(iii) The two functions f, g commute with each other, i.e. $f \circ g = g \circ f$.

Assuming the existence of a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying (3.1), we proceed by proving a number of lemmas containing its illuminating properties.

Lemma 3.2 *If* f *is a solution function of* (3.1)*, then:*

(i) f *is one-to-one (and so* f^{-1} *is well defined on* $R_f \subseteq \mathbb{N}_0$*);*

(ii) f *partitions* $\mathbb{N}_0$ *into (nonempty) equivalence classes via the relation*

$$x \sim y \quad \Longleftrightarrow \quad y = f^{sq}(x) \qquad \textit{for some } s \in \mathbb{Z}.$$

The proof of Lemma 3.2 is simple, so it is left to the reader. We next investigate these equivalence classes.

Lemma 3.3 *For $r, s \in \mathbb{N}_0$ with $r \neq s$, if r and s are not in R_g, then r and s are in different classes.*

Proof Suppose r and s are in the same class. There is $t \in \mathbb{Z}$ such that $r = f^{tq}(s)$. Since $r \neq s$, we see that $t \neq 0$, and by interchanging r, s, we can assume $t > 0$. Thus, $r = f^q\left(f^{(t-1)q}(s)\right) = g(f^{(t-1)q}(s))$, showing that $r \in R_g$, a contradiction. □

Lemma 3.4 *There is a unique nonnegative integer $r \notin R_g$ in each equivalence class, and it is the smallest element in this class.*

Proof Let C be such an equivalence class. If $C \subset R_g$, take an $n_0 \in C$; since g is increasing, we see that $n_0 = g(n_1)$ for some $n_1 \leq n_0$, and since g has no fixed point, we must have $n_1 < n_0$. Moreover, from $f^q(n_1) = g(n_1) = n_0$, we have $n_1 \in C$. Repeating the arguments, we obtain $n_1 = g(n_2)$, $n_2 < n_1$, $n_2 \in C$. Continuing in the same manner, we arrive at a negative integer belonging to C, which is not possible. Thus, C contains an element $r \notin R_g$, which must be unique by Lemma 3.3.

Next we show that r is the smallest element in C. If there exists $u \in C$ such that $u < r$, from the preceding result, $u \in R_g$, i.e. $u = g(u_1)$ for some $u_1 \in \mathbb{N}_0$. This element satisfies $u_1 < u$, $u_1 \in C$, by the same reasoning as above. Continuing in the same manner, we get a negative integer in C, which is again a contradiction. □

Taking into account the results of Lemmas 3.2–3.4, it is convenient to denote each equivalence class by C_r, where $r \notin R_g$ is the (unique) smallest element in the class, and such an element r is the starter of C_r. Clearly, the classes C_r are disjoint, and

$$\mathbb{N}_0 = \bigcup_{r \notin R_g} C_r, \quad C_r = \{g^m(r) \mid m \in \mathbb{Z}\}.$$

Lemma 3.5 *If $r \notin R_g$ is the starter of C_r, then $C_r = \{g^m(r) \mid m \in \mathbb{N}_0\}$.*

Proof If $x \in C_r$ is such that $x = g^{-m}(r)$ for some $m \in \mathbb{N}$, then $r = g^m(x)$. Since g is an increasing function without fixed points, we have

$$r = g^m(x) > g^{m-1}(x) \geq g^{m-2}(x) \geq \cdots \geq x,$$

contradicting the minimality of r. □

Lemma 3.6 *Let r be the starter of C_r. Then:*

(i) *the function f sends the class C_r into a unique class that contains $f(r)$;*

(ii) *after q iterations, f maps a class back into itself, i.e. $f^q(C_r) \subseteq C_r$.*

Proof From Lemma 3.5, each element in C_r is of the form $g^s(r) \in C_r$ $(s \in \mathbb{N}_0)$. Since

$$f(g^s(r)) = f(g(g^{s-1}(r))) = g(f(g^{s-1}(r))) = \cdots = g^s(f(r)),$$

we see that $f(g^s(r))$ and $f(r)$ are in the same class, which shows that all elements of $f(C_r)$ belong to one and only one class, which proves (i). To prove (ii), observe that

$$f^q(g^s(r)) = g(g^s(r)) = g^{s+1}(r) \in C_r.$$

□

If $x \in R_g = R_{f^q}$, then clearly $x \in R_f$, i.e. $R_g \subseteq R_f$, and we now look more closely at R_f.

Lemma 3.7 (i) *There are elements in* $\mathbb{N}_0$ *but not in* R_f.

(ii) *If* $k \in \mathbb{N}_0 \backslash R_f$, *then* $k, f(k), f^2(k), \ldots, f^{q-1}(k)$ *belong to distinct classes and each of them is the starter of its class.*

Proof (i) If $R_f = \mathbb{N}_0$, then f is a bijection (Lemma 3.2) onto its range, and so is f^{-1}. Since g is increasing without fixed points, we get $g(0) \neq 0$, yielding

$$g(0) \geq 1,\ g(1) \geq 2,\ \ldots,\ f^q(k) = g(k) \geq k+1\ (k \in \mathbb{N}_0).$$

This last relation implies that $f^{-q}(l) := u \leq l-1$. Using this repeatedly, we get

$$f^{-2q}(l) = f^{-q}(u) \leq u-1 \leq l-2,\ f^{-3q}(l) = f^{-2q}(u) \leq u-2 \leq l-3,\ \ldots.$$

Thus, for t sufficiently large, we arrive at $f^{-tq}(l) < 0$, which is untenable.

(ii) Take an integer $k \in \mathbb{N}_0 \backslash R_f$, and let

$$f^i(k) \in C_{r_i} \quad (i = 0, 1, 2, \ldots, q-1), \tag{3.2}$$

where the r_is are not necessarily distinct.

We claim that $f^i(k)$ is the smallest element in its class C_{r_i}.

Starting from the last class, $C_{r_{q-1}}$, if the claim is false, then its starter satisfies $r_{q-1} < f^{q-1}(k)$. Since $r_{q-1} = g^0(r_{q-1})$, the definition of the class shows that $f^{q-1}(k) = g^t(r_{q-1})$ for some $t \in \mathbb{N}$. Since r_{q-1} and $f^{q-1}(k)$ are in the same class $C_{r_{q-1}}$, Lemma 3.6(i) indicates that $f(r_{q-1})$ and $f(f^{q-1}(k))$ are in the same class. Since $k \in C_{r_0}$, by Lemma 3.6(ii), $f(f^{q-1}(k)) = f^q(k) \in C_{r_0}$. Thus, $f(r_{q-1}) \in C_{r_0}$. As g is increasing with no fixed point, we have

$$f(r_{q-1}) < g^t(f(r_{q-1})) = f(g^t(r_{q-1})) = f(f^{q-1}(k)) = f^q(k) = g(k).$$

Since there is no element between k and $g(k)$ in C_{r_0} and since $k \notin R_f$, we have $f(r_{q-1}) < k$. After applying $f^q = g$ repeatedly to $f(r_{q-1})$, we must reach k, so that $k \in R_f$, which is a contradiction. Hence, $f^{q-1}(k)$ is the starter in $C_{r_{q-1}}$. The same reasoning also shows that $f^{q-2}(k), \ldots, f(k)$ are the starters of classes $C_{r_{q-2}}, \ldots, C_{r_1}$, respectively.

It remains to show that k is the starter in C_{r_0}. Since $r_0 = g^0(r_0)$, $k = g^t(r_0)$ $(t \in \mathbb{N})$, are both in C_{r_0}, by Lemma 3.6(i), $f(r_0)$ and $f(k)$ are in the same class. Since g is increasing with no fixed point, we have

$$f(r_0) < g^t(f(r_0)) = f(g^t(r_0)) = f(k) \in C_{r_1},$$

yielding $f(r_0) \in C_{r_1}$, which contradicts the fact that $f(k)$ is the smallest in C_{r_1}. By Lemmas 3.3 and 3.4, $k, f(k), f^2(k), \ldots, f^{q-1}(k)$ are in distinct classes. □

Remark. From Lemma 3.7, we see that $r_i = f^i(k)$ $(i = 0, 1, \ldots, q-1)$ and it is clear that for $k \in \mathbb{N}_0 \backslash R_f$, the elements $f(k), f^2(k), \ldots, f^{q-1}(k)$ can be given arbitrary values from $\mathbb{N}_0$.

Keeping the above notation, we now prove Theorem 3.1 by showing that a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying (3.1) must be one of the functions $f_{\mathcal{K}}$ for some $\mathcal{K}$ defined below.

Proof of Theorem 3.1 Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a solution of $f^q(n) = g(n)$. Let $k_0 \in \mathbb{N}_0 \backslash R_f$ and $r_i = f^i(k_0)$ $(i = 0, 1, \dots, q-1)$. By Lemma 3.7(ii), we see that $r_0, r_1, \dots, r_{q-1}$ are starters of q distinct classes C_{r_i} $(0 \le i \le q-1)$. Since r_i is the starter of C_{r_i}, let

$$\mathscr{C}(r_0) = C_{r_0} \cup C_{r_1} \cup \cdots \cup C_{r_{q-1}}$$

and observe that for $h = 0, 1, \dots, q-2$,

$$f : C_{r_h} := \{g^m(r_h) \mid m \in \mathbb{N}_0\} \to C_{r_{h+1}} := \{g^m(r_{h+1}) \mid m \in \mathbb{N}_0\}$$

and

$$f : C_{r_{q-1}} \to C_{r_0}$$

satisfy

$$f(g^m(r_h)) = g^m(f(r_h)) = g^m(r_{h+1})$$

and

$$f(g^m(r_{q-1})) = g^m(f(r_{q-1})) = g^m(r_0).$$

Proceeding generally at the state $i+1$, $(i \ge 0)$, choose

$$r_{iq} \in (S \backslash R_f) \cap \big(\mathbb{N}_0 \backslash \big(\mathscr{C}(r_0) \cup \cdots \cup \mathscr{C}(r_{(i-1)q})\big)\big)$$

and construct

$$\mathscr{C}(r_{iq}) = C_{r_{iq}} \cup \cdots \cup C_{r_{(i+1)q-1}}.$$

Since $\mathscr{C}(r_{iq}) \cap \mathscr{C}(r_{jq}) = \emptyset$ if $i \ne j$, continuing this procedure will eventually exhaust $\mathbb{N}_0$, i.e. we obtain a partition $\mathcal{K}$ of $\mathbb{N}_0$ as

$$\mathbb{N}_0 = \mathscr{C}(r_0) \cup \mathscr{C}(r_q) \cup \mathscr{C}(r_{2q}) \cup \cdots = \bigcup_{i \ge 0} \mathscr{C}(r_{iq}). \tag{3.3}$$

We turn now to obtain an explicit form of the solution f.

Define $f_{\mathcal{K}} : \mathbb{N}_0 \to \mathbb{N}_0$ on each $\mathscr{C}(r_h)$ $(h \in q\mathbb{N}_0)$ by

$$f_{\mathcal{K}} : C_{r_{h+j}} \longrightarrow C_{r_{h+j+1}} \qquad (j = 0, 1, \dots, q-2)$$

via

$$f_{\mathcal{K}}\,(g^m(r_{h+j})) = g^m(r_{h+j+1})$$

and

$$f_{\mathcal{K}} : C_{r_{h+q-1}} \longrightarrow C_{r_h}$$

via

$$f_{\mathcal{K}}\,(g^m(r_{h+q-1})) = g^{m+1}(r_h).$$

The mapping $f_{\mathcal{K}}$ on each $\mathscr{C}(r_h)$ is illustrated by

$$\cdots \xrightarrow{f_{\mathcal{K}}} g^m(r_h) \xrightarrow{f_{\mathcal{K}}} g^m(r_{h+1}) \xrightarrow{f_{\mathcal{K}}} g^m(r_{h+2}) \xrightarrow{f_{\mathcal{K}}} \cdots \xrightarrow{f_{\mathcal{K}}} g^m(r_{h+q-1}) \xrightarrow{f_{\mathcal{K}}} g^{m+1}(r_h)$$

$$\xrightarrow{f_{\mathcal{K}}} g^{m+1}(r_{h+1}) \xrightarrow{f_{\mathcal{K}}} g^{m+1}(r_{h+2}) \xrightarrow{f_{\mathcal{K}}} \cdots \xrightarrow{f_{\mathcal{K}}} g^{m+1}(r_{h+q-1}) \xrightarrow{f_{\mathcal{K}}} g^{m+2}(r_h) \xrightarrow{f_{\mathcal{K}}} \cdots .$$

For $j = 0, 1, \dots, q-2$, we have

$$\begin{aligned}
f_{\mathcal{K}}(g^m(r_{h+j})) &= g^m(r_{h+j+1}),\\
f_{\mathcal{K}}^2(g^m(r_{h+j})) &= f_{\mathcal{K}}(g^m(r_{h+j+1})) = g^m(r_{h+j+2}),\\
&\vdots\\
f_{\mathcal{K}}^{q-1-j}(g^m(r_{h+j})) &= g^m(r_{h+j+(q-1-j)}) = g^m(r_{h+q-1}),\\
f_{\mathcal{K}}^{(q-1-j)+1}(g^m(r_{h+j})) &= f_{\mathcal{K}}(g^m(r_{h+q-1})) = g^{m+1}(r_h),\\
&\vdots\\
f_{\mathcal{K}}^{q}(g^m(r_{h+j})) &= f_{\mathcal{K}}^{(q-1-j)+(1+j)}(g^m(r_{h+j})) = g^{m+1}(r_{h+j}) = g(g^m(r_{h+j})).
\end{aligned}$$

On the set $C_{r_{h+q-1}}$, we have

$$\begin{aligned}
f_{\mathcal{K}}(g^m(r_{h+q-1})) &= g^{m+1}(r_h),\\
f_{\mathcal{K}}^2(g^m(r_{h+q-1})) &= f_{\mathcal{K}}(g^{m+1}(r_h)) = g^{m+1}(r_{h+1}),\\
&\vdots\\
f_{\mathcal{K}}^{q}(g^m(r_{h+q-1})) &= g^{m+1}(r_{h+q-1}) = g(g^m(r_{h+q-1})).
\end{aligned}$$

Hence, f is of the form $f_{\mathcal{K}}$.

Corollary 3.8 *Let $q \in \mathbb{N},\ q \geq 2$ and $g : \mathbb{N}_0 \to \mathbb{N}_0$. Assume that g is an increasing function having no fixed point.*

Let $r \in \mathbb{N}_0$ and $C_r = \{g^m(r) \mid m \in \mathbb{N}_0\}$. Let $I_j = \left\{r_{jq}, r_{jq+1}, \dots, r_{(j+1)q-1}\right\} \subset \mathbb{N}_0$ $(j \in \mathbb{N}_0)$ be a set of q indices $r_{jq} < r_{jq+1} < \cdots < r_{(j+1)q-1}$ such that $C_{r_l} \cap C_{r_t} = \emptyset$ whenever $r_l \neq r_t$ and let $\mathscr{C}_j^{(I_j)} = C_{r_{jq}} \cup C_{r_{jq+1}} \cup \cdots \cup C_{r_{(j+1)q-1}}$.

For each decomposition of $\mathbb{N}_0$ into a countable union of pointwise disjoint $\mathscr{C}_j^{(I_j)}$ $\left(i.e.\ \mathbb{N}_0 = \bigcup_{j\in\mathbb{N}_0} \mathscr{C}_j^{(I_j)}\right)$, denote this partition (I_j) by $\mathcal{K}$.

Define $F_{\mathcal{K}} : \mathbb{N}_0 \to \mathbb{N}_0$ from each class $\mathscr{C}_j^{(I_j)}$ into itself by

$$F_{\mathcal{K}} : C_{r_{jq+l}} := \{g^m(r_{jq+l}) \mid m \in \mathbb{N}_0\} \to C_{r_{jq+l+1}} \qquad (l = 0, 1, \dots, q-2)$$

via

$$F_{\mathcal{K}}(g^m(r_{jq+l})) = g^m(r_{jq+l+1}),$$

and

$$F_{\mathcal{K}} : C_{r_{(j+1)q-1}} := \left\{g^m(r_{(j+1)q-1}) \mid m \in \mathbb{N}_0\right\} \to C_{r_{jq}}$$

via

$$F_{\mathcal{K}}(g^m(r_{(j+1)q-1})) = g^{m+1}(r_{jq}).$$

Then $F_{\mathcal{K}}$ *satisfies* $f^q(n) = g(n)$*.*

Proof Given $q \in \mathbb{N},\ q \geq 2$ and $g : \mathbb{N}_0 \to \mathbb{N}_0$ an increasing function having no fixed point, choose $r_0 \in \mathbb{N}_0$ and construct $C_{r_0} = \{g^m(r_0) \mid m \in \mathbb{N}_0\}$. Choose $r_1 \in \mathbb{N}_0 \backslash C_{r_0}$ and construct $C_{r_1} = \{g^m(r_1) \mid m \in \mathbb{N}_0\}$. Choose $r_2 \in \mathbb{N}_0 \backslash (C_{r_0} \cup C_{r_1})$ and construct $C_{r_2} = \{g^m(r_2) \mid m \in \mathbb{N}_0\}$. Continue in the same manner until we choose $r_{q-1} \in \mathbb{N}_0 \backslash \left(C_{r_0} \cup \cdots \cup C_{r_{q-2}}\right)$ and construct $C_{r_{q-1}} = \{g^m(r_{q-1}) \mid m \in \mathbb{N}_0\}$.

Let $\mathscr{C}(r_0) = \bigcup\limits_{i=0}^{q-1} C_{r_i}$. Let $r_q \in \mathbb{N}_0 \backslash \mathscr{C}(r_0)$. We repeat the above procedure to obtain

$$\mathscr{C}(r_q) = \bigcup_{i=q}^{2q-1} C_{r_i}.$$

Continuing in the same manner, choose

$$r_{jq} \in \mathbb{N}_0 \backslash \left(\mathscr{C}(r_0) \cup \mathscr{C}(r_q) \cup \cdots \cup \mathscr{C}(r_{(j-1)q})\right)$$

and construct

$$\mathscr{C}(r_{jq}) = \bigcup_{i=(j-1)q}^{jq-1} C_{r_i}.$$

Repeat the process until we finally exhaust $\mathbb{N}_0$, and so

$$\mathbb{N}_0 = \mathscr{C}(r_0) \cup \mathscr{C}(r_q) \cup \mathscr{C}(r_{2q}) \cup \cdots; \qquad \mathscr{C}(r_{jq}) \cap \mathscr{C}(r_{kq}) = \emptyset \quad (j \neq k).$$

Denote this partition of $\mathbb{N}_0$ by $\mathcal{K}$.

Define $F_{\mathcal{K}} : \mathbb{N}_0 \to \mathbb{N}_0$ on each class $\mathscr{C}(r_{jq})$ $(j \geq 0)$ (of the partition $\mathcal{K}$) onto itself by

$$\begin{aligned}
F_{\mathcal{K}} &: C_{r_i}\,(\in \mathscr{C}(r_{jq})) \to C_{r_{i+1}} \qquad (i = (j-1)q, (j-1)q+1, \ldots, jq-2),\\
&\quad g^m(r_i) \longmapsto g^m(r_{i+1}),\\
F_{\mathcal{K}} &: C_{r_{jq-1}} \to C_{r_{(j-1)q}},\\
&\quad g^m(r_{jq-1}) \longmapsto g^{m+1}(r_{(j-1)q}).
\end{aligned}$$

We see that for $l \in \{0, 1, \ldots, q-2\}$,

$$\begin{aligned}
F_{\mathcal{K}}\left(g^m(r_{(j-1)q+l})\right) &= g^m(r_{(j-1)q+l+1}),\\
F_{\mathcal{K}}^2\left(g^m(r_{(j-1)q+l})\right) &= F_{\mathcal{K}}\left(g^m(r_{(j-1)q+l+1})\right) = g^m(r_{(j-1)q+l+2}),\\
&\ \,\vdots
\end{aligned}$$

$$\begin{aligned}
F_{\mathcal{K}}^{q-l-1}\left(g^m(r_{(j-1)q+l})\right) &= g^m(r_{(j-1)q+l+(q-l-1)}) = g^m(r_{jq-1}),\\
F_{\mathcal{K}}^{(q-l-1)+1}\left(g^m(r_{(j-1)q+l})\right) &= F_{\mathcal{K}}\left(g^m(r_{jq-1})\right) = g^{m+1}(r_{(j-1)q}),\\
F_{\mathcal{K}}^{q}\left(g^m(r_{(j-1)q+l})\right) &= F_{\mathcal{K}}^{(q-l)+l}\left(g^m(r_{(j-1)q+l})\right) = F_{\mathcal{K}}^{l}\left(g^{m+1}(r_{(j-1)q})\right)\\
&= g^{m+1}(r_{(j-1)q+1}) = g\left(g^m(r_{(j-1)q+1})\right),
\end{aligned}$$

and

$$\begin{aligned}
F_{\mathcal{K}}\left(g^m(r_{jq-1})\right) &= g^{m+1}(r_{(j-1)q}),\\
F_{\mathcal{K}}^{2}\left(g^m(r_{jq-1})\right) &= g^{m+1}(r_{(j-1)q+1}),\\
&\vdots\\
F_{\mathcal{K}}^{q-2}\left(g^m(r_{jq-1})\right) &= g^{m+1}(r_{(j-1)q+(q-3)}) = g^{m+1}(r_{jq-3}),\\
F_{\mathcal{K}}^{q-1}\left(g^m(r_{jq-1})\right) &= F_{\mathcal{K}}^{(q-1)+1}\left(g^m(r_{jq-1})\right) = F_{\mathcal{K}}\left(g^{m+1}(r_{jq-3})\right) = g^{m+1}(r_{jq-2}),\\
F_{\mathcal{K}}^{q}\left(g^m(r_{jq-1})\right) &= F_{\mathcal{K}}^{(q-1)+1}\left(g^m(r_{jq-1})\right) = F_{\mathcal{K}}\left(g^{m+1}(r_{jq-2})\right)\\
&= g^{m+1}(r_{jq-1}) = g\left(g^m(r_{jq-1})\right).
\end{aligned}$$

Hence, $F_{\mathcal{K}}$ satisfies $f^q(n) = g(n)$. □

As an application, we show that the result of Sarkaria mentioned earlier is an immediate consequence of Theorem 3.1.

Corollary 3.9 *Let $\alpha, q \in \mathbb{N},\ q \geq 2$. Then there exists $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying*

$$f^q(n) = n + \alpha$$

if and only if $q \mid \alpha$.

Proof Let $g : \mathbb{N}_0 \to \mathbb{N}_0$ be given by $g(n) = n + \alpha$. Clearly,

$$\mathbb{N}_0 \backslash R_g = \{0, 1, \ldots \alpha - 1\} =: S$$

is the set of all possible starters. The corresponding equivalence classes are

$$\begin{aligned}
C_{r_i}^{(k)} = \{f^i(k), g(f^i(k)), g^2(f^i(k)), \ldots\} &= \{f^i(k), f^i(k) + \alpha, f^i(k) + 2\alpha, \ldots\}\\
&= \left\{l \in \mathbb{N}_0 \mid l \equiv f^i(k) \bmod \alpha\right\} \qquad (3.4)\\
&\quad (k \in S,\ r_i = i \in \{0, 1, \ldots, q-1\}).
\end{aligned}$$

Observe that for the classes (3.4) and the relation (3.3) to be consistent with $\mathbb{N}_0 = \bigcup_{i=0}^{\alpha-1} \{k \in \mathbb{N}_0 \mid k \equiv i \bmod \alpha\}$, i.e. for $f_{\mathcal{K}}$ to be constructible, it is necessary and sufficient that $q \mid \alpha$. □

Example 3.10 *Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfy the iterative functional equation*

$$f^3(n) = g(n) := n^3 + 1. \qquad (3.5)$$

Clearly, $g(n)$ is an increasing function with no fixed points in $\mathbb{N}_0$. The existence of a solution function is guaranteed by Theorem 3.1. The set of all starters is infinite of the form

$$
\begin{aligned}
S &= \mathbb{N}_0 \backslash g(\mathbb{N}_0) = \{m \in \mathbb{N}_0 \mid m-1 \text{ is not a perfect cube in } \mathbb{N}_0\} \\
&= \{0, 3, 4, 5, 6, 7, 8, 10, 11, \ldots\}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& C_0 = \{0, 1, 2, \ldots\}, \ C_3 = \{3, 28, 21953, \ldots\}, \ C_4 = \{4, 65, 274626, \ldots\}, \\
& C_5 = \{5, 126, 2000377, \ldots\}, \ C_6 = \{6, 217, 10218314, \ldots\}, \ C_7 = \{7, 344, 40707585 \ldots\}, \ \ldots.
\end{aligned}
$$

As an example of a partition $\mathcal{K}$ of $\mathbb{N}_0$, choose

$$
I_0 = \{0, 3, 4\}, \ I_1 = \{5, 6, 7\}, \ I_2 = \{8, 10, 11\}, \ \ldots
$$

and let

$$
\mathscr{C}_0^{(I_0)} = C_0 \cup C_3 \cup C_4, \ \mathscr{C}_1^{(I_1)} = C_5 \cup C_6 \cup C_7, \ \mathscr{C}_2^{(I_2)} = C_8 \cup C_{10} \cup C_{11}, \ \ldots.
$$

Then the set

$$
\mathcal{K} = \left\{ \mathscr{C}_0^{(I_0)}, \mathscr{C}_1^{(I_1)}, \mathscr{C}_2^{(I_2)}, \ldots \right\}
$$

forms a partition of $\mathbb{N}_0$. The following diagram shows element maps of one such function $F_{\mathcal{K}}$:

$$
\begin{array}{llll}
F_{\mathcal{K}} : C_0 \to C_3 & F_{\mathcal{K}} : C_5 \to C_6 & F_{\mathcal{K}} : C_8 \to C_{10} & \\
F_{\mathcal{K}} : C_3 \to C_4 & F_{\mathcal{K}} : C_6 \to C_7 & F_{\mathcal{K}} : C_{10} \to C_{11} & \cdots \\
F_{\mathcal{K}} : C_4 \to C_0 & F_{\mathcal{K}} : C_7 \to C_5 & F_{\mathcal{K}} : C_{11} \to C_8 &
\end{array}
$$

$$
\begin{aligned}
& 0 \stackrel{F_{\mathcal{K}}}{\to} 3 \stackrel{F_{\mathcal{K}}}{\to} 4 \stackrel{F_{\mathcal{K}}}{\to} 1 \stackrel{F_{\mathcal{K}}}{\to} 28 \stackrel{F_{\mathcal{K}}}{\to} 65 \stackrel{F_{\mathcal{K}}}{\to} 2 \stackrel{F_{\mathcal{K}}}{\to} 21953 \stackrel{F_{\mathcal{K}}}{\to} 274626 \stackrel{F_{\mathcal{K}}}{\to} \cdots \\
& 5 \to 6 \to 7 \to 126 \to 217 \to 344 \to 2000377 \to 10218314 \to \cdots \\
& 8 \to 10 \to 11 \to 513 \to 1001 \to 1332 \to 135005698 \to \cdots \\
& 12 \to 13 \to 14 \to 1729 \to 2198 \to 2745 \to 5168743490 \to \cdots \\
& \ldots
\end{aligned}
$$

As there are many ways to construct such a partition $\mathcal{K}$, there are infinitely many solutions of (3.5).

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