Turkish Journal of Mathematics

Volume 42 | Number 3 Article 2

1-1-2018

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 752 – 762
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doi:10.3906/mat-1702-89

On biquaternion algebras with orthogonal involution

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Received: 20.02.2017 • **Accepted/Published Online:** 03.07.2017 • **Final Version:** 08.05.2018

Abstract: We investigate the Pfaffians of decomposable biquaternion algebras with involution of orthogonal type. In characteristic two, a classification of these algebras in terms of their Pfaffians and some other related invariants is studied. Also, in arbitrary characteristic, a criterion is obtained for an orthogonal involution on a biquaternion algebra to be metabolic.

Key words: Biquaternion algebra, Pfaffian, orthogonal involution

1. Introduction

A biquaternion algebra is a tensor product of two quaternion algebras. Every biquaternion algebra is a central simple algebra of degree 4 and exponent 2 or 1. A result proved by Albert shows that the converse is also true (see [8, (16.1)]). An *Albert form* of a biquaternion algebra A is a 6-dimensional quadratic form with trivial discriminant whose Clifford algebra is isomorphic to $M_2(A)$. According to [8, (16.3)], two biquaternion algebras over a field F are isomorphic if and only if their Albert forms are similar.

The Albert form of a biquaternion algebra with involution arises naturally as the quadratic form induced by a *Pfaffian* (see [11, (3.3)]). In [11], a Pfaffian of certain modules over Azumaya algebras was defined and used to find a decomposition criterion for involutions on a rank 16 Azumaya algebra, which contains 2 as a unit. A similar criterion for involutions on a biquaternion algebra in arbitrary characteristic was also obtained in [9].

It is known that symplectic involutions on a biquaternion algebra A can be classified, up to conjugation, by their *Pfaffian norms* (see [8, (16.19)]). For orthogonal involutions the situation is a little more complicated. In characteristic $\neq 2$, using [11, (5.3)], one can find a classification of decomposable orthogonal involutions on A in terms of the Pfaffian and the *Pfaffian adjoint* (introduced in [11]). This classification was originally stated in [11] for the more general case where A is an Azumaya algebra that contains 2 as a unit.

In this work we study decomposable biquaternion algebras with orthogonal involution. We start with some general observations on the Pfaffian and the Pfaffian adjoint. For a decomposable orthogonal involution σ we consider the Pfaffian q_σ and certain subsets $\mathrm{Alt}(A,\sigma)^+$ and $\mathrm{Alt}(A,\sigma)^-$ of $\mathrm{Alt}(A,\sigma)$, introduced in [9]. It is shown in (3.8) that the union of $\mathrm{Alt}(A,\sigma)^+$ and $\mathrm{Alt}(A,\sigma)^-$ coincides with the set of all square-central elements in $\mathrm{Alt}(A,\sigma)$. At the end of Section 3, we study in more detail the classification of orthogonal involutions on biquaternion algebras in characteristic $\neq 2$, obtained in [11]. Although this result was already presented in [11],

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2010 *AMS Mathematics Subject Classification:* 11E88, 15A63, 16W10

it is useful to rephrase it to enable comparison with the corresponding result in characteristic 2 (see (3.14) and (4.11)).

The classification problem in characteristic 2 is a little more complicated. Moreover, the results themselves have some substantial differences in this case. For example, the restriction q_σ^+ of q_σ to $\mathrm{Alt}(A,\sigma)^+$ is totally singular in characteristic 2, rather than a regular subform of the Pfaffian q_σ. Considering these remarks, our approach is to study the relation between the form q_σ^+ and the *Pfister invariant* of (A,σ), introduced in [4]. This relation is used in (4.11) to obtain necessary and sufficient conditions for orthogonal involutions to be conjugate to each other.

Finally, we study in Section 5 metabolic involutions on biquaternion algebras. Using some results of previous sections, we obtain various criteria for an orthogonal involution on a biquaternion algebra to be metabolic (see (5.2) and (5.4)). As a final application, we shall see in (5.5) how the Pfaffian can be used to characterize the transpose involution on a split biquaternion algebra.

2. Preliminaries

Let V be a finite dimensional vector space over a field F. A *quadratic form* over F is a map $q: V \to F$ such that (i) $q(av) = a^2q(v)$ for every $a \in F$ and $v \in V$; (ii) the map $\mathfrak{b}_q : V \times V \to F$ defined by $\mathfrak{b}_q(u,v) = q(u+v) - q(u) - q(v)$ is a bilinear form. The map $\mathfrak{b}_q$ is called the *polar form* of q. Note that for every $v \in V$ we have $\mathfrak{b}_q(v,v) = 2q(v)$. In particular, if $\mathrm{char}\, F = 2$, then $\mathfrak{b}_q(v,v) = 0$ for all $v \in V$, i.e. $\mathfrak{b}_q$ is an *alternating* form. The *orthogonal complement* of a subspace $W \subseteq V$ is defined as $W^\perp = \{x \in V \mid b_q(x,y) = 0 \text{ for all } y \in W\}$.

A quadratic form q (resp. a bilinear form $\mathfrak{b}$) on V is called *isotropic* if there exists a nonzero vector $v \in V$ such that $q(v) = 0$ (resp. $\mathfrak{b}(v,v) = 0$). For $\alpha \in F$, we say that q (resp. $\mathfrak{b}$) *represents* α if there exists a nonzero vector $v \in V$ such that $q(v) = \alpha$ (resp. $\mathfrak{b}(v,v) = \alpha$). The sets of all elements of F represented by q and $\mathfrak{b}$ are denoted by $D_F(q)$ and $D_F(\mathfrak{b})$, respectively. For $\alpha \in F^\times$, the *scaled* quadratic form $\alpha \cdot q$ is defined as $\alpha \cdot q(v) = \alpha q(v)$ for every $v \in V$.

For $a_1, \cdots, a_n \in F$, the isometry class of the quadratic form $\sum_{i=1}^n a_i x_i^2$ is denoted by $\langle a_1, \cdots, a_n\rangle_q$. Also, the isometry class of the bilinear form $\sum_{i=1}^n a_i x_i y_i$ is denoted by $\langle a_1, \cdots, a_n\rangle$. Finally, the form $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ is called a *bilinear n-fold Pfister form.*

An *involution* on a central simple F-algebra A is an antiautomorphism σ of A of order 2. We say that σ is *of the first kind* if $\sigma|_F = \mathrm{id}$. An involution σ of the first kind is said to be *symplectic* if over a splitting field of A it becomes adjoint to an alternating bilinear form. Otherwise, σ is called *orthogonal.* The set of *alternating elements* of A is defined as $\mathrm{Alt}(A,\sigma) = \{a - \sigma(a) \mid a \in A\}$. If A is of even degree $2m$, the *discriminant* of an orthogonal involution σ on A is defined as $\mathrm{disc}\,\sigma = (-1)^m \,\mathrm{Nrd}_A(x)F^{\times 2} \in F^\times/F^{\times 2}$, where $x \in \mathrm{Alt}(A,\sigma)$ is a unit and $\mathrm{Nrd}_A(x)$ is the reduced norm of x in A. Note that by [8, (7.1)], the discriminant does not depend on the choice of $x \in \mathrm{Alt}(A,\sigma)$.

A *quaternion algebra* over a field F is a central simple algebra Q of degree 2. The *canonical* involution γ on Q is defined by $\gamma(x) = \mathrm{Trd}_Q(x) - x$ for $x \in Q$, where $\mathrm{Trd}_A(x)$ is the reduced trace of x in A. The canonical involution on Q is the unique involution of symplectic type on Q and it satisfies $\gamma(x)x \in F$ for every $x \in Q$ (see [8, Ch. 2]). The map $N_Q : Q \to F$ defined by $N_Q(x) = \gamma(x)x$ is called the *norm form* of Q. An element $x \in Q$ is called a *pure quaternion* if $\mathrm{Trd}_Q(x) = 0$. The set of all pure quaternions of Q is a

3-dimensional subspace of Q denoted by Q_0. Note that an element $x \in Q$ lies in Q_0 if and only if $\gamma(x) = -x$, or equivalently, $N_Q(x) = -x^2$.

A central simple F-algebra with involution (A, σ) is called *totally decomposable* if it decomposes as a tensor product of σ-invariant quaternion F-algebras. If A is a biquaternion algebra, we will use the term *decomposable* instead of totally decomposable. Note that a biquaternion algebra with orthogonal involution (A, σ) is decomposable if and only if $\operatorname{disc}\sigma$ is trivial (see [9, (3.7)]).

3. The Pfaffian and the Pfaffian adjoint

We begin our discussion by looking at the special cases of [10, (2.1)] and [10, (3.1)].

Theorem 3.1 *Let (A, σ) be a biquaternion algebra with orthogonal involution over a field F and let $d_\sigma \in F^\times$ be a representative of the class $\operatorname{disc}\sigma \in F^\times/F^{\times 2}$. There exists a map $pf_\sigma : \operatorname{Alt}(A, \sigma) \to F$ such that $pf_\sigma(x)^2 = d_\sigma \operatorname{Nrd}_A(x)$ for every $x \in \operatorname{Alt}(A, \sigma)$. The map pf_σ is uniquely determined up to a sign. Moreover, there exists an F-linear map $\pi_\sigma : \operatorname{Alt}(A, \sigma) \to \operatorname{Alt}(A, \sigma)$ such that $x\pi_\sigma(x) = \pi_\sigma(x)x = pf_\sigma(x)$ and $\pi_\sigma^2(x) = d_\sigma x$ for every $x \in \operatorname{Alt}(A, \sigma)$.*

Remark 3.2 *The map π_σ in (3.1) is uniquely determined by pf_σ. Indeed, it is easily seen by scalar extension to a splitting field that $\operatorname{Alt}(A, \sigma)$ has a basis $\mathcal{B}$ consisting of invertible elements. For every $x \in \mathcal{B}$, we must have $\pi_\sigma(x) = x^{-1}pf_\sigma(x)$. As π_σ is F-linear, it is uniquely defined on $\operatorname{Alt}(A, \sigma)$.*

Definition 3.3 *A map pf_σ as in (3.1) is called a Pfaffian of (A, σ). We also call the map π_σ, the Pfaffian adjoint of pf_σ.*

Note that by [11, (3.3)], every Pfaffian of (A, σ) is an Albert form of A.

Notation 3.4 *Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Since $\operatorname{disc}\sigma$ is trivial, by (3.1) there is a unique, up to a sign, Pfaffian pf_σ satisfying $pf_\sigma(x)^2 = \operatorname{Nrd}_A(x)$ for $x \in \operatorname{Alt}(A, \sigma)$. We denote this Pfaffian by q_σ. We also denote by p_σ the Pfaffian adjoint of q_σ; hence,*

$$q_\sigma(x)^2 = \operatorname{Nrd}_A(x), \quad xp_\sigma(x) = p_\sigma(x)x = q_\sigma(x) \quad \text{and} \quad p_\sigma^2(x) = x,$$

for every $x \in \operatorname{Alt}(A, \sigma)$. We also use the following notation:

$$\operatorname{Alt}(A, \sigma)^+ := \{x + p_\sigma(x) \mid x \in \operatorname{Alt}(A, \sigma)\},$$

$$\operatorname{Alt}(A, \sigma)^- := \{x - p_\sigma(x) \mid x \in \operatorname{Alt}(A, \sigma)\}.$$

Note that if $\operatorname{char} F = 2$, then $\operatorname{Alt}(A, \sigma)^+ = \operatorname{Alt}(A, \sigma)^-$. Also, as proved in [11, p. 597] and [9, (3.5)], $\operatorname{Alt}(A, \sigma)^+$ and $\operatorname{Alt}(A, \sigma)^-$ are 3-dimensional subspaces of $\operatorname{Alt}(A, \sigma)$. Since $p_\sigma^2 = \operatorname{id}$, we have $p_\sigma(x) = x$ for every $x \in \operatorname{Alt}(A, \sigma)^+$ and $p_\sigma(x) = -x$ for every $x \in \operatorname{Alt}(A, \sigma)^-$. The converse is also true, i.e.

$$\operatorname{Alt}(A, \sigma)^+ = \{x \in \operatorname{Alt}(A, \sigma) \mid p_\sigma(x) = x\}, \tag{1}$$

$$\operatorname{Alt}(A, \sigma)^- = \{x \in \operatorname{Alt}(A, \sigma) \mid p_\sigma(x) = -x\}. \tag{2}$$

Indeed, if $\operatorname{char} F \neq 2$, then for every $x \in \operatorname{Alt}(A,\sigma)$ with $p_\sigma(x) = x$ we have $x = \frac{1}{2}(x + p_\sigma(x)) \in \operatorname{Alt}(A,\sigma)^+$. Similarly, if $p_\sigma(x) = -x$, then $x = \frac{1}{2}(x - p_\sigma(x)) \in \operatorname{Alt}(A,\sigma)^-$. If $\operatorname{char} F = 2$, then the relation (1) follows from the dimension formula for the image and the kernel of the linear map $p_\sigma + \mathrm{id}$.

The next result is implicitly contained in [8, pp. 249–250].

Lemma 3.5 *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Then p_σ is an isometry of $(\operatorname{Alt}(A,\sigma), q_\sigma)$. Furthermore, $\mathfrak{b}_{q_\sigma}(x,y) = xp_\sigma(y) + yp_\sigma(x)$, for $x, y \in \operatorname{Alt}(A,\sigma)$.*

Proof For every $x \in \operatorname{Alt}(A,\sigma)$ we have $q_\sigma(p_\sigma(x)) = p_\sigma(p_\sigma(x))p_\sigma(x) = xp_\sigma(x) = q_\sigma(x)$. Thus, p_σ is an isometry. The second assertion is easily obtained from the relations $q_\sigma(x) = xp_\sigma(x)$ and $\mathfrak{b}_{q_\sigma}(x,y) = q_\sigma(x+y) - q_\sigma(x) - q_\sigma(y)$. □

Lemma 3.6 *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Then $\operatorname{Alt}(A,\sigma)^+ = (\operatorname{Alt}(A,\sigma)^-)^\perp \subseteq C_A(\operatorname{Alt}(A,\sigma)^-)$.*

Proof Let $\mathfrak{b} = \mathfrak{b}_{q_\sigma}$ and let $x \in \operatorname{Alt}(A,\sigma)^+$. By (3.5), p_σ is an isometry of $(\operatorname{Alt}(A,\sigma), q_\sigma)$, and hence $\mathfrak{b}(x,y) = \mathfrak{b}(p_\sigma(x), p_\sigma(y)) = \mathfrak{b}(x, p_\sigma(y))$ for every $y \in \operatorname{Alt}(A,\sigma)$. Thus, $\mathfrak{b}(x, y - p_\sigma(y)) = 0$, i.e. $\operatorname{Alt}(A,\sigma)^+ \subseteq (\operatorname{Alt}(A,\sigma)^-)^\perp$. By dimension count we obtain $\operatorname{Alt}(A,\sigma)^+ = (\operatorname{Alt}(A,\sigma)^-)^\perp$. Now let $z \in \operatorname{Alt}(A,\sigma)^-$. By (3.5) we have $0 = \mathfrak{b}(x,z) = -xz + zx$. Thus, $xz = zx$, which implies that $\operatorname{Alt}(A,\sigma)^+$ commutes with $\operatorname{Alt}(A,\sigma)^-$, i.e. $\operatorname{Alt}(A,\sigma)^+ \subseteq C_A(\operatorname{Alt}(A,\sigma)^-)$. □

Lemma 3.7 *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F and let $x \in \operatorname{Alt}(A,\sigma)$. If $x^2 \in F$, then $p_\sigma(x) = \pm x$.*

Proof Set $\alpha = x^2 \in F$ and $\beta = q_\sigma(x) \in F$. Then $\beta^2 = q_\sigma(x)^2 = \operatorname{Nrd}_A(x) = \pm\alpha^2$. Thus, $\beta = \lambda\alpha$ for some $\lambda \in F$ with $\lambda^4 = 1$, i.e. $q_\sigma(x) = \lambda x^2$. If $\alpha \neq 0$, then multiplying $xp_\sigma(x) = q_\sigma(x) = \lambda x^2$ on the left by x^{-1} we obtain $p_\sigma(x) = \lambda x$. The relation $p_\sigma^2 = \mathrm{id}$ then implies that $\lambda = \pm 1$ and we are done. Suppose that $\alpha = 0$, i.e. $x^2 = 0$. By (3.5) we have $\mathfrak{b}_{q_\sigma}(p_\sigma(x), x) = p_\sigma(x)^2 + x^2 = p_\sigma(x)^2$, and hence $p_\sigma(x)^2 \in F$. On the other hand, the relations $xp_\sigma(x) = q_\sigma(x) = \lambda x^2 = 0$ show that $p_\sigma(x)$ is not invertible. Thus,

$$p_\sigma(x)^2 = 0. \tag{3}$$

Suppose that $p_\sigma(x) \neq x$; hence, $x \notin \operatorname{Alt}(A,\sigma)^+$. In view of (3.6) one can find $w \in \operatorname{Alt}(A,\sigma)^-$ such that $\mathfrak{b}_{q_\sigma}(x,w) = 1$. By (3.5) we have

$$-xw + wp_\sigma(x) = 1. \tag{4}$$

Multiplying (4) on the left by x we get $xwp_\sigma(x) = x$. Using (4), it follows that $(wp_\sigma(x) - 1)p_\sigma(x) = x$, which yields $p_\sigma(x) = -x$ by (3). This completes the proof (note that if $\operatorname{char} F = 2$, this argument shows that the assumption $p_\sigma(x) \neq x$ leads to the contradiction $p_\sigma(x) = -x$ and hence $p_\sigma(x) = x$). □

The next result follows from (3.7) and the relations (1) and (2) below (3.4).

Proposition 3.8 *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F and let $\operatorname{Alt}(A,\sigma)^0 = \operatorname{Alt}(A,\sigma)^+ \cup \operatorname{Alt}(A,\sigma)^-$. Then $\operatorname{Alt}(A,\sigma)^0 = \{x \in \operatorname{Alt}(A,\sigma) \mid p_\sigma(x) = \pm x\} = \{x \in \operatorname{Alt}(A,\sigma) \mid x^2 \in F\}$.*

Notation 3.9 *For a decomposable biquaternion algebra with involution of orthogonal type* (A,σ) *over a field* F*, we use the notation* $Q(A,\sigma)^+ = F + \mathrm{Alt}(A,\sigma)^+$ *and* $Q(A,\sigma)^- = F + \mathrm{Alt}(A,\sigma)^-$*. We will simply denote* $Q(A,\sigma)^+$ *by* Q^+ *and* $Q(A,\sigma)^-$ *by* Q^-*, if the pair* (A,σ) *is clear from the context.*

Lemma 3.10 ([9]) *Let* (A,σ) *be a decomposable biquaternion algebra with orthogonal involution over a field* F*.*

(1) *If* $\mathrm{char}\, F \neq 2$*, then* Q^+ *and* Q^- *are two* σ*-invariant quaternion subalgebras of* A *with* $Q_0^+ = \mathrm{Alt}(A,\sigma)^+$ *and* $Q_0^- = \mathrm{Alt}(A,\sigma)^-$*. Furthermore, we have* $(A,\sigma) \simeq (Q^+,\sigma|_{Q^+}) \otimes (Q^-,\sigma|_{Q^-})$*, where* $\sigma|_{Q^+}$ *and* $\sigma|_{Q^-}$ *are the canonical involutions of* Q^+ *and* Q^-*, respectively.*

(2) *If* $\mathrm{char}\, F = 2$*, then* $Q^+ = Q^-$ *is a maximal commutative subalgebra of* F *satisfying* $x^2 \in F$ *for every* $x \in Q^+$*.*

Proof Assume first that $\mathrm{char}\, F \neq 2$. As observed in [9, (3.5)], Q^+ is a σ-invariant quaternion subalgebra of A and $\sigma|_{Q^+}$ is of symplectic type. By dimension count and (3.6) we obtain $Q^- = C_A(Q^+)$; hence, $A \simeq Q^+ \otimes_F Q^-$. By [8, (2.23 (1))], $\sigma|_{Q^-}$ is of symplectic type. Finally, since $\mathrm{Trd}_{Q^+}(x) = 0$ for every $x \in \mathrm{Alt}(A,\sigma)^+$, we have $Q_0^+ = \mathrm{Alt}(A,\sigma)^+$. Similarly $Q_0^- = \mathrm{Alt}(A,\sigma)^-$. This proves the first part. The second part follows from [9, (3.6)]. □

Notation 3.11 *Let* (A,σ) *be a decomposable biquaternion algebra with orthogonal involution over a field* F*. We denote by* q_σ^+ *and* q_σ^- *the restrictions of* q_σ *to* $\mathrm{Alt}(A,\sigma)^+$ *and* $\mathrm{Alt}(A,\sigma)^-$*, respectively.*

Lemma 3.12 *Let* (A,σ) *be a decomposable biquaternion algebra with orthogonal involution over a field* F*.*

(1) *Every unit* $u \in \mathrm{Alt}(A,\sigma)^+$ *(resp.* $u \in \mathrm{Alt}(A,\sigma)^-$*) can be extended to a basis* (u,v,w) *of* $\mathrm{Alt}(A,\sigma)^+$ *(resp.* $\mathrm{Alt}(A,\sigma)^-$*) such that* $w = uv$*.*

(2) *Every basis* (u,v,w) *of* $\mathrm{Alt}(A,\sigma)^+$ *(resp.* $\mathrm{Alt}(A,\sigma)^-$*) with* $w = uv$ *is orthogonal with respect to the polar form of* q_σ^+ *(resp.* q_σ^-*).*

(3) *If* $\mathrm{char}\, F \neq 2$*, then* $N_{Q^+} \simeq \langle 1\rangle_q \perp (-1)\cdot q_\sigma^+$ *and* $N_{Q^-} \simeq \langle 1\rangle_q \perp q_\sigma^-$*.*

(4) *If* $\mathrm{char}\, F = 2$ *and* $(A,\sigma) \simeq (Q_1,\sigma_1) \otimes (Q_2,\sigma_2)$ *is a decomposition of* (A,σ)*, then* $q_\sigma^+ \simeq \langle \alpha,\beta,\alpha\beta\rangle_q$*, where* $\alpha \in F^\times$ *and* $\beta \in F^\times$ *are representatives of the classes* $\mathrm{disc}\,\sigma_1 \in F^\times/F^{\times 2}$ *and* $\mathrm{disc}\,\sigma_2 \in F^\times/F^{\times 2}$*, respectively.*

Proof We just prove the result for q_σ^+. The proof for q_σ^- is similar.

(1) Choose an element $u' \in \mathrm{Alt}(A,\sigma)^+ \backslash Fu$ and set $\alpha = u^2 \in F^\times$. By (3.10), $uu' \in Q^+ = F + \mathrm{Alt}(A,\sigma)^+$. Thus, there exist $\lambda \in F$ and $w \in \mathrm{Alt}(A,\sigma)^+$ such that $uu' = \lambda + w$. Set $v = u' - \lambda\alpha^{-1}u \in \mathrm{Alt}(A,\sigma)^+$. Then $uv = w \in \mathrm{Alt}(A,\sigma)^+$. Thus, (u,v,w) is the desired basis.

(2) Let $\mathcal{B} = (u,v,w)$ be a basis of $\mathrm{Alt}(A,\sigma)^+$ with $w = uv$. Then $vu = \sigma(uv) = -uv$. Using (3.5) we obtain $\mathfrak{b}(u,v) = uv + vu = 0$, where $\mathfrak{b}$ is the polar form of q_σ^+. Similarly, $\mathfrak{b}(u,w) = \mathfrak{b}(v,w) = 0$.

(3) Let (u, v, w) be a basis of $\mathrm{Alt}(A, \sigma)^+$ with $w = uv$. By (2), $q_\sigma^+ \simeq \langle \alpha, \beta, -\alpha\beta \rangle_q$, where $\alpha = u^2 \in F$ and $\beta = v^2 \in F$. Since $vu = -uv$, $(1, u, v, w)$ is a quaternion basis of Q^+. Thus, $N_{Q^+} \simeq \langle 1, -\alpha, -\beta, \alpha\beta \rangle_q$ by [5, (9.6)].

(4) Let $u \in \mathrm{Alt}(Q_1, \sigma_1)$ and $v \in \mathrm{Alt}(Q_2, \sigma_2)$ be two units and set $\alpha = u^2 \in F^\times$, $\beta = v^2 \in F^\times$, and $w = uv$. By (3.8) we have $u, v \in \mathrm{Alt}(A, \sigma)^+$. Also, $\operatorname{disc} \sigma_1 = \alpha F^{\times 2} \in F^\times / F^{\times 2}$ and $\operatorname{disc} \sigma_2 = \beta F^{\times 2} \in F^\times / F^{\times 2}$. Since $w \in \mathrm{Alt}(A, \sigma)$ and $w^2 \in F$, by (3.8) we obtain $w \in \mathrm{Alt}(A, \sigma)^+$, and so (u, v, w) is a basis of $\mathrm{Alt}(A, \sigma)^+$ and $q_\sigma^+ \simeq \langle \alpha, \beta, \alpha\beta \rangle_q$. □

Proposition 3.13 (Compare [11, (5.3)]) *Let (A, σ) and (A', σ') be decomposable biquaternion algebras with orthogonal involution over a field F. If $(A, \sigma) \simeq (A', \sigma')$, then either $q_\sigma \simeq q_{\sigma'}$ and $q_\sigma^+ \simeq q_{\sigma'}^+$ or $q_\sigma \simeq (-1) \cdot q_{\sigma'}$ and $q_\sigma^+ \simeq q_{\sigma'}^-$.*

Proof Let $\varphi : (A, \sigma) \xrightarrow{\sim} (A', \sigma')$ be an isomorphism of F-algebras with involution. Then $\varphi(\mathrm{Alt}(A, \sigma)) = \mathrm{Alt}(A', \sigma')$ and

$$q_{\sigma'}(\varphi(x))^2 = \mathrm{Nrd}_{A'}(\varphi(x)) = \mathrm{Nrd}_A(x) = q_\sigma(x)^2, \quad \text{for } x \in \mathrm{Alt}(A, \sigma).$$

Thus, $q'_\sigma \circ \varphi = \pm q_\sigma$. Suppose first that $q'_\sigma \circ \varphi = q_\sigma$. Then φ restricts to an isometry $f : (\mathrm{Alt}(A, \sigma), q_\sigma) \to (\mathrm{Alt}(A', \sigma'), q_{\sigma'})$. Set $h = f \circ p_\sigma \circ f^{-1}$. Then h is an endomorphism of $\mathrm{Alt}(A', \sigma')$. We claim that $h = p_{\sigma'}$. For every $x \in \mathrm{Alt}(A', \sigma')$ we have $h^2(x) = f \circ p_\sigma^2 \circ f^{-1}(x) = f \circ \mathrm{id} \circ f^{-1}(x) = x$ and

$$\begin{aligned} xh(x) &= xf(p_\sigma(f^{-1}(x))) = x\varphi(p_\sigma(f^{-1}(x))) = \varphi(f^{-1}(x))\varphi(p_\sigma(f^{-1}(x))) \\ &= \varphi(f^{-1}(x)p_\sigma(f^{-1}(x))) = \varphi(q_\sigma(f^{-1}(x))) = \varphi(q_{\sigma'}(x)) = q_{\sigma'}(x). \end{aligned}$$

Similarly, we have $h(x)x = q_{\sigma'}(x)$ for every $x \in \mathrm{Alt}(A', \sigma')$. Thus, $h = p_{\sigma'}$ and the claim is proved. It follows that $p_{\sigma'} \circ f = f \circ p_\sigma$. Now, if $x \in \mathrm{Alt}(A, \sigma)^+$, then $p_\sigma(x) = x$, which yields $p_{\sigma'}(f(x)) = f(p_\sigma(x)) = f(x)$. It follows that $f(x) \in \mathrm{Alt}(A', \sigma')^+$, i.e. f restricts to an isometry $q_\sigma^+ \simeq q_{\sigma'}^+$. A similar argument shows that if $q'_\sigma \circ \varphi = -q_\sigma$, then $q_\sigma^+ \simeq q_{\sigma'}^-$. □

The next result complements [11, (5.3)] for biquaternion algebras.

Theorem 3.14 *Let (A, σ) and (A', σ') be two decomposable biquaternion algebras with orthogonal involution over a field F of characteristic different from 2. Let $Q^+ = Q(A, \sigma)^+$, $Q^- = Q(A, \sigma)^-$, $Q'^+ = Q(A', \sigma')^+$, and $Q'^- = Q(A', \sigma')^-$. The following statements are equivalent.*

(1) $(A, \sigma) \simeq (A', \sigma')$.

(2) *Either $q_\sigma \simeq q_{\sigma'}$ and $q_\sigma^+ \simeq q_{\sigma'}^+$ or $q_\sigma \simeq (-1) \cdot q_{\sigma'}$ and $q_\sigma^+ \simeq q_{\sigma'}^-$.*

(3) *$A \simeq A'$ and either $q_\sigma^+ \simeq q_{\sigma'}^+$ or $q_\sigma^+ \simeq q_{\sigma'}^-$.*

(4) *$A \simeq A'$ and either $Q^+ \simeq Q'^+$ or $Q^+ \simeq Q^-$.*

Proof The implication (1) ⇒ (2) follows from (3.13). Since q_σ and $q_{\sigma'}$ are Albert forms of (A, σ) and (A', σ'), respectively, the condition $q_\sigma \simeq q_{\sigma'}$ (resp. $q_\sigma \simeq (-1) \cdot q_{\sigma'}$) implies that $A \simeq A'$, proving (2) ⇒ (3).

The implication (3) $\Rightarrow$ (4) follows from (3.12 (3)) and [12, Ch. III, (2.5)]. To prove (4) $\Rightarrow$ (1) assume first $Q^+ \simeq Q'^+$. By (3.10 (1)) we have $C_A(Q^+) = Q^-$ and $C_{A'}(Q'^+) = Q'^-$. Thus, the isomorphisms $Q^+ \simeq_F Q'^+$ and $A \simeq_F A'$ imply that $Q^- \simeq_F Q'^-$. Since the restrictions of σ to Q^+ and Q^- and the restrictions of σ' to Q'^+ and Q'^- are all symplectic, we obtain

$$\begin{aligned}(A,\sigma) &\simeq_F (Q^+,\sigma|_{Q^+}) \otimes_F (Q^-,\sigma|_{Q^-})\\ &\simeq_F (Q'^+,\sigma'|_{Q'^+}) \otimes_F (Q'^-,\sigma'|_{Q'^-}) \simeq_F (A',\sigma').\end{aligned}$$

A similar argument works if $Q^+ \simeq Q'^-$. □

4. Relation with the Pfister invariant in characteristic two

Throughout this section, F is a field of characteristic 2.

Definition 4.1 *Let A be a finite-dimensional associative F-algebra. The minimum number r such that A can be generated as an F-algebra by r elements is called the minimum rank of A and is denoted by $r_F(A)$.*

Theorem 4.2 ([13]) *Let (A,σ) be a totally decomposable algebra with involution of orthogonal type over F. There exists a symmetric and self-centralizing subalgebra $S \subseteq A$ such that $x^2 \in F$ for every $x \in S$ and $\dim_F S = 2^n$, where $n = r_F(S)$. Furthermore, for every subalgebra S with these properties, we have $S = F + S_0$, where $S_0 = S \cap \mathrm{Alt}(A,\sigma)$. In particular, $S \subseteq F + \mathrm{Alt}(A,\sigma)$. Finally, the subalgebra S is uniquely determined up to isomorphism.*

Proof See [13, (4.6) and (5.10)]. □

Notation 4.3 *We denote the algebra S in (4.2) by $\Phi(A,\sigma)$.*

The next result shows that for biquaternion algebras with orthogonal involution, the subalgebra $\Phi(A,\sigma)$ is unique as a set.

Corollary 4.4 *Let (A,σ) be a decomposable biquaternion algebra with involution of orthogonal type over F. Then $\Phi(A,\sigma) = Q^+$.*

Proof Write $\Phi(A,\sigma) = F + S_0$, where $S_0 = \Phi(A,\sigma) \cap \mathrm{Alt}(A,\sigma)$. Since every element of $\Phi(A,\sigma)$ is square-central, using (3.8) we have $S_0 \subseteq \mathrm{Alt}(A,\sigma)^+$. Then $S_0 = \mathrm{Alt}(A,\sigma)^+$ by dimension count, and hence $\Phi(A,\sigma) = F + \mathrm{Alt}(A,\sigma)^+ = Q^+$. □

Lemma 4.5 *Let (A,σ) be a totally decomposable algebra of degree 2^n with orthogonal involution over F. If there exists a set $\{u_1,\cdots,u_n\} \subseteq \mathrm{Alt}(A,\sigma)$ consisting of pairwise commutative square-central units such that $u_{i_1}\cdots u_{i_l} \in \mathrm{Alt}(A,\sigma)$ for every $1 \le l \le n$ and $1 \le i_1 < \cdots < i_l \le n$, then $\Phi(A,\sigma) \simeq F[u_1,\cdots,u_n]$.*

Proof By [7, (2.2.3)], $S := F[u_1,\cdots,u_n]$ is self-centralizing. The other required properties of S, stated in (4.2), are easily verified. □

Definition 4.6 *A set* $\{u_1, \cdots, u_n\} \subseteq \mathrm{Alt}(A,\sigma)$ *as in (4.5) is called a set of alternating generators of* $\Phi(A,\sigma)$*.*

We recall the following definition from [4].

Definition 4.7 *Let* $(A,\sigma) = (Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n)$ *be a totally decomposable algebra with orthogonal involution over* F*. Let* $\alpha_i \in F^\times$*,* $i = 1,\cdots,n$*, be a representative of the class* $\mathrm{disc}\,\sigma_i \in F^\times/F^{\times 2}$*. The bilinear* n*-fold Pfister form* $\langle\langle\alpha_1,\cdots,\alpha_n\rangle\rangle$ *is called the Pfister invariant of* (A,σ) *and is denoted by* $\mathfrak{Pf}(A,\sigma)$*.*

Note that by [4, (7.5)], $\mathfrak{Pf}(A,\sigma)$ is independent of the decomposition of (A,σ). Also, as observed in [13, pp. 223–224], $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha_1,\cdots,\alpha_n\rangle\rangle$ if and only if there exists a set of alternating generators $\{u_1,\cdots,u_n\}$ of $\Phi(A,\sigma)$ such that $u_i^2=\alpha_i\in F^\times$, $i=1,\cdots,n$.

Lemma 4.8 *Let* $\langle\langle\alpha,\beta\rangle\rangle$ *be an isotropic bilinear Pfister form over* F*. If* $\alpha\beta\neq 0$*, then* $\langle\langle\alpha,\beta\rangle\rangle\simeq\langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle$ *for every* $\lambda\in F$*.*

Proof Since $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic, by [5, (4.14)] either $\alpha\in F^{\times 2}$ or $\beta\in D_F\langle 1,\alpha\rangle$. If $\alpha\in F^{\times 2}$, using [5, (4.15 (2))] and [5, (4.15 (1))] we obtain

$$\begin{aligned}\langle\langle\alpha,\beta\rangle\rangle &\simeq \langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta\rangle\rangle \simeq \langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta(\alpha^{-1}\lambda^2-(\beta+\alpha^{-1}\lambda^2))\rangle\rangle\\ &\simeq \langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta^2\rangle\rangle \simeq \langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle.\end{aligned}$$

If $\beta\in D_F\langle 1,\alpha\rangle$, then there exist $b,c\in F$ such that $\beta=b^2+c^2\alpha$. Let $s=\alpha^{-1}\beta^{-1}\lambda\in F$. Using [5, (4.15 (1))] we obtain

$$\begin{aligned}\langle\langle\alpha,\beta\rangle\rangle &\simeq \langle\langle\alpha,\beta((1+cs\alpha)^2-(bs)^2\alpha)\rangle\rangle \simeq \langle\langle\alpha,\beta(1+c^2s^2\alpha^2+b^2s^2\alpha)\rangle\rangle\\ &\simeq \langle\langle\alpha,\beta+s^2\alpha\beta(c^2\alpha+b^2)\rangle\rangle \simeq \langle\langle\alpha,\beta+s^2\alpha\beta^2\rangle\rangle \simeq \langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle.\end{aligned}$$

□

Lemma 4.9 *Let* (A,σ) *be a decomposable biquaternion algebra with involution of orthogonal type over* F *and let* $\alpha,\beta\in F^\times$*. Then* $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$ *if and only if* $q_\sigma^+\simeq\langle\alpha,\beta,\alpha\beta\rangle_q$*.*

Proof If $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$, then there exists a set of alternating generators $\{u,v\}$ of $\Phi(A,\sigma)$ such that $u^2=\alpha$ and $v^2=\beta$. By (4.4) and (3.12 (2)), (u,v,uv) is an orthogonal basis of $\mathrm{Alt}(A,\sigma)^+$ and hence $q_\sigma^+\simeq\langle\alpha,\beta,\alpha\beta\rangle_q$.

To prove the converse, choose a basis (x,y,z) of $\mathrm{Alt}(A,\sigma)^+$ with $x^2=\alpha$, $y^2=\beta$, and $z^2=\alpha\beta$. Consider the element $xy\in\Phi(A,\sigma)$. By (4.4), $\Phi(A,\sigma)=F+\mathrm{Alt}(A,\sigma)^+$. Thus, there exist $a,b,c,d\in F$ such that

$$xy=a+bx+cy+dz. \tag{5}$$

If $a=0$ then $xy=bx+cy+dz\in\mathrm{Alt}(A,\sigma)^+$, which implies that $\{x,y\}$ is a set of alternating generators of $\Phi(A,\sigma)$. As $x^2=\alpha$ and $y^2=\beta$ we obtain $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$. Suppose that $a\neq 0$. By squaring both sides of (5), we obtain $\alpha\beta=a^2+b^2\alpha+c^2\beta+d^2\alpha\beta$, which yields

$$1+(ba^{-1})^2\alpha+(ca^{-1})^2\beta+((d+1)a^{-1})^2\alpha\beta=0.$$

Therefore, the form $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic. Set $y' = y + \alpha^{-1}ax \in \mathrm{Alt}(A,\sigma)^+$. By (5) we have $xy' = xy + a = bx + cy + dz \in \mathrm{Alt}(A,\sigma)^+$; hence, $\{x, y'\}$ is a set of alternating generators of $\Phi(A,\sigma)$. As $x^2 = \alpha$ and $y'^2 = \beta + \alpha^{-1}a^2$, we obtain $\mathfrak{Pf}(A,\sigma) \simeq \langle\langle\alpha, \beta + \alpha^{-1}a^2\rangle\rangle$. Thus, $\mathfrak{Pf}(A,\sigma) \simeq \langle\langle\alpha,\beta\rangle\rangle$ by (4.8). □

Using (4.9) and (3.12 (4)), we obtain the following relation between the Pfister invariant and the quadratic form q_σ^+.

Proposition 4.10 *Let (A,σ) and (A',σ') be decomposable biquaternion algebras with orthogonal involution over F. Then $q_\sigma^+ \simeq q_{\sigma'}^+$ if and only if $\mathfrak{Pf}(A,\sigma) \simeq \mathfrak{Pf}(A',\sigma')$.*

The following result is analogous to (3.14).

Theorem 4.11 *Let (A,σ) and (A',σ') be decomposable biquaternion algebras with orthogonal involution over F. Then the following statements are equivalent:*

(1) $(A,\sigma) \simeq (A',\sigma')$.

(2) $q_\sigma \simeq q_{\sigma'}$ *and* $q_\sigma^+ \simeq q_{\sigma'}^+$.

(3) $A \simeq A'$ *and* $q_\sigma^+ \simeq q_{\sigma'}^+$.

(4) $A \simeq A'$ *and* $\mathfrak{Pf}(A,\sigma) \simeq \mathfrak{Pf}(A',\sigma')$.

Proof The implications $(1) \Rightarrow (2)$ follow from (3.13).

$(2) \Rightarrow (3)$: Since q_σ and $q_{\sigma'}$ are Albert forms of (A,σ) and (A',σ'), respectively, $q_\sigma \simeq q_{\sigma'}$ implies that $A \simeq A'$ by [8, (16.3)].

The implication $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$ follows from (4.10) and [13, (6.5)], respectively. □

Lemma 4.12 *If $\langle\langle\alpha,\beta\rangle\rangle$ is an anisotropic bilinear Pfister form over F, then $\langle\langle\alpha,\beta\rangle\rangle \not\simeq \langle\langle\alpha+1,\beta\rangle\rangle$.*

Proof As proved in [1, p. 16], two bilinear Pfister forms are isometric if and only if their pure subforms are isometric. Thus, it is enough to show that the pure subform of $\langle\langle\alpha,\beta\rangle\rangle$ does not represent $\alpha+1$. If $\alpha + 1 \in D_F(\langle\alpha,\beta,\alpha\beta\rangle)$, then there exist $a,b,c \in F$ such that $a^2\alpha + b^2\beta + c^2\alpha\beta = \alpha + 1$. Thus, $1 + (a+1)^2\alpha + b^2\beta + c^2\alpha\beta = 0$, i.e. $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic, which contradicts the assumption. □

Definition 4.13 *For $\alpha \in F^\times$, define an involution $T_\alpha : M_2(F) \to M_2(F)$ via*

$$T_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c\alpha^{-1} \\ b\alpha & d \end{pmatrix}.$$

Note that T_α is of orthogonal type and $\operatorname{disc} T_\alpha = \alpha F^{\times 2} \in F^\times/F^{\times 2}$.

The following example shows that if $\operatorname{char} F = 2$, the conditions $A \simeq_F A'$ and $Q^+ \simeq_F Q'^+$ do not necessarily imply that $(A,\sigma) \simeq (A',\sigma')$ (compare (3.14)).

Example 4.14 *Let $\langle\langle \alpha, \beta \rangle\rangle$ be an anisotropic Pfister form over a field F of characteristic 2 and let $A = M_4(F)$. Consider the involutions $\sigma = T_\alpha \otimes T_\beta$ and $\sigma' = T_{\alpha+1} \otimes T_\beta$ on A. Then $\mathfrak{Pf}(A, \sigma) \simeq \langle\langle \alpha, \beta \rangle\rangle$ and $\mathfrak{Pf}(A, \sigma') \simeq \langle\langle \alpha + 1, \beta \rangle\rangle$, and hence $\mathfrak{Pf}(A, \sigma) \not\simeq \mathfrak{Pf}(A, \sigma')$ by (4.12). Using (4.11), we obtain $(A, \sigma) \not\simeq (A, \sigma')$.*

On the other hand, there exists a set of alternating generators $\{u, v\}$ (resp. $\{u', v'\}$) of $\Phi(A, \sigma)$ (resp. $\Phi(A, \sigma')$) such that $u^2 = \alpha$ and $v^2 = \beta$ (resp. ${u'}^2 = \alpha + 1$ and ${v'}^2 = \beta$). Then $\Phi(A, \sigma) \simeq F[u, v]$ and $\Phi(A, \sigma') \simeq F[u', v']$. The linear map $f : F[u, v] \to F[u', v']$ induced by $f(1) = 1$, $f(u) = u' + 1$, $f(v) = v'$, and $f(uv) = (u' + 1)v'$ is an F-algebra isomorphism. Thus, $\Phi(A, \sigma) \simeq \Phi(A, \sigma')$, which implies that $Q(A, \sigma)^+ \simeq Q(A, \sigma')^+$ by (4.4).

5. Metabolic involutions

Let (A, σ) be an algebra with involution over a field F of arbitrary characteristic. An idempotent $e \in A$ is called *hyperbolic* (resp. *metabolic*) with respect to σ if $\sigma(e) = 1 - e$ (resp. $\sigma(e)e = 0$ and $(1 - e)(1 - \sigma(e)) = 0$). The pair (A, σ) is called *hyperbolic* (resp. *metabolic*) if A contains a hyperbolic (resp. metabolic) idempotent with respect to σ. Every hyperbolic involution σ is metabolic but the converse is not always true. If σ is symplectic or $\operatorname{char} F \neq 2$, the involution σ is metabolic if and only if it is hyperbolic (see [3, (4.10)] and [2, (A.3)]).

Lemma 5.1 *Let (A, σ) be a central simple algebra with orthogonal involution over a field F. If $e \in A$ is a metabolic idempotent, then $(e - \sigma(e))^2 = 1$.*

Proof This follows from the relations $(1 - e)(1 - \sigma(e)) = 0$ and $\sigma(e)e = 0$. □

Theorem 5.2 *Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. The following statements are equivalent:*

(1) *(A, σ) is metabolic.*

(2) *Q^+ or Q^- splits.*

(3) *$1 \in D_F(q^+_\sigma)$ or $-1 \in D_F(q^-_\sigma)$.*

(4) *q^+_σ or q^-_σ is isotropic.*

Proof If $\operatorname{char} F \neq 2$, by (3.10 (1)) we have $(A, \sigma) \simeq (Q^+, \sigma|_{Q^+}) \otimes (Q^-, \sigma|_{Q^-})$, where $\sigma|_{Q^+}$ and $\sigma|_{Q^-}$ are the canonical involutions of Q^+ and Q^-, respectively. Thus, the equivalence $(1) \Leftrightarrow (2)$ follows from [6, (3.1)]. The equivalences $(2) \Leftrightarrow (3)$ and $(2) \Leftrightarrow (4)$ both follow from (3.12 (3)) and [12, Ch. III, (2.7)].

Now, let $\operatorname{char} F = 2$. Then the equivalence $(1) \Leftrightarrow (2)$ follows from [13, (6.6)].

$(1) \Rightarrow (3)$: Let e be a metabolic idempotent with respect to σ and let $x = e - \sigma(e)$. By (5.1), we have $x^2 = 1$. Since $x \in \operatorname{Alt}(A, \sigma)$, (3.8) implies that $x \in \operatorname{Alt}(A, \sigma)^+$ and hence $q^+_\sigma(x) = 1$.

$(3) \Rightarrow (4)$: Suppose that $q^+_\sigma(u) = 1$ for some $u \in \operatorname{Alt}(A, \sigma)^+$. By (3.12 (1)) and (3.12 (2)), the element u extends to an orthogonal basis (u, v, w) of $\operatorname{Alt}(A, \sigma)^+$ with $w = uv$. According to (3.10 (2)), Q^+ is commutative. Thus, $q^+_\sigma(v + w) = (v + w)^2 = v^2 + (uv)^2 = 0$, i.e. q^+_σ is isotropic.

(4) $\Rightarrow$ (2): If q_σ^+ is isotropic, then there exists a nonzero $x \in \mathrm{Alt}(A,\sigma)^+ \subseteq Q^+$ such that $x^2 = 0$ and hence Q^+ splits. □

Corollary 5.3 *Let (A,σ) be a central simple algebra with involution over a field F. If σ is metabolic, then* $\operatorname{disc}\sigma$ *is trivial.*

Proof The result follows from (5.1) if $\operatorname{char} F = 2$ and [2, (2.3)] if $\operatorname{char} F \neq 2$. □

Proposition 5.4 *Let (A,σ) be a biquaternion algebra with involution of orthogonal type over a field F. Then σ is metabolic if and only if there exists $u \in \mathrm{Alt}(A,\sigma)$ such that $u^2 = 1$.*

Proof If σ is metabolic, then by (5.3), $\operatorname{disc}\sigma$ is trivial. Thus, σ is decomposable and the result follows from (5.2). Conversely, suppose that there exists $u \in \mathrm{Alt}(A,\sigma)$ such that $u^2 = 1$. Then $\operatorname{disc}\sigma = \mathrm{Nrd}_A(u)F^{\times 2}$ is trivial, so (A,σ) is decomposable by [9, (3.7)]. Since $u^2 = 1 \in F$ and $u \in \mathrm{Alt}(A,\sigma)$, by (3.8) we have $u \in \mathrm{Alt}(A,\sigma)^+ \cup \mathrm{Alt}(A,\sigma)^-$. Therefore, either $u \in \mathrm{Alt}(A,\sigma)^+$ (i.e. $q_\sigma^+(u) = 1$) or $u \in \mathrm{Alt}(A,\sigma)^-$ (i.e. $q_\sigma^-(u) = -1$). By (5.2), σ is metabolic. □

Proposition 5.5 *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Then $(A,\sigma) \simeq (M_4(F),t)$ if and only if $q_\sigma^+ \simeq \langle -1,-1,-1\rangle_q$ and $q_\sigma^- \simeq \langle 1,1,1\rangle_q$.*

Proof If $\operatorname{char} F = 2$, the result follows from [13, (5.7)] and (4.9). Suppose that $\operatorname{char} F \neq 2$. As observed in [7, p. 235], $Q(M_4(F),t)^+$ has an F-basis $(1,u,v,w)$ subject to the relations $u^2 = -1$, $v^2 = -1$ and $w = uv = -vu$. By (3.12 (2)) we obtain $q_t^+ \simeq \langle -1,-1,-1\rangle_q$. A similar argument shows that $q_t^- \simeq \langle 1,1,1\rangle_q$. Thus, the result follows from (3.14). □

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