

1-1-2018

## The local and semilocal convergence analysis of new Newton-like iteration methods

VATAN KARAKAYA

KADRI DOĞAN

YUNUS ATALAN

NOUR EL HOUDA BOUZARA

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

KARAKAYA, VATAN; DOĞAN, KADRI; ATALAN, YUNUS; and BOUZARA, NOUR EL HOUDA (2018) "The local and semilocal convergence analysis of new Newton-like iteration methods," *Turkish Journal of Mathematics*: Vol. 42: No. 3, Article 1. <https://doi.org/10.3906/mat-1505-16>  
Available at: <https://journals.tubitak.gov.tr/math/vol42/iss3/1>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact [academic.publications@tubitak.gov.tr](mailto:academic.publications@tubitak.gov.tr).

## The local and semilocal convergence analysis of new Newton-like iteration methods

Vatan KARAKAYA<sup>1</sup>, Kadri DOĞAN<sup>2,\*</sup>, Yunus ATALAN<sup>3</sup>  
Nour El Houda BOUZARA<sup>4</sup>

<sup>1</sup>Department of Mathematical Engineering, Yıldız Technical University, Davutpaşa Campus, Esenler, Istanbul, Turkey

<sup>2</sup>Department of Computer Engineering, Artvin Çoruh University, City Campus, Artvin, Turkey

<sup>3</sup>Department of Mathematics, Aksaray University, Aksaray, Turkey

<sup>4</sup>Mathematical Faculty, University of Science and Technology, Houari Boumedine, Bab-Ezzouar, Algiers, Algeria

Received: 06.05.2015

Accepted/Published Online: 22.05.2017

Final Version: 08.05.2018

**Abstract:** The aim of this paper is to find new iterative Newton-like schemes inspired by the modified Newton iterative algorithm and prove that these iterations are faster than the existing ones in the literature. We further investigate their behavior and finally illustrate the results by numerical examples.

**Key words:** Modified Newton algorithm, Picard S hybrid iteration, semilocal convergence, local convergence

### 1. Introduction

Throughout this paper, the following notations will be used:

- $C$  is an open convex subset of a Banach space  $X$ ,
- $F$  is a Fréchet differentiable operator at each point of  $C$  with values in a Banach space  $Y$ ,
- $B_r[x] = \{y \in X : \|y - x\| \leq r\}$ , for any  $x \in X$  and  $r > 0$ ,
- $B_r(x) = \{y \in X : \|y - x\| < r\}$ , for any  $x \in X$  and  $r > 0$ ,
- $B(Y, X)$  is the space of all bounded linear operators from  $Y$  to  $X$ ,
- $\mathbb{N}$  denotes all positive integers including zero.

Many problems that arise in engineering and scientific disciplines can be modeled by the following nonlinear operator equation:

$$F(x) = 0. \quad (1)$$

Several problems about studying the solvability of (1) are brought forward (see [3, 13]). To solve this equation, the iterative approximation method is considered as one of the main tools in fixed point theory. Therefore, many iterative methods have been defined and studied by numerous mathematicians (see [7, 10, 11, 14]).

\*Correspondence: dogankadri@hotmail.com

2010 AMS Mathematics Subject Classification: 47H10, 47H09, 49M15

In 2011 Sahu et al. [17] introduced the normal-S iteration process for finding solutions of constrained minimization problems and split feasibility problems as follows:

Let  $E$  be a normed space,  $B$  be a nonempty, convex subset of  $E$  and  $A : B \rightarrow B$  be an operator. Then, for an arbitrary  $x_0 \in B$ ,

$$x_{n+1} = A((1 - \alpha_n)x_n + \alpha_n Ax_n) \quad n \in \mathbb{N}, \tag{2}$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$ .

Gürsoy [9] introduced Picard-S iterative process as follows:

Let  $B$  be a closed convex subset of a Banach space  $X$  and  $T : B \rightarrow B$  be an operator. Then, for an arbitrary  $x_0 \in B$

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1 - \alpha_n)Tx_n + \alpha_n Tz_n \\ z_n = (1 - \beta_n)x_n + \beta_n Tx_n \quad (n \in \mathbb{N}), \end{cases} \tag{3}$$

where  $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty \in [0,1]$ .

For solving nonlinear operator equation (1), many authors present several generalizations of the Newton method (see [3] and [21]). The Newton method is given as follows:

$$\begin{cases} x_0 \in C \\ x_{n+1} = x_n - F'_{x_n}{}^{-1}F(x_n), \quad \forall n \in \mathbb{N} \end{cases} \tag{4}$$

where  $F'_x$  denotes the Fréchet derivative of  $F$  at the point  $x \in C$ .

In the Newton method (4), the functional value of the inverse of the derivative is required at each step. A natural question is how to modify the Newton iteration process (4) so that the computation of the inverse of the derivative at each step in the Newton method (4) can be avoided. Argyros [1], Bartle [4], Dennis [6], and Rheinboldt [16] discussed the following modified Newton method,

$$x_{n+1} = x_n - F'_{x_0}{}^{-1}F(x_n), \quad n \in \mathbb{N}. \tag{5}$$

Let  $x^* \in C$  be a solution of (1) such that  $F'_{x^*}{}^{-1} \in B(Y, X)$ . For some  $x_0 \in C$ , assume that  $F'_{x^*}{}^{-1}$  and  $F$  satisfy the following:

$$\|F'_x - F'_{x_0}\| \leq K_0 \|x - x_0\|, \quad \forall x \in C \text{ and for some } K_0 > 0, \tag{6}$$

$$\|F'_{x^*}{}^{-1}(F'_x - F'_{x_0})\| \leq K_1 \|x - x_0\|, \quad \forall x \in C \text{ and for some } K_1 > 0, \tag{7}$$

and

$$\|F'_{x^*}{}^{-1}(F'_x - F'_{x^*})\| \leq K_2 \|x - x^*\|, \quad \forall x \in C \text{ and for some } K_2 > 0. \tag{8}$$

In [2] and [15], the authors proved theorems for semilocal and local convergence analysis of (5) to solve the operator equation (4).

Recently, Sahu et al. [18] introduced the following Newton-like S-iteration processes (SIP) for solving equation (1):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F'_{x_0^{-1}}F(y_n) \\ y_n = (1 - \alpha)x_n + \alpha z_n \\ z_n = x_n - F'_{x_0^{-1}}F(x_n) \quad n \in \mathbb{N}, \end{cases} \tag{9}$$

and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F'_{y_0^{-1}}F(y_n) \\ y_n = (1 - \alpha)x_n + \alpha z_n \\ z_n = x_n - F'_{x_0^{-1}}F(x_n) \quad n \in \mathbb{N}, \end{cases} \tag{10}$$

where  $\alpha \in (0, 1)$ .

In the following theorems, they also proved semilocal as well as local convergence analysis of (9) and (10) and obtained that these iterative algorithms are faster than (5).

**Theorem 1** [18] *Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $C$  of a Banach space  $X$  with values in a Banach space  $Y$ . For some  $x_0 \in C$ , let  $F'_{x_0^{-1}} \in B(Y, X)$  and the operator  $F$  satisfy (6) and the following conditions:*

- i)  $\|F'_{x_0^{-1}}F(x_0)\| \leq \eta$ , for some  $\eta > 0$ ;
- ii)  $\|F'_{x_0^{-1}}\| \leq \beta$ , for some  $\beta > 0$ .

Assume that  $\alpha \in (0, 1)$ ,  $h = \eta\beta K_0 < \frac{1}{2}$  and  $B_r[x_0] \subseteq C$  such that  $r = \frac{1 - \sqrt{1 - 2h}}{h}\eta$ . Then, under the above restrictions, the following assertions are true:

- a) The operator  $A : B_r[x_0] \rightarrow X$ , defined by

$$Ax = x - F'_{x_0^{-1}}F(x), x \in B_r[x_0],$$

is a contraction self-operator on  $B_r[x_0]$  with Lipschitz constant  $\beta r K_0$  and the operator of Equation (1) has a unique solution in  $B_r[x_0]$ .

- b) The S-operator  $A_\alpha : B_r[x_0] \rightarrow X$  generated by  $\alpha$  and  $A$  is a contraction self-operator on  $B_r[x_0]$  with Lipschitz constant  $\beta r K_0(1 - \alpha + \alpha\beta r K_0)$ .

**Theorem 2** [18] *Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $C$  of a Banach space  $X$  with values in a Banach space  $Y$ . Suppose that  $\lambda \in (0, 1]$  and  $x^* \in C$  is a solution of (1) such that  $F'_{x^*} \in B(Y, X)$ . For some  $x_0 \in C$ , let  $F'_{x^*}$  and  $F$  satisfy the conditions (7) and (8). Assume that  $B_{r_1}(x^*) \subseteq C$ , where  $r_1 = \frac{2}{K_2}$ . For  $x_0 \in B_r(x^*)$  with  $r = \frac{2}{2K_2 + 3K_1}$ , let  $A_\lambda$  be an operator defined by*

$$A_\lambda(x) = x - \lambda F'_{x_0^{-1}}F(x), \forall x \in B_r(x^*).$$

Then we have the following:

i) For  $x \in B_r(x^*)$ , we have

$$\|A_\lambda(x) - x^*\| \leq (\lambda\delta_x + 1 - \lambda)\|x - x^*\|,$$

where  $\delta_x = \frac{K_1}{2(1-rK_2)} (\|x - x^*\| + 2\|x_0 - x^*\|)$ .

ii)  $A_\lambda$  is a quasi-contraction and self-operator on  $B_r(x^*)$  with constant  $1 - (1 - \delta)\lambda$ , where  $\delta = \sup_{x \in B_r(x^*)} \{\delta_x\}$ .

Motivated by the above studies, we introduce new Newton-like iteration processes as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F'_{x_0^{-1}}F(y_n) \\ y_n = (1 - \alpha)[x_n - F'_{x_0^{-1}}F(x_n)] + \alpha[z_n - F'_{x_0^{-1}}F(z_n)] \\ z_n = (1 - \theta)x_n + \theta[x_n - F'_{x_0^{-1}}F(x_n)] \quad n \in \mathbb{N}, \end{cases} \tag{11}$$

and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F'_{y_0^{-1}}F(y_n) \\ y_n = (1 - \alpha)[x_n - F'_{x_0^{-1}}F(x_n)] + \alpha[z_n - F'_{z_0^{-1}}F(z_n)] \\ z_n = (1 - \theta)x_n + \theta[x_n - F'_{x_0^{-1}}F(x_n)] \quad n \in \mathbb{N}, \end{cases} \tag{12}$$

where  $\alpha, \theta \in (0, 1)$ .

In the present study, we obtain semilocal and local convergence results of (11) and (12). Moreover, we compare the rates of convergence of the modified Newton method (5), the SIP of Newton-like iterative processes (9)–(10), and our Newton-like iterative processes (11)–(12).

## 2. Preliminaries

**Definition 1** Let  $C$  be a nonempty subset of normed space  $X$ . A mapping  $T : C \rightarrow X$  is said to be:

i) *Contraction* if there exists a constant  $\delta \in (0, 1)$  such that

$$\|Tx - Ty\| \leq \delta \|x - y\|, \quad \forall x, y \in C.$$

ii) *Quasi-contraction* [19] if there exists a constant  $\delta \in (0, 1)$  and  $F_T = \{x \in C : Tx = x\} \neq \emptyset$  such that

$$\|Tx - p\| \leq \delta \|x - p\|, \quad \forall x \in C \text{ and } p \in F_T.$$

**Definition 2** [17] Let  $C$  be a nonempty convex subset of a normed space  $X$  and  $T : C \rightarrow C$  an operator. The operator  $G : C \rightarrow C$  is said to be *S-operator* generated by  $\alpha \in (0, 1)$ ,  $T$ , and identity mapping  $I$  if

$$G = T[(1 - \alpha)I + \alpha T].$$

**Definition 3** [5] Let  $\{a_n\}_{n=0}^\infty$  and  $\{b_n\}_{n=0}^\infty$  be nonnegative real convergent sequences with limits  $a$  and  $b$ , respectively. Then  $\{a_n\}_{n=0}^\infty$  converges faster than  $\{b_n\}_{n=0}^\infty$  if

$$\lim_{n \rightarrow \infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.$$

**Lemma 1** [1] Let  $R$  be a bounded linear operator on a Banach space  $X$ . Then the following assumptions are equivalent:

i) There is a bounded linear operator  $S$  on  $X$  such that  $S^{-1}$  exists and

$$\|S - R\| = \frac{1}{\|S^{-1}\|}.$$

ii)  $R^{-1}$  exists.

Furthermore, if  $R^{-1}$  exists, then

$$\|R^{-1}\| \leq \frac{\|S^{-1}\|}{1 - \|1 - S^{-1}R\|} \leq \frac{\|S^{-1}\|}{1 - \|S^{-1}\| \|S - R\|}.$$

**Lemma 2** [19] Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Let  $x^* \in D$  be a solution of (1) such that  $F'_{x^*}{}^{-1} \in B(Y, X)$  and the operator  $F$  satisfies the conditions (8). Assume that  $B_r(x^*) \subseteq D$ , where  $r = \frac{1}{K_2}$ . Then, for any  $x \in B_r(x^*)$ ,  $F_x$  is invertible and the following estimate holds:

$$\left\| \left( F'_{x^*}{}^{-1} F'_x \right)^{-1} \right\| \leq \frac{1}{1 - K_2 \|x - x^*\|}.$$

**Lemma 3** [12] Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  a contraction mapping. Then  $T$  has a unique fixed point in  $X$ .

**Lemma 4** [20] Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Then, for all  $x, y \in D$ , we have

$$Fx - Fy = \int_0^1 F'_{y+t(x-y)}(x - y) dt.$$

### 3. Main results

#### 3.1. Semilocal convergence analysis

In this subsection, we give semilocal convergence analysis of algorithm (11).

**Theorem 3** Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . For some  $x_0 \in D$  let  $F'_{x_0}{}^{-1} \in B(Y, X)$ . Assume that  $F'_{x_0}{}^{-1}$  and  $F$  satisfy (6), (7), and (8) with the following conditions:

i)  $\|F'_{x_0}{}^{-1} F(x_0)\| \leq \eta$ , for some  $\eta > 0$ .

ii)  $\|F'_{x_0}{}^{-1}\| \leq \beta$ , for some  $\beta > 0$ .

Assume that  $\gamma, \alpha, \theta \in (0, 1)$ ,  $h = \eta\beta K_0 < \frac{1}{2}$ , and  $B_r[x_0] \subseteq D$  such that  $r = \frac{1-\sqrt{1-2h}}{h}\eta$ . Then, under the above restrictions, the following assertions are true.

- a) The  $SP$ -operator  $K_\alpha : B_r[x_0] \rightarrow X$  generated by  $\alpha$ ,  $\theta$ ,  $A$ , and  $A_\theta$  is a contraction self-operator on  $B_r[x_0]$  with Lipschitz constant  $\gamma^2 [1 - \alpha (1 + \gamma\theta) (1 - \gamma)]$  such that

$$K_\alpha x = A[(1 - \alpha) Ax + \alpha AA_\theta x].$$

- b) Equation (1) has a unique solution  $x^* \in B_r[x_0]$ .  
 c) The sequence  $\{x_n\}$  generated by Algorithm (11) is in  $B_r[x_0]$  and it converges strongly to  $x^*$ .  
 d) The following error estimate holds:

$$\|x_{n+1} - x^*\| \leq \kappa^{2(n+1)} \|x_0 - x^*\|, \forall n \in \mathbb{N},$$

where  $\kappa = \gamma [1 - \alpha (1 + \gamma\theta) (1 - \gamma)]$  and  $\gamma = \beta r K_0$ .

**Proof** By Theorem (1) and Theorem (2), we know that the following inequalities are provided for (12) and (11), and we have

$$\|Ax - Ay\| \leq \gamma \|x - y\|,$$

$$\|A_\lambda x - x^*\| \leq (1 - \lambda (1 - \delta_x)) \|x - x^*\|,$$

and

$$\|A_\theta x - A_\theta y\| \leq \gamma (1 - \theta (1 - \gamma)) \|x - y\|.$$

- a) We show that  $K_\alpha$  is a contraction as follows:

$$\begin{aligned} \|K_\alpha x - K_\alpha y\| &= \|A[(1 - \alpha) Ax + \alpha AA_\theta x] - A[(1 - \alpha) Ay + \alpha AA_\theta y]\| \\ &\leq \gamma \|(1 - \alpha) Ax + \alpha AA_\theta x - (1 - \alpha) Ay - \alpha AA_\theta y\| \\ &\leq \gamma (1 - \alpha) \|Ax - Ay\| + \gamma \alpha \|AA_\theta x - AA_\theta y\| \\ &\leq \gamma^2 (1 - \alpha) \|x - y\| + \gamma^3 \alpha (1 - \theta (1 - \gamma)) \|x - y\| \\ &= \gamma^2 [(1 - \alpha) + \gamma \alpha (1 - \theta (1 - \gamma))] \|x - y\| \\ &= \gamma^2 [1 - \alpha (1 + \gamma\theta) (1 - \gamma)] \|x - y\|. \end{aligned}$$

- b) It is clear from Theorem (1).  
 c) From (11), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|K_\alpha x_n - K_\alpha x^*\| \\ &\leq \gamma^2 [1 - \alpha (1 + \gamma\theta) (1 - \gamma)] \|x_n - x^*\|. \end{aligned}$$

By induction, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \gamma^2 [1 - \alpha(1 + \gamma\theta)(1 - \gamma)] \|x_n - x^*\| \\ \|x_n - x^*\| &\leq \gamma^4 [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^2 \|x_{n-1} - x^*\| \\ \|x_{n-1} - x^*\| &\leq \gamma^6 [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^3 \|x_{n-2} - x^*\| \\ &\vdots \\ \|x_{n+1} - x^*\| &\leq \gamma^{2(n+1)} [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^{n+1} \|x_0 - x^*\|. \end{aligned}$$

This implies that  $x \rightarrow x^*$  as  $n \rightarrow \infty$ .

d) Since  $\kappa = \gamma [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]$  and  $\gamma = \beta r K_0$ , we have

$$\|x_{n+1} - x^*\| \leq \kappa^{2(n+1)} \|x_0 - x^*\|, \forall n \in \mathbb{N}.$$

□

### 3.2. Local convergence analysis

The following theorems present results about the local convergence analysis of (11) and (12).

**Theorem 4** *Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Assume that  $\alpha, \theta \in (0, 1)$  and  $x^* \in D$  is a solution of (1). For some  $x_0 \in D$  let  $F'_{x_0} \in B(Y, X)$ , and  $F'_{x_0}$  and  $F$  satisfy (7) and (8). Assuming that  $B_{r_1}(x^*) \subseteq D$ , where  $r_1 = \frac{2}{K_2}$ , then we have the following:*

- i) *For initial  $x_0 \in B_r(x^*)$  with  $r = \frac{2}{2K_2 + 3K_1}$ , the sequence  $\{x_n\}$  generated by algorithm (11) is in  $B_r(x^*)$  and converges strongly to the unique solution  $x^*$  in  $B_{r_1}(x^*)$ .*
- ii) *The following error estimate holds:*

$$\|x_{n+1} - x^*\| \leq (\kappa')^{2(n+1)} \|x_0 - x^*\|, \forall n \in \mathbb{N},$$

where  $\kappa' = \delta_0 [1 - \alpha(1 + \delta_0\theta)(1 - \delta_0)]$  and  $\delta_0 = \frac{\|x_0 - x^*\|}{r}$ .

#### Proof

- i) First we show that  $x^*$  is a unique solution of (1) in  $B_{r_1}(x^*)$ . For contradiction, suppose that  $y^*$  is another solution of (1) in  $B_{r_1}(x^*)$ . Then we have

$$0 = F(x^*) - F(y^*) = \int_0^1 F'_{y^* + t(x^* - y^*)} (x^* - y^*) dt.$$

Define an operator  $L$  such that

$$L(h) = \int_0^1 F'_{y^* + t(x^* - y^*)} h dt, \forall h \in X.$$



Hence, we have

$$\begin{aligned} \|F'_{x^*} - L\| &= \left\| \int_0^1 F'_{x^*} dt - \int_0^1 F'_{y^*+t(x^*-y^*)} dt \right\| \\ &= \left\| \int_0^1 (F'_{x^*} - F'_{y^*+t(x^*-y^*)}) dt \right\| \end{aligned}$$

and

$$\begin{aligned} \|I - F'^{-1}_{x^*} L\| &= \left\| \int_0^1 F'^{-1}_{x^*} (F'_{x^*} - F'_{y^*+t(x^*-y^*)}) dt \right\| \\ &\leq \int_0^1 \|F'^{-1}_{x^*} (F'_{x^*} - F'_{y^*+t(x^*-y^*)})\| dt \\ &\leq \frac{K_2}{2} \|x^* - y^*\| \\ &< \frac{r_1 K_2}{2} = 1. \end{aligned}$$

By Lemma 1,  $L$  is an invertible operator and hence  $x^* = y^*$  is a contradiction. It implies that  $x^*$  is the unique solution of (1) in  $B_{r_1}(x^*)$ .

Now we examine that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ . By Theorem 3, we know that  $K_\alpha$  is a quasi-contraction self-operator on mapping  $B_r(x^*)$ . Therefore,  $x_n \in B_r(x^*)$  and

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|K_\alpha(x_n) - K_\alpha(x^*)\| \\ &\leq \delta_{x_n}^2 [1 - \alpha(1 + \delta_{x_n}\theta)(1 - \delta_{x_n})] \|x_n - x^*\|, \end{aligned} \tag{13}$$

where  $\delta_{x_n}$  is defined in Theorem 2-b. Since  $\delta_{x_n} < 1$ , then for all  $n \in \mathbb{N}_0$  we obtain

$$\|x_{n+1} - x^*\| \leq \|x_0 - x^*\|.$$

From the definition of  $\delta_x$ , we have

$$\begin{aligned} \delta_{x_n} &= \frac{K_1}{2(1 - rK_2)} (\|x_n - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{3K_1 \|x_0 - x^*\|}{2(1 - rK_2)} \\ &\leq \frac{\|x_0 - x^*\|}{r} \\ &= \delta_0. \end{aligned}$$

Hence, from (13), we obtain

$$\|x_{n+1} - x^*\| \leq (\delta_0^2 [1 - \alpha(1 + \delta_0\theta)(1 - \delta_0)])^{n+1} \|x_0 - x^*\|, \tag{14}$$

which implies that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ . This complete the proof.

ii) We conclude from (14) that for all  $n \in \mathbb{N}$ , we have

$$\|x_{n+1} - x^*\| \leq (\kappa')^{2(n+1)} \|x_0 - x^*\|.$$

□

**Theorem 5** Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Assume that  $\alpha, \theta \in (0, 1)$  and  $x^* \in D$  is solution of (1). For some  $x_0 \in D$ , let  $F'_{x^*} \in B(Y, X)$ , and  $F'_{x^*}$  and  $F$  satisfy the conditions (7) and (8), and for all  $x \in D$ , some  $K_3 > 0$ ,

$$\left\| F'_{x^*}{}^{-1} (F'_x - F'_{y_0}) \right\| \leq K_3 \|x - y_0\|, \tag{15}$$

such that  $K_3 \leq K_1$ . Assume that  $B_{r_1}(x^*) \subseteq D$ , where  $r_1 = \frac{2}{K_2}$  and  $x_0, y_0, z_0 \in B_r(x^*)$ , where  $r = \frac{2}{2K_2 + 3K_1}$ .

For all  $x \in B_r(x^*)$ , consider three operators such that

$$V_{y_0}(x) = x - F'_{y_0}{}^{-1} F(x),$$

$$V_{z_0}(x) = x - F'_{z_0}{}^{-1} F(x),$$

and

$$\mathfrak{S}_\alpha(x) = V_{y_0} [(1 - \alpha) A(x) + \alpha V_{z_0} A_\theta(x)].$$

Then we have the following:

i) For all  $x \in B_r(x^*)$ , we obtain

$$\|V_{y_0}(x) - x^*\| \leq \delta'_x \|x - x^*\|,$$

where  $\delta'_x = \frac{K_3}{2(1-rK_2)} (\|x - x^*\| + 2\|y_0 - x^*\|)$ .

ii)  $V_{y_0}$  is a quasi-contraction and self-operator on  $B_r(x^*)$  with constant  $\delta'$ , where  $\delta' = \sup_{x \in B_r(x^*)} \{\delta'_x\}$ .

iii) For all  $x \in B_r(x^*)$ , we obtain

$$\|V_{z_0}(x) - x^*\| \leq \delta''_x \|x - x^*\|,$$

where  $\delta''_x = \frac{K_3}{2(1-rK_2)} (\|x - x^*\| + 2\|z_0 - x^*\|)$ .

iv)  $V_{z_0}$  is a quasi-contraction and self-operator on  $B_r(x^*)$  with constant  $\delta''$ , where  $\delta'' = \sup_{x \in B_r(x^*)} \{\delta''_x\}$ .

v) The sequence  $\{x_n\}$  generated by Algorithm 12 is in  $B_r(x^*)$  and it converges strongly to the a unique solution  $x^*$  in  $B_{r_1}(x^*)$ .

vi) The following error estimate holds:

$$\|x_{n+1} - x^*\| \leq (\kappa'')^{(n+1)} \|x_0 - x^*\|,$$

where  $\kappa'' = \delta'_0 ((1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0)))$  and  $\delta'_0 = \delta''_0 = \frac{3K_3 \|x_0 - x^*\|}{2(1-rK_2)}$ .

**Proof**

i) For all  $x \in B_r(x^*)$ , by Lemma 2 and (15), we obtain

$$\begin{aligned} \|V_{y_0}(x) - x^*\| &= \left\| x - F_{y_0}'^{-1}F(x) - x^* \right\| \\ &= \left\| F_{y_0}'^{-1} [F(x) - F(x^*) - F_{y_0}'(x - x^*)] \right\| \\ &\leq \left\| (F_{x^*}'^{-1}F_{y_0}')^{-1} \right\| \int_0^1 \left\| F_{x^*}'^{-1} (F_{x^*+t(x-x^*)}' - F_{y_0}'(x - x^*)) \right\| dt \\ &\leq \frac{K_3 \|x - x^*\| [\|x - x^*\| + 2 \|y_0 - x^*\|]}{2(1 - rK_2)} \\ &\leq \delta'_x \|x - x^*\|. \end{aligned}$$

ii)

$$\begin{aligned} \delta' &= \sup_{x \in B_r(x^*)} \{\delta'_x\} = \frac{K_3 \left[ \sup_{x \in B_r(x^*)} \|x - x^*\| + 2 \|y_0 - x^*\| \right]}{2(1 - rK_2)} \\ &\leq \frac{K_3 [r + 2 \|y_0 - x^*\|]}{2(1 - rK_2)} \\ &\leq \frac{K_3 [r + 2r]}{2(1 - rK_2)} \\ &< \frac{3rK_1}{2(1 - rK_2)} \\ &= 1. \end{aligned}$$

Thus, the operator  $V_{y_0}$  is a quasi-contraction self-mapping on  $B_r(x^*)$ .

iii) Similarly, for all  $x \in B_r(x^*)$ , by Lemma 2 and (15), we obtain

$$\begin{aligned} \|V_{z_0}(x) - x^*\| &= \left\| x - F_{z_0}'^{-1}F(x) - x^* \right\| \\ &= \left\| F_{z_0}'^{-1} [F(x) - F(x^*) - F_{z_0}'(x - x^*)] \right\| \\ &\leq \left\| (F_{x^*}'^{-1}F_{z_0}')^{-1} \right\| \int_0^1 \left\| F_{x^*}'^{-1} (F_{x^*+t(x-x^*)}' - F_{z_0}'(x - x^*)) \right\| dt \\ &\leq \frac{K_3 \|x - x^*\| [\|x - x^*\| + 2 \|z_0 - x^*\|]}{2(1 - rK_2)} \\ &\leq \delta''_x \|x - x^*\|. \end{aligned}$$

iv)

$$\begin{aligned} \delta'' &= \sup_{x \in B_r(x^*)} \{\delta''_x\} = \frac{K_3 \left[ \sup_{x \in B_r(x^*)} \|x - x^*\| + 2 \|z_0 - x^*\| \right]}{2(1 - rK_2)} \\ &\leq \frac{K_3 [r + 2 \|z_0 - x^*\|]}{2(1 - rK_2)} \\ &\leq \frac{K_3 [r + 2r]}{2(1 - rK_2)} \\ &< \frac{3rK_1}{2(1 - rK_2)} \\ &= 1. \end{aligned}$$

Thus, the operator  $V_{z_0}$  is a quasi-contraction self-mapping on  $B_r(x^*)$ .

v) If we define an operator  $A_\theta$  by

$$A_\theta(x) = x - \theta F_{x_0}^{-1} F(x), \quad \forall x \in B_r(x^*),$$

then  $A_\theta$  is a quasi-contraction self-map on  $B_r(x^*)$  and the following satisfies:

$$\|A_\theta(x) - x^*\| \leq (1 - \theta(1 - \delta_x)) \|x - x^*\|, \quad \forall x \in B_r(x^*).$$

As in Theorem 4,  $x^*$  is the unique solution of (1) in  $B_{r_1}(x^*)$ . Therefore, by Algorithm 12, we obtain

$$y_0 = (1 - \alpha) Ax + \alpha V_{z_0} A_\theta x_0 \in B_r(x^*)$$

and

$$z_0 = A_\theta x_0 \in B_r(x^*).$$

Also, we can rearrange Algorithm 12 as follows:

$$x_{n+1} = \mathfrak{S}_\alpha(x_n) = V_{y_0} [(1 - \alpha) A(x_n) + \alpha V_{z_0} A_\theta(x_n)].$$

Now we show that the sequence  $\{x_n\}$  generated by Algorithm 12 is in  $B_r(x^*)$  and it converges strongly to the unique solution  $x^*$  in  $B_{r_1}(x^*)$ . For all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\mathfrak{S}_\alpha(x_n) - x^*\| = \|V_{y_0} [(1 - \alpha) A(x_n) + \alpha V_{z_0} A_\theta(x_n)] - x^*\| \\ &\leq \delta'_{y_n} \|(1 - \alpha) A(x_n) + \alpha V_{z_0} A_\theta(x_n) - x^*\| \\ &\leq \delta'_{y_n} (1 - \alpha) \|A(x_n) - x^*\| + \delta'_{y_n} \alpha \|V_{z_0} A_\theta(x_n) - x^*\| \\ &\leq \delta'_{y_n} (1 - \alpha) \delta_x \|x_n - x^*\| + \delta'_{y_n} \alpha \delta''_{z_n} \|A_\theta(x_n) - x^*\| \\ &\leq \delta'_{y_n} (1 - \alpha) \delta_{x_n} \|x_n - x^*\| + \delta'_{y_n} \alpha \delta''_{z_n} (1 - \theta(1 - \delta_{x_n})) \|x_n - x^*\| \\ &= \delta'_{y_n} [(1 - \alpha) \delta_{x_n} + \alpha \delta''_{z_n} (1 - \theta(1 - \delta_{x_n}))] \|x_n - x^*\|. \end{aligned} \tag{16}$$

From the definitions of  $\delta''_x$  and  $\delta'_x$ , we obtain

$$\begin{aligned} \delta''_{z_n} &= \frac{K_3}{2(1-rK_2)} (\|A_\theta(x_n) - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{K_3}{2(1-rK_2)} ((1-\theta(1-\delta_{x_n}))\|x_n - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{K_3}{2(1-rK_2)} (\|x_0 - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{3K_1\|x_0 - x^*\|}{2(1-rK_2)} \\ &= \delta''_0 \\ &\leq \frac{\|x_0 - x^*\|}{r} \\ &= \delta_0. \end{aligned}$$

Similarly,

$$\begin{aligned} \delta'_{y_n} &= \frac{K_3}{2(1-rK_2)} (\|(1-\alpha)A(x_n) + \alpha V_{z_0}U_\theta(x_n) - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{K_3}{2(1-rK_2)} ((1-\alpha)\delta_{x_n} + \alpha\delta''_{z_n}(1-\theta(1-\delta_{x_n}))\|x_n - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{K_3}{2(1-rK_2)} (\|x_0 - x^*\| + 2\|x_0 - x^*\|) \\ &\leq \frac{3K_1\|x_0 - x^*\|}{2(1-rK_2)} \\ &= \delta'_0 \\ &\leq \frac{\|x_0 - x^*\|}{r} \\ &= \delta_0. \end{aligned}$$

Hence, from (16), we obtain

$$\|x_{n+1} - x^*\| \leq [\delta'_0((1-\alpha)\delta_0 + \alpha\delta''_0(1-\theta(1-\delta_0)))]^{n+1} \|x_0 - x^*\|, \tag{17}$$

and it implies that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ .

vi) For all  $n \in \mathbb{N}_0$ , it is concluded from (17) that  $\|x_{n+1} - x^*\| \leq (\kappa'')^{(n+1)} \|x_0 - x^*\|$ .

□

### 3.3. The rate of convergence

**Theorem 6** *Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . For some  $x_0 \in D$  let  $F'_{x_0} \in B(Y, X)$ . Assume that  $F'_{x_0}$  and  $F$  satisfy (6) with the following conditions:*

i)  $\|F'_{x_0}{}^{-1}F(x_0)\| \leq \eta$ , for some  $\eta > 0$ .

ii)  $\|F'_{x_0}{}^{-1}\| \leq \beta$ , for some  $\beta > 0$ .

Assume that  $\gamma, \alpha, \theta \in (0, 1)$ ,  $h = \eta\beta K_0 < \frac{1}{2}$ , and  $B_r[x_0] \subseteq D$  such that  $r = \frac{1-\sqrt{1-2h}}{h}\eta$ .

Under the above restrictions, for given  $u_0 = x_0 \in D$ , consider the iterative sequences  $\{x_n\}_{n=0}^\infty$  and  $\{u_n\}_{n=0}^\infty$  defined by (11) and (9), respectively. Then  $\{x_n\}_{n=0}^\infty$  converges to  $x^*$  faster than  $\{u_n\}_{n=0}^\infty$  does in  $B_r[x_0]$ .

**Proof** Respectively, by Theorems 1 and Theorem 3, we have the following inequalities:

$$\|u_{n+1} - x^*\| \leq \gamma^{n+1} (1 - \alpha(1 - \gamma))^{n+1} \|u_0 - x^*\|,$$

and

$$\|x_{n+1} - x^*\| \leq \gamma^{2(n+1)} [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^{n+1} \|x_0 - x^*\|.$$

Define

$$a_n = \gamma^{2(n+1)} [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^{n+1} \|x_0 - x^*\|,$$

$$b_n = \gamma^{n+1} (1 - \alpha(1 - \gamma))^{n+1} \|u_0 - x^*\|,$$

and

$$\begin{aligned} \wp_n &= \frac{a_n}{b_n} = \frac{\gamma^{2(n+1)} [1 - \alpha(1 + \gamma\theta)(1 - \gamma)]^{n+1} \|x_0 - x^*\|}{\gamma^{n+1} (1 - \alpha(1 - \gamma))^{n+1} \|u_0 - x^*\|} \\ &= \gamma^{n+1} \left( \frac{1 - \alpha(1 + \gamma\theta)(1 - \gamma)}{1 - \alpha(1 - \gamma)} \right)^{n+1}. \end{aligned}$$

Since  $\gamma, \alpha, \theta \in (0, 1)$ , we have

$$\left[ \gamma \left( \frac{1 - \alpha(1 + \gamma\theta)(1 - \gamma)}{1 - \alpha(1 - \gamma)} \right) \right] < 1.$$

Therefore,  $\lim_{n \rightarrow \infty} \wp_n = 0$ . From Definition 3, we obtain that  $\{x_n\}_{n=0}^\infty$  converges faster than  $\{u_n\}_{n=0}^\infty$ .  $\square$

**Theorem 7** Let  $F$  be a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Assume that  $\alpha, \theta \in (0, 1)$  and  $x^* \in D$  is a solution of (1). For some  $x_0 \in D$ , let  $F'_{x^*}{}^{-1} \in B(Y, X)$ , and  $F'_{x^*}{}^{-1}$  and  $F$  satisfy the conditions (7) and (8), and for all  $x \in D$ , some  $K_3 > 0$ ,

$$\|F'_{x^*}{}^{-1}(F'_x - F'_{y_0})\| \leq K_3 \|x - y_0\|, \tag{18}$$

such that  $K_3 \leq K_1$ . Assume that  $B_{r_1}(x^*) \subseteq D$ , where  $r_1 = \frac{2}{K_2}$  and  $x_0, y_0, z_0 \in B_r(x^*)$ , where  $r = \frac{2}{2K_2 + 3K_1}$ .

Under the above restrictions, for given  $u_0 = x_0 \in D$ , consider the iterative sequences  $\{x_n\}_{n=0}^\infty$  and  $\{u_n\}_{n=0}^\infty$  defined by (12) and (10), respectively. Then  $\{x_n\}_{n=0}^\infty$  converges to  $x^*$  faster than  $\{u_n\}_{n=0}^\infty$  does in  $B_{r_1}(x^*)$ .

**Proof** Respectively, by Theorems 2 and Theorem 5, we have the following inequalities:

$$\|u_{n+1} - x^*\| \leq \left(\delta'_0 (1 - \alpha (1 - \delta_0))\right)^{n+1} \|u_0 - x^*\|,$$

and

$$\|x_{n+1} - x^*\| \leq [\delta'_0 ((1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0)))]^{n+1} \|x_0 - x^*\|.$$

Define

$$a_n = [\delta'_0 ((1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0)))]^{n+1} \|x_0 - x^*\|,$$

$$b_n = \left(\delta'_0 (1 - \alpha (1 - \delta_0))\right)^{n+1} \|u_0 - x^*\|,$$

and

$$\begin{aligned} \wp_n &= \frac{a_n}{b_n} = \frac{[\delta'_0 ((1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0)))]^{n+1} \|x_0 - x^*\|}{(\delta'_0 (1 - \alpha (1 - \delta_0)))^{n+1} \|u_0 - x^*\|} \\ &= \frac{[(1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0))]^{n+1}}{(1 - \alpha (1 - \delta_0))^{n+1}}. \end{aligned}$$

Since

$$\delta''_0 \leq \delta_0$$

and  $\gamma, \alpha, \theta \in (0, 1)$ , we have

$$\frac{[(1 - \alpha) \delta_0 + \alpha \delta''_0 (1 - \theta (1 - \delta_0))]^{n+1}}{(1 - \alpha (1 - \delta_0))^{n+1}} < 1.$$

Therefore,  $\lim_{n \rightarrow \infty} \wp_n = 0$ . From Definition 3, we obtain that  $\{x_n\}_{n=0}^\infty$  converges faster than  $\{u_n\}_{n=0}^\infty$ .  $\square$

**Example 1** [18] Let  $X = \mathbb{R}$ ,  $C = (-1, 1)$ , and  $F : C \rightarrow \mathbb{R}$  an operator defined by

$$F(x) = e^x - 1, \quad \forall x \in C.$$

Then  $F$  is Fréchet differentiable and its Fréchet derivative  $F'_x$  at any point  $x \in C$  is given by

$$F'_x = e^x.$$

For  $x_0 = 0, 26$ , we get

$$F'^{-1}_{x_0} = \frac{1}{e^{0,26}}.$$

It is easy to see that  $\beta = 0.771051585803566, \eta = 0.228948414196434$ , and  $K_0 = 2.718281828459046$ . Then

$$h = \eta \beta K_0 < \frac{1}{2}.$$

Therefore, we have:

i)  $\|F'^{-1}_{x_0} F(x_0)\| \leq \eta_0,$

ii)  $\|F'_{x_0}{}^{-1}\| \leq \beta_0,$

iii)  $\|F'_x - F'_{x_0}\| \leq K_0 \|x - x_0\|.$

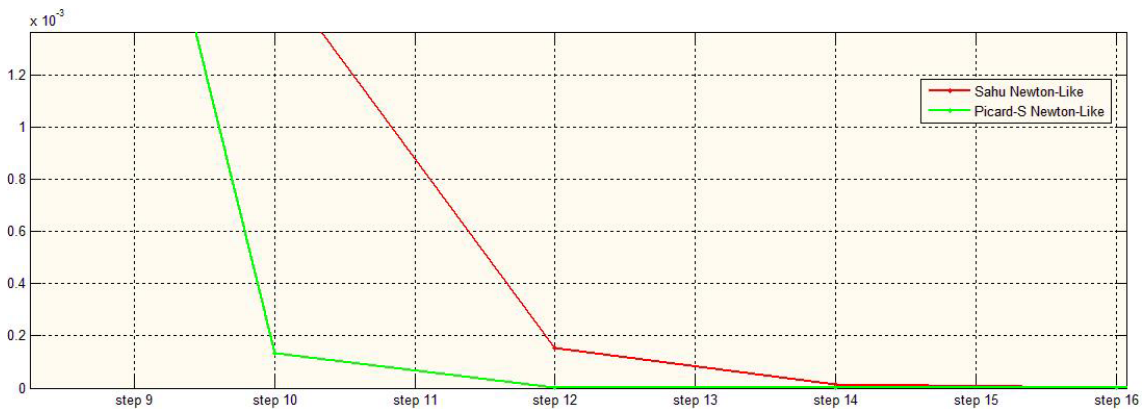
Consequently, all conditions of Theorem 3 hold. Therefore, the sequence  $\{x_n\}$  generated by (11) is in  $B_r[x_0]$  and converges to  $x^* \in B_r[x_0]$ . For given  $u_0 = x_0 \in C$ , Figure 1, Figure 2, and the Table show that the sequence  $\{x_n\}$  defined by (11) is faster than the sequence  $\{u_n\}$  generated by (9).

**Table.** Comparison rate of convergence

$x_n$	Sahu Newton type	Picard-S Newton type
$x_0$	0.2600000000000000	0.2600000000000000
$x_1$	0.017117996475976	0.004551903139995
$x_2$	0.001613569978360	0.000134045775368
$x_3$	0.000155439797062	0.000003977964746
$x_4$	0.000015004477715	0.000000118077485
$x_5$	0.000001448654022	0.000000003504904
$x_6$	0.000000139867460	0.000000000104036
$x_7$	0.000000013504220	0.000000000003088
$x_8$	0.000000001303834	0.000000000000092
$x_9$	0.000000000125885	0.000000000000003
$x_{10}$	0.000000000012154	0.000000000000000
$x_{11}$	0.000000000001174	0.000000000000000
$x_{12}$	0.000000000000113	0.000000000000000
$x_{13}$	0.000000000000011	0.000000000000000
$x_{14}$	0.000000000000001	0.000000000000000
$x_{15}$	0.000000000000000	0.000000000000000

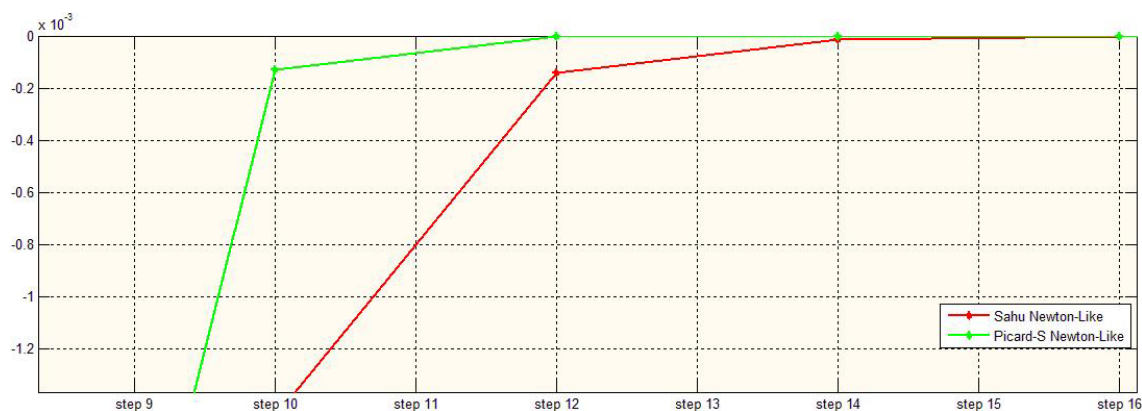
The Table shows that our iteration reaches a fixed point at the 10th step while the Sahu Newton-like iteration reaches it at the 15th step.

Figures 1 and 2 are the graphical presentations of the Table.



**Figure 1.** Comparison of the iteration processes defined by (1.9) and (1.11).



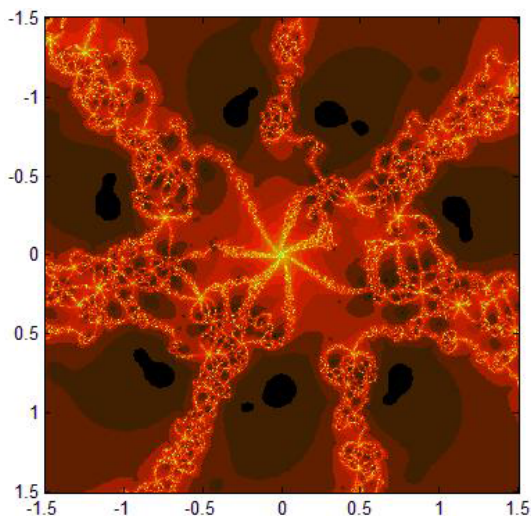


**Figure 2.** Derivative comparison of the iteration processes defined by (1.9) and (1.11).

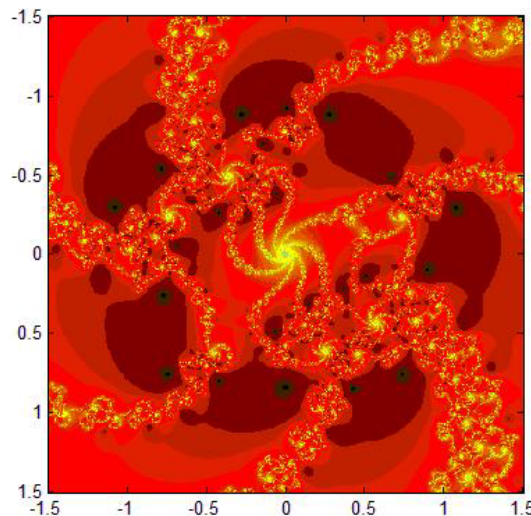
**Corollary 1** (9) is shown to be faster than the iteration method of (5). In Example 1, (11) iterations were shown to be faster than (9). For this reason, it is proved that the (11) iteration method is faster than the (5) iteration method.

An interesting result is that Newton’s method works for complex valued functions. Having seen that Newton’s method behaves differently for different starting points, converging to different roots or possibly not converging at all, what happens at problem areas in the complex plane? For example, consider the case of starting points that are equidistant to multiple different roots. Using the Newton method one can attempt to visualize the convergence of each possible starting complex number, resulting in a fractal pattern.

**Example 2** [8] Figures 3 and 4 present examples of fractal polynomiography obtained using the two iteration methods 9 and 11 by taking the following parameters:  $P(z) = z^7 + z^2 - 1$ ,  $M = 15$ , standard convergence test with  $\epsilon = 0.001$ ,  $A = [-1.5, 1.5] \times [-1.5, 1.5]$ , and fixing the coefficients as follows: Picard-S:  $\alpha = 0.9 + i$ ,  $\beta = 0.6 - 0.5i$ , Sahu  $\alpha = 0.9 + i$ .



**Figure 3.** Picard-S-Newton-like.



**Figure 4.** Sahu-Newton-like.

We notice that in each polynomiograph we find seven main areas with different forms and colors. The fractal boundaries represent the seven solutions of the polynomial. Further, the colors in Figures 3 and 4 show how long it took the starting point to reach the convergence point. In the darker colored spots the convergence is faster than in lighter colored ones.

### References

- [1] Argyros IK. Computational Theory of Iterative Methods, Series: Studies in Computational Mathematics. New York, NY, USA: Elsevier, 2007.
- [2] Argyros IK. Convergence and Applications of Newton-Type Iterations. Berlin, Germany: Springer, 2008.
- [3] Argyros IK. On a class of Newton-like methods for solving nonlinear equations. *J Comput Appl Math* 2009; 228: 115-122.
- [4] Bartle RG. Newton's method in Banach spaces. *P Am Math Soc* 1955; 6: 827-831.
- [5] Berinde V. Iterative Approximation of Fixed Points. Berlin, Germany: Springer, 2007.
- [6] Dennis JE. On the Kantorovich hypothesis for Newton's method. *SIAM J Numer Anal* 1969; 6: 493-507.
- [7] Dogan K, Karakaya V. On the convergence and stability results for a new general iterative process. *Scientific World Journal* 2014; 2014: 852475.
- [8] Gdawiec K, Kotarski W, Lisowska A. Polynomiography based on the nonstandard Newton-like root finding methods. *Abstr Appl Anal* 2015; 2015: 797594.
- [9] Gürsoy F. A Picard-S iterative method for approximating fixed point of weak-contraction mappings. 2016; 30: 2829-2845.
- [10] Ishikawa S. Fixed point by a new iteration method. *P Am Math Soc* 1974; 44: 147-150.
- [11] Karakaya V, Doğan K, Gürsoy F, Ertürk M. Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces. *Abstr Appl Anal* 2013; 2013: 560258.
- [12] Kreyszig E. Introductory Functional Analysis with Applications. New York, NY, USA, Wiley: 1978.
- [13] Li Q, Mo Z, Qi L. Numerical Methods for Solving Non-linear Equations. Beijing, China: Science Press, 1997.
- [14] Picard E. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. *J Math Pures Appl* 1890; 6: 145-210 (in French).
- [15] Ren H, Argyros IK. On convergence of the modified Newtons method under Hölder continuous Fréchet derivative. *Appl Math Comput* 2009; 213: 440-448.
- [16] Rheinboldt WC. A unified convergence theory for a class of iterative processes. *SIAM J Numer Anal* 1968; 5: 42-63.
- [17] Sahu DR. Applications of the S-iteration process to constrained minimization problems and split feasibility problems. *Fixed Point Theory* 2011; 12: 187-204.
- [18] Sahu DR, Singh KK, Singh VK. Some Newton-like methods with sharper error estimates for solving operator equations in Banach spaces. *Fixed Point Theory A* 2012; 1: 1-20.
- [19] Scherzer O. Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems. *J Math Anal Appl* 1995; 194: 911-933.
- [20] Suhubi ES. Functional Analysis. London, UK: Kluwer Academic Publishers, 2003.
- [21] Wu Q, Zhao Y. Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space. *Appl Math Comput* 2006; 175: 1515-1524.