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# Coframe bundle and problems of lifts on its cross-sections

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**Abstract:** The main purpose of this paper is to study the complete and horizontal lifts of vector and tensor fields of type (1,1) on cross-sections in the coframe bundle. Explicit formulas of these lifts are obtained. Finally, complete lifts of almost complex structures restricted to almost analytic cross-sections are investigated.

**Key words:** Coframe bundle, cross-section, Tachibana operator, Nijenhuis–Shirokov tensor, almost complex structure

## 1. Introduction

Let $M$ be an $n-$dimensional differentiable manifold of class $C^\infty$ and $F^*M$ its coframe bundle. The differential geometry of the cotangent bundle has been studied by many authors (see, for example, [2, 3, 9–11]).

When a field of global coframes is given on $M$, its defines a cross-section $\sigma : M \to F^*M$ in the coframe bundle. In this paper, we study the behavior on this cross-section of lifts of tensor fields from $M$ to $F^*M$ .

In 2 we briefly describe the definitions and results that are needed later, after which the complete and horizontal lifts of affinor fields (tensor fields of type $(1,1)$) are constructed in 3. In 4 and 5 we consider, respectively, the complete and horizontal lifts of the vector and affinor fields along the $n-$dimensional submanifold $\sigma(M)$ of $F^*M$ defined by cross-section $\sigma$. In 6 we study the particular case of an almost complex structure on $\sigma(M)$.

All results in this paper can be closely compared with those of the corresponding theory for cross-sections in the cotangent bundle [12]. A similar approach was applied in [1], when studying lifts on cross-sections of the bundle of frames by means of the tangent bundle.

## 2. Preliminaries

Manifolds, tensor fields, and linear connections under consideration are all assumed to be differentiable and of class $C^\infty$. Indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$ and indices $A, B, C, ...$ run from $1$ to $n + n^2$. We put $h_\alpha = \alpha \cdot n + h$. Summation over repeated indices is always implied. Entries of matrices are written as $A^i_j, A_{ij}$ or $A^{ij}$, and in all cases $i$ is the row index while $j$ is the column index.

Let $M$ be an $n-$dimensional differentiable manifold of class $C^\infty$. Coordinate systems in $M$ are denoted by $(U, x^i)$, where $U$ is the coordinate neighborhood and $x^i$ are the coordinate functions. We denote the Lie derivative by $L_X$, and by $\Im^r_s(M)$ the set of all differentiable tensor fields of type $(r, s)$ on $M$.

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Let $T_x^*M$ be the cotangent space at a point $x \in M$, $(X^\alpha) = (X^1, ..., X^n)$ a coframe at $x$ and $F^*M$ the coframe bundle over $M$, that is, the set of all coframes at all points of $M$ (see [4]). Let $\pi : F^*M \to M$ be the canonical projection of $F^*M$ onto $M$. For the coordinate system $(U, x^i)$ in $M$ we put $F^*U = \pi^{-1}(U)$. A coframe $(X^\alpha)$ at $x$ can be expressed uniquely in the form $X^\alpha = X_i^\alpha (dx^i)_x$. The induced coordinate system in $F^*U$ is $\left\{ F^*U, (x^i, X_i^\alpha) \right\}$. We shall denote $\frac{\partial}{\partial x^i}$ by $\partial_i$ and $\frac{\partial}{\partial X_i^\alpha}$ by $\partial_{i_\alpha}$. The matrix $(X_i^\alpha)$ is nonsingular and its inverse will be written as $(X_\alpha^i)$. We denote by $\nabla$ the linear connection on $M$ with components $\Gamma_{ij}^k$.

Let $V$ be a vector field on $M$, and let $V^i$ be its components in $U$. Then the complete lift ${}^CV$ and horizontal lift ${}^HV$ of $V$ to $F^*M$ are given by (see [4])

$$ {}^CV = V^i \partial_i - X_j^\alpha (\partial_i V^j) \partial_{i_\alpha}, \tag{2.1} $$

$$ {}^HV = V^i \partial_i + X_j^\alpha \Gamma_{ki}^j V^k \partial_{i_\alpha}, \tag{2.2} $$

respectively.

## 3. Lifts of affinor fields to the coframe bundle

Let $\varphi$ be an affinor field on $M$ and let $\varphi_i^j$ be its local components in $U$.

The following Theorem 1 holds.

**Theorem 1** *If we put*

$$ \begin{cases} \tilde{\varphi}_j^i = \varphi_j^i, & \tilde{\varphi}_{j_\beta}^i = 0, \\ \tilde{\varphi}_j^{i_\alpha} = X_k^\alpha (\partial_j \varphi_i^k - \partial_i \varphi_j^k), & \tilde{\varphi}_{j_\beta}^{i_\alpha} = \delta_\beta^\alpha \varphi_i^j, \end{cases} \tag{3.1} $$

*then we get an affinor field $\tilde{\varphi}$ on $F^*M$ whose components are $\tilde{\varphi}_J^I$ with respect to the coordinate system $\left\{F^*U, (x^i, X_i^\alpha)\right\}$, where $\delta_\beta^\alpha$ is the Kronecker delta.*

**Proof** We shall show that under the coordinate transformation

$$ \begin{cases} x^{i'} = x^{i'}(x^1, ..., x^n), \\ X_{i'}^\alpha = A_{i'}^i X_i^\alpha \end{cases} \tag{3.2} $$

on $F^*U \bigcap F^*U'$, the equation

$$ \tilde{\varphi}_{J'}^{I'} = A_I^{I'} A_{J'}^J \tilde{\varphi}_J^I \tag{3.3} $$

holds good, where $A_{i'}^i = \frac{\partial x^i}{\partial x^{i'}}$ are elements of the Jacobian matrix of the inverse transformation $x^i = x^i(x^{1'}, ..., x^{n'})$, and $A_I^{I'}$ are elements of the Jacobian matrix of the transformation (3.2), i.e.

$$ (A_I^{I'}) = \begin{pmatrix} A_i^{i'} & 0 \\ X_j^\alpha \partial_i A_{i'}^j & A_{i'}^i \delta_\beta^\alpha \end{pmatrix}. \tag{3.4} $$

On the other hand, the Jacobian matrix $(A_{J'}^J)$ of the inverse transformation has the structure

$$ (A_{J'}^J) = \begin{pmatrix} A_{j'}^j & 0 \\ X_{k'}^\alpha \partial_{j'} A_j^{k'} & A_j^{j'} \delta_\beta^\alpha \end{pmatrix}. \tag{3.5} $$

In the case where $I' = i', J' = j'$, we can easily verify that the right-hand side of (3.3) reduces to

$$A_I^{i'} A_{j'}^J \tilde{\varphi}_J^I = A_i^{i'} A_{j'}^j \tilde{\varphi}_j^i + A_{i_\gamma}^{i'} A_{j'}^j \tilde{\varphi}_j^{i_\gamma} + A_i^{i'} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^i$$

$$+A_{i_\gamma}^{i'} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^{i_\gamma} = A_i^{i'} A_{j'}^j \varphi_j^i = \varphi_{j'}^{i'} = \tilde{\varphi}_{j'}^{i'}.$$

In the case where $I' = i', J' = j'_\beta$ or $I' = i'_\alpha, J' = j'_\beta$, it follows that (3.3) holds good by the same manner as before. In the case where $I' = i'_\alpha, J' = j'$, the left-hand side of (3.3) reduces to

$$\tilde{\varphi}_{j'}^{i'_\alpha} = X_{k'}^\alpha (\partial_{j'} \varphi_{i'}^{k'} - \partial_{i'} \varphi_{j'}^{k'}),$$

which is the sum of the following six terms $a_1, a_2, ..., a_6$ :

$$a_1 = X_{k'}^\alpha (\partial_{j'} A_m^{k'}) A_{i'}^i \varphi_i^m, a_2 = X_{k'}^\alpha A_m^{k'} (\partial_{j'} A_{i'}^i) \varphi_i^m,$$

$$a_3 = X_{k'}^\alpha A_m^{k'} A_{i'}^i (\partial_{j'} \varphi_i^m), a_4 = -X_{k'}^\alpha (\partial_{i'} A_m^{k'}) A_{j'}^j \varphi_j^m,$$

$$a_5 = -X_{k'}^\alpha A_m^{k'} (\partial_{i'} A_{j'}^j) \varphi_j^m, a_6 = -X_{k'}^\alpha A_m^{k'} A_{j'}^j (\partial_{i'} \varphi_j^m).$$

On the other hand, the right-hand side of (3.3) can be written as

$$A_I^{i'_\alpha} A_{j'}^J \tilde{\varphi}_J^I = A_i^{i'_\alpha} A_{j'}^j \tilde{\varphi}_j^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^j \tilde{\varphi}_j^{i_\gamma} + A_i^{i'_\alpha} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^{i_\gamma}.$$

The last expression is the sum of the following four terms $b_1, ..., b_4$ :

$$b_1 = X_k^\alpha (\partial_i A_{i'}^k) A_{j'}^j \varphi_j^i, b_2 = X_k^\alpha A_{i'}^i A_{j'}^j (\partial_j \varphi_i^k),$$

$$b_3 = -X_k^\alpha A_{i'}^i A_{j'}^j (\partial_i \varphi_j^k), b_4 = X_{k'}^\alpha A_{i'}^i (\partial_{j'} A_j^{k'}) \varphi_i^j.$$

After some calculations we get the following relations:

$$a_1 = b_4, \quad a_3 = b_2, \quad a_4 = b_1, \quad a_2 + a_5 = 0, \quad a_6 = b_3. \tag{3.6}$$

Hence, by virtue of (3.6), we see that (3.3) holds good. Consequently, $\tilde{\varphi}$ is an affinor field on $F^*M$. An affinor field $\tilde{\varphi}$ is called a complete lift of $\varphi$ to $F^*M$. □

**Theorem 2** *If we put*

$$\begin{cases} \bar{\varphi}_j^i = \varphi_j^i, & \bar{\varphi}_{j_\beta}^i = 0, \\ \bar{\varphi}_j^{i_\alpha} = X_k^\alpha (\varphi_j^m \Gamma_{mi}^k - \varphi_i^m \Gamma_{jm}^k), & \bar{\varphi}_{j_\beta}^{i_\alpha} = \delta_\beta^\alpha \varphi_i^j, \end{cases} \tag{3.7}$$

*then we get an affinor field* $\bar{\varphi}$ *on* $F^*M$ *whose components are* $\bar{\varphi}_J^I$ *with respect to the coordinate system* $\{F^*U, (x^i, X_i^\alpha)\}$.

**Proof** We shall show that under the coordinate transformation (3.2) the equation

$$\bar{\varphi}_{J'}^{I'} = A_I^{I'} A_{J'}^J \bar{\varphi}_J^I \tag{3.8}$$

holds good.

In the case $I' = i', J' = j'$, we can easily verify that the right-hand side of (3.8) reduces to

$$A_I^{i'} A_{j'}^J \bar{\varphi}_J^I = A_i^{i'} A_{j'}^j \bar{\varphi}_j^i + A_{i_\gamma}^{i'} A_{j'}^j \bar{\varphi}_j^{i_\gamma} + A_i^{i'} A_{j'}^{j_\lambda} \bar{\varphi}_{j_\lambda}^i$$

$$+ A_{i_\gamma}^{i'} A_{j'}^{j_\lambda} \bar{\varphi}_{j_\lambda}^{i_\gamma} = A_i^{i'} A_{j'}^j \varphi_j^i = \varphi_{j'}^{i'} = \bar{\varphi}_{j'}^{i'}.$$

In the cases $I' = i', J' = j'_\beta$ and $I' = i'_\alpha, J' = j'_\beta$, it follows that (3.8) holds good by the same manner as before. In the case where $I' = i'_\alpha, J' = j'$, the left-hand side of (3.8) reduces to

$$\bar{\varphi}_{j'}^{i'_\alpha} = X_{k'}^\alpha (\varphi_{j''}^{m'} \Gamma_{m'i'}^{k'} - \varphi_{i'}^{m'} \Gamma_{j'm'}^{k'}),$$

which is the sum of the following four terms $c_1, ..., c_4$ :

$$c_1 = X_{k'}^\alpha \varphi_{j'}^{m'} A_k^{k'} A_{m'}^m A_{i'}^i \Gamma_{mi}^k, c_2 = X_{k'}^\alpha \varphi_{j'}^{m'} A_k^{k'} (\partial_{m'} A_{i'}^k),$$

$$c_3 = -X_{k'}^\alpha \varphi_{i'}^{m'} A_k^{k'} A_{m'}^m A_{j'}^j \Gamma_{jm}^k, c_4 = -X_{k'}^\alpha \varphi_{i'}^{m'} A_k^{k'} (\partial_{j'} A_{m'}^k).$$

On the other hand, the right-hand side of (3.8) can be written as

$$A_I^{i'_\alpha} A_{j'}^J \bar{\varphi}_J^I = A_i^{i'_\alpha} A_{j'}^j \bar{\varphi}_j^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^j \bar{\varphi}_j^{i_\gamma} + A_i^{i'_\alpha} A_{j'}^{j_\lambda} \bar{\varphi}_{j_\lambda}^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^{j_\lambda} \bar{\varphi}_{j_\lambda}^{i_\gamma}.$$

The last expression is the sum of the following four terms $d_1, ..., d_4$ :

$$d_1 = X_k^\alpha (\partial_i A_{i'}^k) A_{j'}^j \varphi_j^i, d_2 = X_k^\alpha A_{i'}^i A_{j'}^j \varphi_j^m \Gamma_{mi}^k,$$

$$d_3 = -X_k^\alpha A_{i'}^i A_{j'}^j \varphi_i^m \Gamma_{jm}^k, d_4 = X_{k'}^\alpha A_{i'}^i (\partial_{j'} A_j^{k'}) \varphi_i^j.$$

After some calculations we get the following relations:

$$c_1 = d_2, \quad c_2 = d_1, \quad c_3 = d_3, \quad c_4 = d_4. \tag{3.9}$$

Hence, by virtue of (3.9), we see that (3.8) holds good. It means that $\bar{\varphi}$ is an affinor field on $F^*M$. An affinor field $\bar{\varphi}$ is called a horizontal lift of $\varphi$ to $F^*M$. □

## 4. Lifts of vector fields on cross-sections

Let $\sigma$ be a cross-section of the coframe bundle $F^*M$, that is $\sigma : M \to F^*M$ a mapping of class $C^\infty$ such that $\pi \circ \sigma = Id_M$. Then $\sigma$ defines a field of global coframes on $M$, that is, at each point $x \in M, \sigma(x) = (\sigma^1(x), ..., \sigma^n(x))$ is a linear coframe at $x$. If we put $\sigma = (\sigma^1, ..., \sigma^n)$ then each $\sigma^\alpha$ is a covector field globally defined on $M$. Assume that $\sigma^\alpha$ has local components $\sigma_h^\alpha(x)$ with respect to a coordinate system $(U, x^i)$ in $M$, that is $\sigma^\alpha = \sigma_h^\alpha(x) dx^h$ in $U$. Then $\sigma(M)$, which will be called a cross-section determined by $\sigma$, is the $n-$dimensional submanifold of $F^*M$ locally expressed in $F^*U$ by

$$\begin{cases} x^h = x^h, \\ X_h^\alpha = \sigma_h^\alpha(x). \end{cases} \tag{4.1}$$

Thus tangent vectors $B_i^H = \partial_i x^H$ to the cross-section $\sigma(M)$ have components

$$B_i^H = \left(\frac{\partial x^H}{\partial x^i}\right) = \begin{pmatrix} \delta_i^h \\ \partial_i \sigma_h^\alpha \end{pmatrix}. \tag{4.2}$$

On the other hand, the fiber being represented by

$$\begin{cases} x^h = const, \\ X_h^\alpha = X_h^\alpha, \end{cases} \tag{4.3}$$

the tangent vectors $C_{i_\beta}^H = \partial_{i_\beta} x^H$ to the fiber have components

$$C_{i_\beta}^H = C^{i_\beta H} = \begin{pmatrix} 0 \\ \delta_h^i \delta_\beta^\alpha \end{pmatrix}. \tag{4.4}$$

The vectors $B_i^H$ and $C_{i_\beta}^H$, being linearly independent, form a frame $E_I^H = (B_i^H, C_{i_\beta}^H)$ along the cross-section $\sigma(M)$. We call this the frame $(B, C)$ along the cross-section. The coframe $\tilde{E}_H^J = (\tilde{B}_H^j, \tilde{C}_H^{j_\gamma})$ corresponding to this frame is given by

$$\tilde{B}_H^j = (\delta_h^j, 0), \tilde{C}_H^{j_\gamma} = (-\partial_h \sigma_j^\gamma, \delta_j^h \delta_\alpha^\gamma). \tag{4.5}$$

Let $V$ be a vector field on $M$ and ${}^C V$ its complete lift to $F^*M$, which is locally given by (2.1):

$${}^C V = {}^C V^h \partial_h + {}^C V^{h_\alpha} \partial_{h_\alpha} = V^h \partial_h - X_j^\alpha (\partial_h V^j) \partial_{h_\alpha}. \tag{4.6}$$

On the other hand, the complete lift ${}^C V$ has the following decomposition with respect to the $(B, C)$-frame along the cross-section $\sigma(M)$:

$${}^C V = \tilde{V}^i B_i + \tilde{V}^{i_\beta} C_{i_\beta}. \tag{4.7}$$

Thus, from (4.6) and (4.7) we have

$$\begin{aligned} {}^C V^h \partial_h + {}^C V^{h_\alpha} \partial_{h_\alpha} &= \tilde{V}^i B_i + \tilde{V}^{i_\beta} C_{i_\beta} = \tilde{V}^i B_i^h \partial_h + \tilde{V}^i B_i^{h_\alpha} \partial_{h_\alpha} \\ &+ \tilde{V}^{i_\beta} C_{i_\beta}^h \partial_h + \tilde{V}^{i_\beta} C_{i_\beta}^{h_\alpha} \partial_{h_\alpha} = \left(\tilde{V}^i B_i^h + \tilde{V}^{i_\beta} C\,{}_{i_\beta}^{h}\right) \partial_h \\ &+ \left(\tilde{V}^i B_i^{h_\alpha} + \tilde{V}^{i_\beta} C\,{}_{i_\beta}^{h_\alpha}\right) \partial_{h_\alpha}. \end{aligned} \tag{4.8}$$

By using (4.2) and (4.4), from (4.8) we obtain:

$${}^C V^h = \tilde{V}^i B_i^h + \tilde{V}^{i_\beta} C_{i_\beta}^h = \tilde{V}^i \delta_i^h = \tilde{V}^h,$$

$${}^C V^{h_\alpha} = -\sigma_j^\alpha \partial_h V^j = \tilde{V}^i \partial_i \sigma_h^\alpha + \tilde{V}^{i_\beta} C_{i_\beta}^{h_\alpha} = V^i \partial_i \sigma_h^\alpha + \tilde{V}^{i_\beta} \delta_h^i \delta_\beta^\alpha.$$

Thus the complete lift ${}^C V$ of a vector field $V$ in $M$ to $F^*M$, having components (2.1) with respect to the natural frame, has components

$$\begin{pmatrix} V^h \\ -L_V \sigma_h^\alpha \end{pmatrix}$$

with respect to the frame $(B, C)$ along the cross-section $\sigma(M)$.

This means that

$$^{C}V = V^{h} B_{h}^{A} - (L_{V} \sigma_{h}^{\alpha})\, C_{h_{\alpha}}^{A}.$$

From here follows

**Theorem 3** *The complete lift $^{C}V$ of a vector field $V$ in $M$ to $F^{*}M$ is tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^{1}, ..., \sigma^{n})$ if and only if the Lie derivative of each $\sigma^{\alpha}$ with respect to $V$ vanishes, i.e. $L_{V}\sigma^{\alpha} = 0,\ 1 \leq \alpha \leq n$.*

By analogy, the horizontal lift $^{H}V$ of a vector field $V$ in $M$ to $F^{*}M$, having components (2.2) with respect to the natural frame, has components

$$\begin{pmatrix} V^{h} \\ -\nabla_{V} \sigma_{h}^{\alpha} \end{pmatrix}$$

with respect to the frame $(B, C)$ along the cross-section $\sigma(M)$, where $\nabla_{V}$ is a covariant derivative along a vector field $V$ in an affine connection $\nabla$. Therefore

$$^{H}V = V^{h} B_{h}^{A} - (\nabla_{V} \sigma_{h}^{\alpha})\, C_{h_{\alpha}}^{A},$$

from which follows

**Theorem 4** *The horizontal lift $^{H}V$ of a vector field $V$ in $M$ to $F^{*}M$ is tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^{1}, ..., \sigma^{n})$ if and only if the covariant derivative of each $\sigma^{\alpha}$ with respect to $V$ vanishes, i.e. $\nabla_{V}\sigma^{\alpha} = 0,\ 1 \leq \alpha \leq n$.*

## 5. Lifts of affinor fields on cross-sections

Let $\varphi$ be an affinor field on $M$ and $^{C}\varphi$ its complete lift to $F^{*}M$, which is locally given by (3.1) with respect to the natural frame, i.e.

$$^{C}\varphi = \begin{pmatrix} \varphi_{i}^{h} & 0 \\ X_{k}^{\alpha}(\partial_{i}\varphi_{h}^{k} - \partial_{h}\varphi_{i}^{k}) & \varphi_{h}^{i}\delta_{\beta}^{\alpha} \end{pmatrix}. \tag{5.1}$$

If $^{C}\tilde{\varphi}_{J}^{I}$ are components of the complete lift $^{C}\varphi$ with respect to the $(B, C)$-frame along the cross-section $\sigma(M)$, then we have

$$^{C}\varphi_{I}^{J} = {}^{C}\tilde{\varphi}_{H}^{A} E_{A}^{J} \tilde{E}_{I}^{H}. \tag{5.2}$$

By using (4.2), (4.4), (4.5), and (5.1) we have

$$1)^{C}\varphi_{i}^{j} = \varphi_{i}^{j} = {}^{C}\tilde{\varphi}_{h}^{a}\delta_{a}^{j}\delta_{i}^{h} + {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a}\delta_{a}^{j}(-\partial_{i}\sigma_{h}^{\alpha}) = {}^{C}\tilde{\varphi}_{i}^{j} - {}^{C}\tilde{\varphi}_{h_{\alpha}}^{j}(\partial_{i}\sigma_{h}^{\alpha}), \tag{5.3}$$

$$2)^{C}\varphi_{i_{\beta}}^{j} = 0 = {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a} E_{a}^{j} \tilde{E}_{i_{\beta}}^{h_{\alpha}} = {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a}\delta_{a}^{j}\delta_{h}^{i}\delta_{\beta}^{\alpha},$$

from which it follows that

$$^{C}\tilde{\varphi}_{h_{\alpha}}^{a} = 0. \tag{5.4}$$

Using (5.4), from (5.3) we get

$$^{C}\tilde{\varphi}_{h}^{a} = \varphi_{h}^{a}.$$

3) ${}^{C}\varphi_{i_\beta}^{j_\gamma}=\varphi_j^i\delta_\beta^\gamma={}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_{i_\beta}^{h_\alpha}={}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_h^i\delta_\beta^\alpha$, consequently

$${}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}=\varphi_a^h\delta_\alpha^\tau.$$

$$4)\,{}^{C}\varphi_i^{j_\gamma}=\sigma_k^\gamma\partial_i\varphi_j^k-\sigma_k^\gamma\partial_j\varphi_i^k={}^{C}\tilde{\varphi}_h^aE_a^{j_\gamma}\tilde{E}_i^h+{}^{C}\tilde{\varphi}_h^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^h+{}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^{h_\alpha}$$

$$=\varphi_h^a\partial_a\sigma_j^\gamma\delta_i^h+{}^{C}\tilde{\varphi}_h^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_i^h+\varphi_a^h\delta_\alpha^\tau\delta_j^a\delta_\tau^\gamma(-\partial_i\sigma_h^\alpha)$$

or

$${}^{C}\tilde{\varphi}_h^{a_\sigma}\delta_j^a\delta_\sigma^\gamma\delta_i^h=\sigma_k^\gamma\partial_i\varphi_j^k-\sigma_k^\gamma\partial_j\varphi_i^k-\varphi_i^k\partial_k\sigma_j^\gamma+\varphi_j^h\partial_i\sigma_h^\gamma,$$

from which we obtain

$${}^{C}\tilde{\varphi}_h^{a_\tau}=\sigma_k^\tau\partial_h\varphi_a^k-\sigma_k^\tau\partial_a\varphi_h^k-\varphi_h^k\partial_k\sigma_a^\tau+\varphi_a^k\partial_h\sigma_k^\tau=-(\varphi_h^k\partial_k\sigma_a^\tau$$

$$-\varphi_a^k\partial_h\sigma_k^\tau-\sigma_k^\tau\partial_h\varphi_a^k+\sigma_k^\tau\partial_a\varphi_h^k)=-(\Phi_\varphi\sigma^\tau)_{ha},$$

where $\Phi_\varphi\sigma^\tau$ is the Tachibana operator applied to $\sigma^\tau$ (see [7]).
Thus we have

**Theorem 5** *The complete lift ${}^{C}\varphi$ having components (5.1) with respect to the natural frame has the nonzero components*

$${}^{C}\tilde{\varphi}_h^a=\varphi_h^a,\quad {}^{C}\tilde{\varphi}_h^{a_\tau}=-(\Phi_\varphi\sigma^\tau)_{ha},\quad {}^{!}\tilde{\varphi}_{h_\alpha}^{a_\tau}=\varphi_a^h\delta_\alpha^\tau$$

*with respect to the frame $(B,C)$ along the cross-section $\sigma(M)$.*

Now we assume that ${}^{H}\varphi$ is the horizontal lift of the affinor field $\varphi$ to $F^*M$, given by (3.7) with respect to the natural frame, i.e.

$${}^{H}\varphi=\begin{pmatrix}\varphi_i^h & 0\\ X_k^\alpha(\varphi_i^m\Gamma_{mh}^k-\varphi_h^m\Gamma_{im}^k) & \varphi_h^i\delta_\beta^\alpha\end{pmatrix}.\tag{5.5}$$

On the other hand, the horizontal lift ${}^{H}\varphi$ has the following decomposition with respect to the $(B,C)$-frame along the cross-section $\sigma(M)$:

$${}^{H}\varphi_I^J={}^{H}\tilde{\varphi}_H^AE_A^J\tilde{E}_I^H.\tag{5.6}$$

Using (3.7), (3.2), (3.4), and (5.5) we find

$$1)\,{}^{H}\varphi_i^j=\varphi_i^j={}^{H}\tilde{\varphi}_h^a\delta_a^j\delta_i^h+{}^{H}\tilde{\varphi}_{h_\alpha}^a\delta_a^j(-\partial_i\sigma_h^\alpha)={}^{H}\tilde{\varphi}_i^j-{}^{H}\tilde{\varphi}_{h_\alpha}^j(\partial_i\sigma_h^\alpha).\tag{5.7}$$

2) ${}^{H}\varphi_{i_\beta}^j=0={}^{H}\tilde{\varphi}_{h_\alpha}^aE_a^j\tilde{E}_{i_\beta}^{h_\alpha}={}^{H}\tilde{\varphi}_{h_\alpha}^a\delta_a^j\delta_h^i\delta_\beta^\alpha$, consequently

$${}^{H}\tilde{\varphi}_{h_\alpha}^a=0.\tag{5.8}$$

Based on equality (5.8), from (5.7) we get

$${}^{H}\tilde{\varphi}_h^a=\varphi_h^a.$$

$$3)\,{}^{H}\varphi_{i_\beta}^{j_\gamma}=\varphi_j^i\delta_\beta^\gamma={}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_{i_\beta}^{h_\alpha}={}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_h^i\delta_\beta^\alpha,$$

from which it follows that

$$^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau} = \varphi_a^h \delta_\alpha^\tau.$$

$$4)^{H}\varphi_i^{j_\gamma} = \sigma_k^\gamma \varphi_i^m \Gamma_{mj}^k - \sigma_k^\gamma \varphi_j^m \Gamma_{mi}^k = {}^{H}\tilde{\varphi}_h^a E_a^{j_\gamma} \tilde{E}_i^h + {}^{H}\tilde{\varphi}_h^{a_\tau} E_{a_\tau}^{j_\gamma} \tilde{E}_i^h$$

$$+{}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau} E_{a_\tau}^{j_\gamma} \tilde{E}_i^{h_\alpha} = \varphi_h^a \partial_a \sigma_j^\gamma \delta_i^h + {}^{H}\tilde{\varphi}_h^{a_\tau} \delta_j^a \delta_\tau^\gamma \delta_i^h + \varphi_a^h \delta_\alpha^\tau \delta_j^a \delta_\tau^\gamma (-\partial_i \sigma_h^\alpha)$$

or

$$^{H}\tilde{\varphi}_h^{a_\tau} \delta_j^a \delta_\tau^\gamma \delta_i^h = \sigma_k^\gamma \varphi_i^m \Gamma_{mj}^k - \sigma_k^\gamma \varphi_j^m \Gamma_{mi}^k - \varphi_i^k \partial_k \sigma_j^\gamma + \varphi_j^h \partial_i \sigma_h^\gamma,$$

from which we obtain

$$^{H}\tilde{\varphi}_h^{a_\tau} = \sigma_k^\tau \varphi_h^m \Gamma_{ma}^k - \sigma_k^\tau \varphi_a^m \Gamma_{mh}^k - \varphi_h^k \partial_k \sigma_a^\tau + \varphi_a^k \partial_h \sigma_k^\tau$$

$$= -\varphi_h^k (\partial_k \sigma_a^\tau - \Gamma_{ka}^m \sigma_m^\tau) + \varphi_a^k (\partial_h \sigma_k^\tau - \Gamma_{kh}^m \sigma_m^\tau) = -\varphi_h^k \nabla_k \sigma_a^\tau + \varphi_a^k \nabla_h \sigma_k^\tau$$

$$= -(\varphi_h^k \nabla_k \sigma_a^\tau - \varphi_a^k \nabla_h \sigma_k^\tau) = -(\tilde{\Phi}_\varphi \sigma^\tau)_{ha},$$

where $\tilde{\Phi}_\varphi \sigma^\tau$ is the Vishnevskii operator applied to $\sigma^\tau$ (see [7]).
Thus we have

**Theorem 6** *The horizontal lift $^{H}\varphi$ having the nonzero components (5.5) with respect to the natural frame has the nonzero components*

$$^{H}\tilde{\varphi}_h^a = \varphi_h^a, \quad {}^{H}\tilde{\varphi}_h^{a_\tau} = -(\tilde{\Phi}_\varphi \sigma^\tau)_{ha}\,, \quad {}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau} = \varphi_a^h \delta_\alpha^\tau$$

*with respect to the frame $(B, C)$ along the cross-section $\sigma(M)$.*

## 6. Complete lift of almost complex structure on cross-sections

Suppose that the manifold $M$ has an almost complex structure $F$. Its mean that $F^2 = -I$. We have

**Theorem 7** *Let $M$ be a differentiable manifold with an almost complex structure $F$. Then the complete lift $^{C}F$ of $F$ to $F^*M$ is an almost complex structure if and only if $X_k^\beta Q(F,F)_{ij}^k = 0$, where $Q(F,F)-$ the Nijenhuis–Shirokov tensor of $F$ (see [5]).*

**Proof** From (5.1) we have

$$1)^{C}F_i^{H\,C}F_H^j = {}^{C}F_i^{h\,C}F_h^j + {}^{C}F_i^{h_\gamma\,C}F_{h_\gamma}^j = F_i^h F_h^j = -\,\delta_i^j = -{}^{C}I_i^j,$$

$$2)^{C}F_{i_\alpha}^{H\,C}F_H^j = {}^{C}F_{i_\alpha}^{h\;C}F_h^j + {}^{C}F_{i_\alpha}^{h_\gamma\,C}F_{h_\gamma}^j = 0 = -{}^{C}I_{i_\alpha}^j,$$

$$3)^{C}F_{i_\alpha}^{H\,C}F_H^{j_\beta} = {}^{C}F_{i_\alpha}^{h\;C}F_h^{j_\beta} + {}^{C}F_{i_\alpha}^{h_\gamma\,C}F_{h_\gamma}^{j_\beta} = F_h^i \delta_\alpha^\gamma F_j^h \delta_\gamma^\beta = -\delta_j^i \delta_\alpha^\beta =$$

$$= -{}^{C}I_{i_\alpha}^{j_\beta}, \tag{6.1}$$

$$4)^{C}F_i^{H\,C}F_H^{j_\beta} = {}^{C}F_i^{h\,C}F_h^{j_\beta} + {}^{C}F_i^{h_\gamma\,C}F_{h_\gamma}^{j_\beta} = F_i^h X_k^\beta (\partial_h F_j^k - \partial_j F_h^k)$$

$$+X_k^\gamma (\partial_i F_h^k - \partial_h F_i^k) F_j^h \delta_\gamma^\beta = X_k^\beta (F_i^h \partial_h F_j^k - F_i^h \partial_j F_h^k + F_j^h \partial_i F_h^k$$

$$-F_j^h\partial_h F_i^k) = X_k^\beta(\partial_i(F_j^h F_h^k) - \partial_j(F_i^h F_h^k)) + X_k^\beta(F_i^h\partial_h F_j^k$$

$$-F_j^h\partial_h F_i^k - F_h^k\partial_i F_j^h + F_h^k\partial_j F_i^h) = -{}^C I_i^{j\beta} + X_k^\beta Q(F,F)_{ij}^k.$$

From (6.1) we obtain

$$({}^C F)^2 = {}^C(F^2) + \gamma(X \circ Q(F,F)), \tag{6.2}$$

where

$$\gamma(X\circ Q(F,F)) = \begin{pmatrix} 0 & 0 \\ X_k^\beta Q(F,F)_{ij}^k & 0 \end{pmatrix}.$$

Equation (6.2) completes the proof of Theorem 7. □

The complete lift ${}^C F$ having the components (5.1) with respect to the natural frame has the components

$$\begin{pmatrix} F_i^h & 0 \\ \sigma_k^\alpha(\partial_i F_h^k - \partial_h F_i^k) - F_i^k\partial_k\sigma_h^\alpha + F_h^k\partial_k\sigma_i^\alpha & F_h^i\delta_\beta^\alpha \end{pmatrix} \tag{6.3}$$

with respect to the frame $(B,C)$ along the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^1,...,\sigma^n)$.

It is well known that for an arbitrary almost analytic 1-form (or almost analytic covector field) $\sigma$ on a differentiable manifold $M$ with an almost complex structure $F$, we have the relation

$$\sigma \circ N_F = 0$$

(see [8]), where $N_F$ is the Nijenhuis tensor for $F$ ([6, p. 38]).

Now by using (6.3) along the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^1,...,\sigma^n)$ on $M$, similarly to (6.1) we obtain

$$({}^C F)^2 = {}^C(F^2) + \gamma(\sigma^\beta \circ N_F), \tag{6.4}$$

where

$$\gamma(\sigma^\beta\circ N_\varphi) = \begin{pmatrix} 0 & 0 \\ \sigma_k^\beta N_{ij}^k & 0 \end{pmatrix}.$$

Thus from (6.4) we have

**Theorem 8** *Let $M$ be a differentiable manifold with an almost complex structure $F$. Then the complete lift ${}^C F \in \Im_1^1(F^*M)$ of $F$, which is restricted to the cross-section $\sigma(M)$ determined by an almost analytic covector field $\sigma^1,...,\sigma^n$ on $M$, is an almost complex structure.*

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