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Second Hankel determinant for certain subclasses of bi-univalent functions involving Chebyshev polynomials

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Abstract: In this paper our purpose is to find the upper bound estimate for the second Hankel determinant $|a_2a_4 - a_3^2|$ for functions defined by convolution belonging to the class $\mathcal{N}_\sigma^{\mu,\delta}(\lambda, t)$ by using Chebyshev polynomials.

Key words: Univalent function, bi-univalent function, coefficient bounds, Chebyshev polynomial, Hankel determinant, convolution

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{n\geq 2} a_n z^n. \tag{1.1}$$

Because of the Koebe one-quarter theorem it is well known that every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{U}) \to \mathbb{U}$ satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w, \quad (|w| < 1/4).$$

Moreover, it is easy to check that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ..., \quad w \in f(\mathbb{U}). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse $g = f^{-1}$ are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history of functions in the class σ, and also various other properties of the bi-univalent function, one can see recent works [2, 8, 12, 21, 26] and the references therein.

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Some of the prominent and well-examined subclasses of univalent functions of class $\mathcal{S}$ are those of the class $\mathcal{S}^\star(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. Moreover, by means of the analytic descriptions, we have

$$\mathcal{S}^\star(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha; \quad z \in \mathbb{U}; \quad 0 \leq \alpha < 1 \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha; \quad z \in \mathbb{U}; \quad 0 \leq \alpha < 1 \right\}.$$

For $0 \leq \alpha < 1$, a function $f \in \sigma$ is in the class $\mathcal{S}^\star_\sigma(\alpha)$ of bi-starlike function of order α, or $\mathcal{K}_\sigma(\alpha)$ of bi-convex function of order α if both f and its inverse f^{-1} are, respectively, starlike or convex functions of order α.

We say that $f \in \mathcal{A}$ is subordinate to the function $g \in \mathcal{A}$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$.

For $f(z)$ given by (1.1) and $\Theta(z)$ defined by

$$\Theta(z) = z + \sum_{n\geq 2} \theta_n z^n, \quad (\theta_n \geq 0), \tag{1.3}$$

the Hadamard product (or convolution) $(f * \Theta)(z)$ of the functions $f(z)$ and $\Theta(z)$ is defined by

$$(f * \Theta)(z) = z + \sum_{n\geq 2} a_n \theta_n z^n = (\Theta * f)(z). \tag{1.4}$$

Next, we consider the function

$$\begin{aligned} f_\delta(z) &= \int_0^z \left(\frac{1+r}{1-r}\right)^\delta \frac{1}{1-r^2} dr \\ &= z + \delta z^2 + \frac{1}{3}(2\delta^2 + 1)z^3 + \ldots \\ &= z + \sum_{n\geq 2} b_n(\delta) z^n, \quad (\delta > 0, \quad z \in \mathbb{U}). \end{aligned} \tag{1.5}$$

It is worth mentioning that if $\delta < 1$, then $zf'_\delta(z)$ is starlike with two slits. Moreover, we can see that since $zf'_1(z)$ is the Koebe function, all the functions f_δ are univalent and convex in $\mathbb{U}$. For more detail about the function $f_\delta(z)$ one can refer to [27]. If we put the function $f_\delta(z)$ defined by (1.5) in for the function $\Theta(z)$ given by (1.3) in the equality (1.4), we have

$$h_\delta(z) = (f * f_\delta)(z) = z + \sum_{n\geq 2} a_n b_n(\delta) z^n = (f_\delta * f)(z). \tag{1.6}$$

In 1976, Noonan and Thomas [20] defined the qth Hankel determinant of f given by (1.1) for integers $n \geq 1$ and $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

This determinant has been investigated by several authors in the literature [13, 20]. For instance, this determinant is useful in showing that a function of bounded characteristic in $\mathbb{U}$, i.e. a function that is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [9]. Moreover, it is important to mention that the Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$ are well known as Fekete–Szegö and second Hankel determinant functionals, respectively. In 1969, the Fekete–Szegö problem for the classes $\mathcal{S}^\star$ and $\mathcal{K}$ was investigated by Keogh and Merkes [17]. Recently, many authors have discussed upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions (see [1, 11, 18] and the references therein). Very recently, the upper bounds of $H_2(2)$ for the classes $\mathcal{S}^\star_\sigma(\alpha)$ and $\mathcal{K}_\sigma(\alpha)$ were investigated by Deniz et al. [12]. Later, the works were extended by Orhan et al. [23, 24] and Altınkaya and Yalçın [4].

Chebyshev polynomials, which are used by us in this study, play an important role in many branches of mathematics, especially in numerical analysis (see [10]). We know that there are several kinds of Chebyshev polynomials. In particular, we shall introduce the first and second kind of polynomials, $T_n(x)$ and $U_n(x)$. For a brief history of the Chebyshev polynomials of first kind $T_n(x)$ and second kind $U_n(x)$ and their numerous uses in different applications, one can refer [3, 14, 15].

The most remarkable kinds of the Chebyshev polynomials are the first and second kinds, and in the case of real variable x on $(-1, 1)$ they are defined by

$$T_n(x) = \cos(n \arccos x) \quad U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)} = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}.$$

Now we consider the function that is the generating function of a Chebyshev polynomial:

$$G(t,z) = \frac{1}{1-2tz+z^2}, \quad t \in (\frac{1}{2}, 1), z \in \mathbb{U}.$$

It is well known that if $t = \cos\theta$, $\quad t = (-\pi/3, \pi/3)$, then

$$\begin{aligned} G(t,z) =& 1 + \sum_{n\geq 1} \frac{\sin(n+1)\theta}{\sin\theta} z^n \\ =& 1 + 2\cos\theta z + (3\cos^2\theta - \sin^2\theta)z^2 + ..., \quad (z \in \mathbb{U}). \end{aligned}$$

That is, in view of [28], we can write

$$G(t,z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + ..., \quad t \in (\frac{1}{2}, 1), z \in \mathbb{U}, \tag{1.7}$$

where $U_n(t)$ stands for the second kind of Chebyshev polynomials. From the definition of the second kind of Chebyshev polynomials, we easily arrive at $U_1(t) = 2t$. Also, it is well known that we have the following recurrence relation:

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t),$$

for all $n \in \mathbb{N}$. From here, we can easily obtain

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, ... \tag{1.8}$$

Definition 1 *For $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 1$ and $t \in (1/2, 1]$, a function $h_\delta \in \sigma$ given by (1.6) is said to be in class $\mathcal{N}_\sigma^{\mu,\delta}(\lambda, t)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:*

$$(1-\lambda)(\frac{h_\delta(z)}{z})^\mu + \lambda h'_\delta(z)(\frac{h_\delta(z)}{z})^{\mu-1} \prec G(z,t) \tag{1.9}$$

and

$$(1-\lambda)(\frac{k_\delta(w)}{w})^\mu + \lambda k'_\delta(w)(\frac{k_\delta(w)}{w})^{\mu-1} \prec G(w,t), \tag{1.10}$$

where the function $k_\delta = h_\delta^{-1}$ is defined by (1.2).

Obviously, for $\delta = 1$, we get that $\mathcal{N}_\sigma^{\mu,1}(\lambda, t) = \mathcal{N}_\sigma^{\mu}(\lambda, t)$. It is important to mention that the class $\mathcal{N}_\sigma^{\mu}(\lambda, t)$ was introduced and investigated by Bulut et al. [6]. They also discussed initial coefficient estimates and Fkete–Szegö bounds for the class $\mathcal{N}_\sigma^{\mu}(\lambda, t)$ and its subclasses, given in the following remark.

Remark 1 *(i) For $\delta = 1$ and $\mu = 1$, we get the class $\mathcal{N}_\sigma^{1,1}(\lambda, t) = \mathcal{B}_\sigma(\lambda, t)$ consisting of functions $f \in \sigma$ satisfying the condition*

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec G(z,t)$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec G(w,t)$$

where the function $g = f^{-1}$ is defined by (1.2). This class was introduced and studied by Bulut et al. [7] (see also [19]).

(ii) For $\delta = 1$ and $\lambda = 1$, we obtain the class $\mathcal{N}_\sigma^{\mu,1}(1, t) = \mathcal{B}_\sigma^{\mu}(t)$ consisting of bi-Bazilevič functions:

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec G(z,t)$$

and

$$g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec G(w,t),$$

where the function $g = f^{-1}$ is defined by (1.2). This class was introduced and studied by Altınkaya and Yalçın [5].

(iii) For $\delta = 1$, $\mu = 1$, and $\lambda = 1$, we have the class $\mathcal{N}_\sigma^{1,1}(1, t) = \mathcal{B}_\sigma(t)$ consisting of functions f satisfying the condition

$$f'(z) \prec G(z,t)$$

and

$$g'(w) \prec G(w,t),$$

where the function $g = f^{-1}$ is defined by (1.2).

(iv) For $\delta = 1$, $\lambda = 1$, *and* $\mu = 0$, *we have the class* $\mathcal{N}_\sigma^{0,1} = \mathcal{S}_\sigma^\star(t)$ *satisfying the condition*

$$\frac{zf'(z)}{f(z)} \prec G(z,t)$$

and

$$\frac{wg'(z)}{g(w)} \prec G(w,t),$$

where the function $g = f^{-1}$ *is defined by* (1.2).

Let us take a look at some lemmas that are very useful in building our main results.

Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{U}$ such that $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$, $z \in \mathbb{U}$. We know that this class is usually called the Carathéodory class.

Lemma 1 *(see [25]) If the function* $p \in \mathcal{P}$ *is given by the following series:*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{1.11}$$

then the sharp estimate given by

$$|c_n| \leq 2 \quad (n = 1, 2, 3, ...) \tag{1.12}$$

holds true.

Lemma 2 *[16] If the function* $p \in \mathcal{P}$ *is given by the series* (1.11), *then*

$$\begin{aligned} 2c_2 =& c_1^2 + x(4 - c_1^2), \\ 4c_3 =& c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \end{aligned}$$

for some x *and* z *with* $|x| \leq 1$ *and* $|y| \leq 1$.

In the present investigation, we seek the upper bound for the second Hankel determinant for functions h_δ belonging to the class $\mathcal{N}_\sigma^{\mu,\delta}(\lambda, t)$ by making use of the Chebyshev polynomial expansions and the Hadamard product. Also, we give some remarkable consequences related to the class $\mathcal{N}_\sigma^{\mu,\delta}(\lambda, t)$.

2. Main results

Theorem 1 *Let* $h_\delta \in \sigma$ *of the form* (1.6) *be in* $\mathcal{N}_\sigma^{\mu,\delta}(\lambda; t)$. *Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \varphi(2^-, t), & \chi_1 \geq 0 \quad \chi_2 \geq 0 \\ \frac{36t^2}{(2\delta^2+1)^2(2\lambda+\mu)^2}, & \chi_1 \leq 0 \quad \chi_2 \leq 0 \\ \max\left\{\frac{36t^2}{(2\delta^2+1)^2(2\lambda+\mu)^2}, \quad \varphi(2^-, t)\right\}, & \chi_1 > 0 \quad \chi_2 < 0 \\ \max\left\{\varphi(c_0, t), \varphi(2^-, t)\right\}, & \chi_1 < 0 \quad \chi_2 > 0 \end{cases},$$

where

$$\varphi(2^-,t) = \frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2} + \frac{\chi_1+9\chi_2}{6\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4},$$

$$\varphi(c_0,t) = -\frac{27\chi_2^2}{8\chi_1\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4} + \frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2}, \qquad c_0 = \sqrt{\frac{-18\chi_2}{\chi_1}}$$

and

$$\chi_1 = 18(\lambda+\mu)^3\Big(3\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu) - (2\delta^2+1)^2(2\lambda+\mu)^2\Big)U_1^2(t) + (2\lambda+\mu)^2U_1(t)\left|\Omega_{\lambda,\mu,\delta}(t)\right| - 9(\lambda+\mu)^2(2\lambda+\mu)U_1(t)\Big((3\lambda+\mu)(8\delta^4-4\delta^2+5)U_1^2(t) + 4(2\delta^2+1)^2(2\lambda+\mu)(\lambda+\mu)U_2(t)\Big),$$

$$\chi_2 = \Big[(2\lambda+\mu)(3\lambda+\mu)\left(8\delta^4-4\delta^2+5\right)U_1^3(t) + 4(2\delta^2+1)^2(\lambda+\mu)(2\lambda+\mu)^2U_1(t)U_2(t) + (\lambda+\mu)U_1^2(t)\big(2(2\delta^2+1)^2(2\lambda+\mu)^2 - 12\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)\big)\Big](\lambda+\mu)^2,$$

$$\Omega_{\lambda,\mu,\delta}(t) = 18(2\delta^2+1)^2(\lambda+\mu)^3U_3(t) - U_1^3(t)(3\lambda+\mu)\Big(3(2\delta^2+1)^2(\mu^2+3\mu-4) + 54\delta(\delta^3+2\delta)\Big).$$

Proof Let $h_\delta \in \mathcal{N}_\sigma^{\mu,\delta}(\lambda,t)$. Then we have

$$(1-\lambda)(\frac{h_\delta(z)}{z})^\mu + \lambda h'_\delta(z)(\frac{h_\delta(z)}{z})^{\mu-1} = G(t,u(z)) \tag{2.1}$$

and

$$(1-\lambda)(\frac{k_\delta(w)}{w})^\mu + \lambda k'_\delta(w)(\frac{k_\delta(w)}{w})^{\mu-1} = G(t,v(w)) \tag{2.2}$$

where $p_1, p_2 \in \mathcal{P}$ and defined by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + ... \tag{2.3}$$

and

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1w + d_2w^2 + d_3w^3 + ... \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + ...\right] \tag{2.5}$$

and

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[d_1 w + \left(d_2 - \frac{d_1^2}{2}\right)w^2 + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)w^3 + ...\right]. \tag{2.6}$$

Using (2.5) together with (2.6), and taking $G(z,t)$ as given in (1.7), we get that

$$\begin{aligned} G(t,u(z)) =& 1 + \frac{U_1(t)}{2}c_1 z + \left[\frac{U_1(t)}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{U_2(t)}{4}c_1^2\right]z^2 \\ &+ \left[\frac{U_1(t)}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{U_2(t)}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{U_3(t)}{8}c_1^3\right]z^3 + ... \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} G(t,v(w)) =& 1 + \frac{U_1(t)}{2}d_1 w + \left[\frac{U_1(t)}{2}\left(d_2 - \frac{d_1^2}{2}\right) + \frac{U_2(t)}{4}d_1^2\right]w^2 \\ &+ \left[\frac{U_1(t)}{2}\left(d_3 - d_1d_2 + \frac{d_1^3}{4}\right) + \frac{U_2(t)}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{U_3(t)}{8}d_1^3\right]w^3 + ... \end{aligned} \tag{2.8}$$

By considering (2.1), (2.7) and (2.2), (2.8), when some elementary calculations are done, we get that

$$(\lambda+\mu)a_2b_2(\delta) = \frac{U_1(t)}{2}c_1, \tag{2.9}$$

$$(2\lambda+\mu)\left[a_3b_3(\delta) + (\mu-1)\frac{a_2^2b_2^2(\delta)}{2}\right] = \frac{U_1(t)}{2}(c_2 - \frac{c_1^2}{2}) + \frac{U_2(t)}{4}c_1^2, \tag{2.10}$$

$$\begin{aligned} (3\lambda+\mu)&\left[a_4b_4(\delta) + (\mu-1)a_2a_3b_2(\delta)b_3(\delta) + (\mu-1)(\mu-2)\frac{a_2^3b_2^3(\delta)}{6}\right] = \\ &= \frac{U_1(t)}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{U_2(t)}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{U_3(t)}{8}c_1^3, \end{aligned} \tag{2.11}$$

and

$$-(\lambda+\mu)a_2b_2(\delta) = \frac{U_1(t)}{2}d_1, \tag{2.12}$$

$$(2\lambda+\mu)\left[(\mu+3)\frac{a_2^2b_2^2(\delta)}{2} - a_3b_3(\delta)\right] = \frac{U_1(t)}{2}(d_2 - \frac{d_1^2}{2}) + \frac{U_2(t)}{4}d_1^2, \tag{2.13}$$

$$\begin{aligned} (3\lambda+\mu)&\left[(\mu+4)a_2a_3b_2(\delta)b_3(\delta) - (\mu+4)(\mu+5)\frac{a_2^3b_2^3(\delta)}{6} - a_4b_4(\delta)\right] \\ &= \frac{U_1(t)}{2}\left(d_3 - d_1d_2 + \frac{d_1^3}{4}\right) + \frac{U_2(t)}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{U_3(t)}{8}d_1^3. \end{aligned} \tag{2.14}$$

Using (2.9) along with (2.12), we find that

$$c_1 = -d_1 \tag{2.15}$$

and

$$a_2 = \frac{U_1(t)}{2\delta(\lambda+\mu)} c_1. \tag{2.16}$$

Now, from (2.10), (2.13), and (2.16), we obtain that

$$a_3 = \frac{3}{2\delta^2+1}\left[\frac{U_1^2(t)}{4(\lambda+\mu)^2}c_1^2 + \frac{U_1(t)}{4(2\lambda+\mu)}(c_2-d_2)\right]. \tag{2.17}$$

Also, subtracting (2.14) from (2.11) and using (2.16) together with (2.17), we get that

$$\begin{aligned} a_4 =& \frac{3}{\delta^3+2\delta}\Bigg[\left(\frac{U_1(t)-2U_2(t)+U_3(t)}{8(3\lambda+\mu)} - \frac{(\mu^2+3\mu-4)U_1^3(t)}{48(\lambda+\mu)^3}\right)c_1^3 + \frac{U_1(t)}{4(3\lambda+\mu)}(c_3-d_3) \\ &+ \frac{5U_1^2(t)}{16(\lambda+\mu)(2\lambda+\mu)}c_1(c_2-d_2) + \frac{U_2(t)-U_1(t)}{4(3\lambda+\mu)}c_1(c_2+d_2)\Bigg]. \end{aligned} \tag{2.18}$$

Thus, we can easily determine that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| =& \Bigg|\frac{U_1(t)\Delta_{\lambda,\mu,\delta}(t)}{96\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(\lambda+\mu)^4}c_1^4 - \frac{9U_1^2(t)}{16(2\delta^2+1)^2(2\lambda+\mu)^2}(c_2-d_2)^2 \\ &+ \frac{3U_1(t)\left[U_2(t)-U_1(t)\right]}{8\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}c_1^2(c_2+d_2) + \frac{3U_1^2(t)}{8\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}c_1(c_3-d_3) \\ &+ \frac{U_1^3(t)\left[15(2\delta^2+1)^2 - 36\delta(\delta^3+2\delta)\right]}{32\delta(\delta^3+2\delta)(2\delta^2+1)^2(\lambda+\mu)^2(2\lambda+\mu)}c_1^2(c_2-d_2)\Bigg|, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \Delta_{\lambda,\mu,\delta}(t) =& 18(2\delta^2+1)^2(\lambda+\mu)^3((U_1(t)-2U_2(t)+U_3(t)) \\ &- U_1^3(t)(3\lambda+\mu)\big(3(2\delta^2+1)^2(\mu^2+3\mu-4)+54\delta(\delta^3+2\delta)\big). \end{aligned}$$

In view of Lemma 2 and (2.15), we write

$$c_2 - d_2 = \frac{4-c_1^2}{2}(x-y), \tag{2.20}$$

$$c_2 + d_2 = c_1^2 + \frac{4-c_1^2}{2}(x+y), \tag{2.21}$$

$$c_3 - d_3 = \frac{c_1^3}{2} + \frac{(4-c_1^2)c_1}{2}(x+y) - \frac{(4-c_1^2)c_1}{4}(x^2+y^2) + \frac{4-c_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right], \tag{2.22}$$

for some x, y and z, w with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$. Next, we will need to plug the last three equations given by (2.20), (2.21), and (2.22) into the Hankel functional given by equation (2.19). Also, with the help of an application of the triangle inequality, we have

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| &\leq \frac{U_1(t)\left|\Omega_{\lambda,\mu,\delta}(t)\right|}{96\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(\lambda+\mu)^4}c_1^4+\frac{3U_1^2(t)c_1(4-c_1^2)}{8\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\\
&+\left[\frac{\left(15(2\delta^2+1)^2-36\delta(\delta^3+2\delta)\right)U_1^3c_1^2(4-c_1^2)}{64\delta(\delta^3+2\delta)(2\delta^2+1)^2(\lambda+\mu)^2(2\lambda+\mu)}+\frac{3U_1(t)U_2(t)c_1^2(4-c_1^2)}{16\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\right](|x|+|y|)\\
&+\left[\frac{3U_1^2(t)c_1^2(4-c_1^2)}{32\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}-\frac{3U_1^2(t)c_1(4-c_1^2)}{16\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\right](|x|^2+|y|^2)\\
&+\frac{9U_1^2(t)(4-c_1^2)^2}{64(2\delta^2+1)^2(2\lambda+\mu)^2}(|x|+|y|)^2,
\end{aligned}$$

where

$$\Omega_{\lambda,\mu,\delta}(t)=18(2\delta^2+1)^2(\lambda+\mu)^3U_3(t)-U_1^3(t)(3\lambda+\mu)\left(3(2\delta^2+1)^2(\mu^2+3\mu-4)+54\delta(\delta^3+2\delta)\right).$$

Since class $\mathcal{P}$ is invariant under the rotations, by (1.12) we may suppose without loss of generality that $c_1:=c\in[0,2]$. Therefore, for $\eta=|x|\leq 1$ and $\zeta=|y|\leq 1$, we obtain

$$\left|a_2a_4-a_3^2\right|\leq\kappa_1+\kappa_2(\gamma_1+\gamma_2)+\kappa_3(\gamma_1^2+\gamma_2^2)+\kappa_4(\gamma_1+\gamma_2)^2=\psi(\gamma_1,\gamma_2)$$

where

$$\begin{aligned}
\kappa_1 &=\frac{U_1(t)\left|\Omega_{\lambda,\mu,\delta}(t)\right|}{96\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(\lambda+\mu)^4}c^4+\frac{3U_1^2(t)c(4-c^2)}{8\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\geq 0,\\
\kappa_2 &=\left[\frac{\left(15(2\delta^2+1)^2-36\delta(\delta^3+2\delta)\right)U_1^3c^2(4-c^2)}{64\delta(\delta^3+2\delta)(2\delta^2+1)^2(\lambda+\mu)^2(2\lambda+\mu)}+\frac{3U_1(t)U_2(t)c^2(4-c^2)}{16\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\right]\geq 0,\\
\kappa_3 &=\frac{3U_1^2(t)c(c-2)(4-c^2)}{32\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)}\leq 0,\\
\kappa_4 &=\frac{9U_1^2(t)(4-c^2)^2}{64(2\delta^2+1)^2(2\lambda+\mu)^2}\geq 0,\quad \frac{1}{2}<t<1.
\end{aligned}$$

Now let us consider the closed square $\mathbb{S}=\{(\gamma_1,\gamma_2):0\leq\gamma_1\leq 1,0\leq\gamma_2\leq 1\}$. In that case, all that we need to do is to maximize the function $\psi(\gamma_1,\gamma_2)$ in the closed square $\mathbb{S}$ for $c\in[0,2]$. Since $\kappa_3\leq 0$ and $\kappa_3+2\kappa_4\geq 0$ for all $t\in(\frac{1}{2},1)$ and $c\in(0,2)$, we conclude that

$$\psi_{\gamma_1\gamma_1}\psi_{\gamma_2\gamma_2}-(\psi_{\gamma_1\gamma_2})^2<0,\quad\text{for all}\quad \gamma_1,\gamma_2\in\mathbb{S}.$$

Thus, the function ψ cannot have a local maximum in the interior of the square $\mathbb{S}$. Now we investigate the maximum of ψ on the boundary of the square $\mathbb{S}$.

For $\gamma_1=0$ and $0\leq\gamma_2\leq 1$ (similarly $\gamma_2=0$ and $0\leq\gamma_1\leq 1$), we get

$$\psi(0,\gamma_2)=\phi(\gamma_2)=\kappa_1+\kappa_2\gamma_2+(\kappa_3+\kappa_4)\gamma_2^2.$$

Next, we are going to be dealing with the following two cases separately.

Case 1 Let $\kappa_3+\kappa_4 \geq 0$. In this case for $0<\gamma_2<1$, and any fixed c with $0 \leq c < 2$ and for all t with $\frac{1}{2}<t<1$, it is clear that $\phi'(\gamma_2) = 2(\kappa_3+\kappa_4)\gamma_2+\kappa_2>0$, that is, $\phi(\gamma_2)$ is an increasing function. Hence, for fixed $c\in[0,2)$ and $t\in(1/2,1)$, the function $\phi(\gamma_2)$ attains a maximum at $\gamma_2=1$ and

$$\max\left(\phi(\gamma_2)\right) = \phi(1) = \kappa_1+\kappa_2+\kappa_3+\kappa_4.$$

Case 2 Let $\kappa_3+\kappa_4<0$. Since $\kappa_2+2(\kappa_3+\kappa_4)\geq 0$ for $0<\gamma_2<1$, any fixed c with $0\leq c<2$, and all $t\in(1/2,1)$, it is clear that $\kappa_2+2(\kappa_3+\kappa_4)<2(\kappa_3+\kappa_4)\gamma_2+\kappa_2<\kappa_2$ and so $\phi'(\gamma_2)>0$. Hence, for fixed $c\in[0,2)$ and $t\in(\frac{1}{2},1)$, the function $\phi(\gamma_2)$ attains a maximum at $\gamma_2=1$.

For $\gamma_1=1$ and $0\leq\gamma_2\leq 1$ (similarly $\gamma_2=1$ and $0\leq\gamma_1\leq 1$), we get

$$\psi(1,\gamma_2)=\Psi(\gamma_2)=(\kappa_3+\kappa_4)\gamma_2^2+(\kappa_2+2\kappa_4)\gamma_2+\kappa_1+\kappa_2+\kappa_3+\kappa_4.$$

Thus, from the above cases of $\kappa_3+\kappa_4$, we get that

$$\max\Psi(\gamma_2)=\Psi(1)=\kappa_1+2\kappa_2+2\kappa_3+4\kappa_4.$$

Since $\phi(1)\leq\Psi(1)$ for $c\in(0,2)$ and $t\in(\frac{1}{2},1)$, we obtain

$$\max(\psi(\gamma_1,\gamma_2))=\psi(1,1)$$

on the boundary of the square $\mathbb{S}$. Thus, the maximum of ψ occurs at $\gamma_1=1$ and $\gamma_2=1$ in the closed square $\mathbb{S}$.

Let a function $\varphi:[0,2]\to\mathbb{R}$ defined by

$$\varphi(c,t)=\max\left(\psi(\gamma_1,\gamma_2)\right)=\psi(1,1)=\kappa_1+2\kappa_2+2\kappa_3+4\kappa_4 \tag{2.23}$$

for fixed values of t. Substituting the values of $\kappa_1,\kappa_2,\kappa_3$, and κ_4 in the function φ defined by (2.23) yields

$$\varphi(c,t)=\frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2}+\frac{\chi_1c^4+36\chi_2c^2}{96\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4},$$

where

$$\begin{aligned}\chi_1=&18(\lambda+\mu)^3\Big(3\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)-(2\delta^2+1)^2(2\lambda+\mu)^2\Big)U_1^2(t)\\&+(2\lambda+\mu)^2U_1(t)\left|\Omega_{\lambda,\mu,\delta}(t)\right|-9(\lambda+\mu)^2(2\lambda+\mu)U_1(t)\Big((3\lambda+\mu)(8\delta^4-4\delta^2+5)U_1^2(t)\\&+4(2\delta^2+1)^2(2\lambda+\mu)(\lambda+\mu)U_2(t)\Big),\\\chi_2=&\Big[(2\lambda+\mu)(3\lambda+\mu)\left(8\delta^4-4\delta^2+5\right)U_1^3(t)+4(2\delta^2+1)^2(\lambda+\mu)(2\lambda+\mu)^2U_1(t)U_2(t)\\&+(\lambda+\mu)U_1^2(t)\big(2(2\delta^2+1)^2(2\lambda+\mu)^2-12\delta(\delta^3+2\delta)(\lambda+\mu)(3\lambda+\mu)\big)\Big](\lambda+\mu)^2.\end{aligned}$$

Assuming that $\varphi(c,t)$ has a maximum value in an interior of $c \in [0,2]$, by elementary calculation, we obtain that

$$\varphi'(c,t) = \frac{\chi_1 c^3 + 18\chi_2 c}{24\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4}.$$

We will examine the sign of the function $\varphi'(c,t)$ depending on the different cases of the signs of χ_1 and χ_2 as follows:

1. Let $\chi_1 \geq 0$ and $\chi_2 \geq 0$, and then $\varphi'(c,t) \geq 0$, so $\varphi(c,t)$ is an increasing function. Therefore,

$$\max\{\varphi(c,t) : c \in (0,2)\} = \varphi(2^-,t) = \frac{\chi_1 + 9\chi_2}{6\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4} + \frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2}. \tag{2.24}$$

   That is, $\max\{\max\{\psi(\gamma_1,\gamma_2) : 0 \leq \gamma_1,\gamma_2 \leq 1\} : 0 < c < 2\} = \varphi(2^-,t)$.

2. Let $\chi_1 \leq 0$ and $\chi_2 \leq 0$, and then $\varphi'(c,t) \leq 0$, so $\varphi(c,t)$ is an decreasing function on the interval $(0,2)$. Therefore,

$$\max\{\varphi(c,t) : c \in (0,2)\} = \varphi(0^+,t) = 4\kappa_4 = \frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2}. \tag{2.25}$$

3. Let $\chi_1 > 0$ and $\chi_2 < 0$, and then $c_0 = \sqrt{\frac{-18\chi_2}{\chi_1}}$ is a critical point of the function $\varphi(c,t)$. We suppose that $c_0 \in (0,2)$, and since $\varphi''(c_0,t) > 0$, c_0 is a local minimum point of the function $\varphi(c,t)$. That is, the function $\varphi(c,t)$ cannot have a local maximum.

4. Let $\chi_1 < 0$ and $\chi_2 > 0$, and then c_0 is a critical point of the function $\varphi(c,t)$. We assume that $c_0 \in (0,2)$. Since $\varphi''(c_0,t) < 0$, c_0 is a local maximum point of the function $\varphi(c,t)$ and the maximum value occurs at $c = c_0$. Therefore,

$$\max\{\varphi(c,t) : c \in (0,2)\} = \varphi(c_0,t) \tag{2.26}$$

   where

$$\varphi(c_0,t) = \frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+\mu)^2} - \frac{27\chi_2^2}{8\chi_1\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+\mu)(2\lambda+\mu)^2(\lambda+\mu)^4}.$$

Thus, from (2.24) to (2.26), the proof of Theorem 1 is completed. □

Now we would like to draw attention to some remarkable results obtained for some values of λ, μ, and δ in Theorem 1.

Corollary 1 *Let $h_\delta \in \sigma$ of the form (1.6) be in $\mathcal{B}_\sigma^\delta(\lambda,t)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \varphi(2^-,t), & \chi_3 \geq 0 \quad \chi_4 \geq 0 \\ \frac{36t^2}{(2\delta^2+1)^2(2\lambda+1)^2}, & \chi_3 \leq 0 \quad \chi_4 \leq 0 \\ \max\left\{\frac{36t^2}{(2\delta^2+1)^2(2\lambda+1)^2}, \quad \varphi(2^-,t)\right\}, & \chi_3 > 0 \quad \chi_4 < 0 \\ \max\{\varphi(c_0,t), \varphi(2^-,t)\}, & \chi_3 < 0 \quad \chi_4 > 0 \end{cases},$$

where

$$\varphi(2^-,t)=\frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+1)^2}+\frac{\chi_3+9\chi_4}{6\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+1)(2\lambda+1)^2(\lambda+1)^4},$$

$$\varphi(c_0,t)=\frac{9U_1^2(t)}{(2\delta^2+1)^2(2\lambda+1)^2}-\frac{27\chi_4^2}{8\chi_3\delta(\delta^3+2\delta)(2\delta^2+1)^2(3\lambda+1)(2\lambda+1)^2(\lambda+1)^4},$$

and $c_0=\sqrt{\frac{-18\chi_4}{\chi_3}}$, $\chi_3=\chi_1(\lambda,\mu=1,\delta;t)$, $\chi_4=\chi_2(\lambda,\mu=1,\delta;t)$.

Taking $\lambda=1$ in Theorem 1, we get the following result.

Corollary 2 *Let* $h_\delta\in\sigma$ *of the form* (1.6) *be in* $\mathcal{B}_\sigma^{\mu,\delta}(t)$. *Then*

$$\left|a_2a_4-a_3^2\right|\le\begin{cases}\varphi(2^-,t), & \chi_5\ge 0\quad \chi_6\ge 0\\ \frac{36t^2}{(2\delta^2+1)^2(2+\mu)^2}, & \chi_5\le 0\quad \chi_6\le 0\\ \max\left\{\frac{36t^2}{(2\delta^2+1)^2(2+\mu)^2},\quad \varphi(2^-,t)\right\}, & \chi_5>0\quad \chi_6<0\\ \max\left\{\varphi(c_0,t),\varphi(2^-,t)\right\}, & \chi_5<0\quad \chi_6>0\end{cases},$$

where

$$\varphi(2^-,t)=\frac{9U_1^2(t)}{(2\delta^2+1)^2(2+\mu)^2}+\frac{\chi_5+9\chi_6}{6\delta(\delta^3+2\delta)(2\delta^2+1)^2(3+\mu)(2+\mu)^2(1+\mu)^4},$$

$$\varphi(c_0,t)=\frac{9U_1^2(t)}{(2\delta^2+1)^2(2+\mu)^2}-\frac{27\chi_6^2}{8\chi_5\delta(\delta^3+2\delta)(2\delta^2+1)^2(3+\mu)(2+\mu)^2(1+\mu)^4},$$

and $c_0=\sqrt{\frac{-18\chi_6}{\chi_5}}$, $\chi_5=\chi_1(\lambda=1,\mu,\delta;t)$, $\chi_6=\chi_2(\lambda=1,\mu,\delta;t)$.

Putting $\lambda=1$ and $\mu=1$ in Theorem 1, we find the following result.

Corollary 3 *If* $h_\delta\in\mathcal{B}_\sigma^{\delta}(t)$ *is of the form* (1.6), *then*

$$\left|a_2a_4-a_3^2\right|\le\begin{cases}\varphi(2^-,t), & \frac{1}{2}<t\le t_0\\ \varphi(c_0,t), & t_0<t<1\end{cases},$$

where

$$\varphi(2^-,t)=\frac{3t^2\big((2\delta^2+1)^2-t(5\delta^4+2\delta^2+2)\big)}{(\delta^2+2)(2\delta^3+\delta)^2},$$

$$\varphi(c_0,t)=-\frac{t(\nabla^{(3,1,-1)}_{(4,20,-3);(22,40,64)})^2}{\delta(2\delta^2+1)^2(\delta^3+2\delta)\big(\nabla^{(3,1,-1)}_{(-4,4,-3);(22,40,64)}+6t\left|t(5\delta^4+2\delta^2+2)-(2\delta^2+1)^2\right|\big)}+\frac{4t^2}{(2\delta^2+1)^2},$$

$\nabla^{(m,n,r)}_{(a,b,c);(d,e,f)}(\delta;t)=\nabla^{(m,n,r)}_{(a,b,c);(d,e,f)}=m(1+2\delta^2)^2+nt(a+bt^2+ct^4)+rt^2(d+et^2+ft^4)$. *Moreover, the value of* t_0 *is the root of equation* $\chi_1=0$ *for* $\lambda=\mu=1$ *and* $\frac{1}{2}<t<1$.

Setting $\delta=1$ in Corollary 3, we obtain the following result.

Corollary 4 *If $h_\delta \in \mathcal{B}_\sigma(t)$ is of the form (1.6), then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} t^2(1-t^2), & \frac{1}{2} < t \leq t_{0_1} \\ \frac{t(260t^4+84t^3-139t^2-18t+9)}{8(18t^3+42t^2-17t-9)}, & t_{0_1} < t < 1 \end{cases},$$

where the value of t_{0_1}, which is approximately $t_{0_1} = 0.603615$, is the root of equation $\chi_1 = 0$ for $\lambda = \mu = \delta = 1$ and $\frac{1}{2} < t < 1$.

Next, taking $\lambda = 1$ and $\mu = 0$ in Theorem 1, we arrive at the following result.

Corollary 5 *If $h_\delta \in \mathcal{S}_\sigma^{\star,\delta}(t)$ is of the form (1.6), then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \varphi(2^-, t), & \frac{1}{2} < t \leq t_{0_2} \\ \varphi(c_0, t), & t_{0_2} < t < 1 \end{cases},$$

where

$$\varphi(2^-, t) = \frac{8t^2\big((2\delta^2+1)^2 - 6t^2(\delta^2-1)^2\big)}{(2\delta^3+\delta)^2(\delta^2+2)},$$

$$\varphi(c_0, t) = -\frac{t(\nabla^{(-2,-1,1)}_{(-2,10,1);(23,20,56)})^2}{\delta(2\delta^2+1)^2(\delta^3+2\delta)\big(\nabla^{(-4,1,2)}_{(4,-2,7);(23,20,56)} - 8t\left|6t^2(\delta^2-1)^2 - (2\delta^2+1)^2\right|\big)} + \frac{9t^2}{(2\delta^2+1)^2}.$$

Moreover, the value of t_{0_2} is the root of equation $\chi_1 = 0$ for $\lambda = 1$, $\mu = 0$, and $\frac{1}{2} < t < 1$.

Now, taking $\delta = 1$ in Corollary 5, we attain the following result.

Corollary 6 *If $h_\delta \in \mathcal{S}_\sigma^{\star}(t)$ is of the form (1.6), then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{8t^2}{3}, & \frac{1}{2} < t \leq \frac{7+\sqrt{401}}{44} \\ t^2 + \frac{t(2+t-11t^2)^2}{3(-4-7t+22t^2)}, & \frac{7+\sqrt{401}}{44} < t < 1 \end{cases}.$$

When $\delta = 1$, we note that the results given above coincide with the results in [22].

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