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## Approximation by integral functions of finite degree in variable exponent Lebesgue spaces on the real axis

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**Abstract:** We obtain several inequalities of approximation by integral functions of finite degree in generalized Lebesgue spaces with variable exponent defined on the real axis. Among them are direct, inverse, and simultaneous estimates of approximation by integral functions of finite degree in  $L^{p(\cdot)}$ . An equivalence of modulus of continuity with Peetre's  $K$ -functional is established. A constructive characterization of Lipschitz class is also obtained.

**Key words:** Direct theorem, inverse theorem, modulus of continuity, simultaneous approximation, Lipschitz class

### 1. Introduction

In recent years, variable exponent function spaces and approximation problems in variable exponent Lebesgue spaces  $L^{p(x)}$  have attracted more attention (see Cruz-Uribe and Fiorenza [7], Diening et al. [9], and Sharapudinov [42]). Many authors have obtained analogues of classical results in function space with variable exponents because of their applications in elasticity theory [51], fluid mechanics [35, 36], differential operators [10, 36], nonlinear Dirichlet boundary value problems [25], nonstandard growth [27, 51], and variational calculus. Starting from the work of Orlicz [32], the theory of variable exponents and  $L^{p(x)}$  was developed in the late 1900s. In fact,  $L^{p(x)}$  is a modular space [14, 28] and under the condition  $p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}} p(x) < \infty$ ,  $L^{p(x)}$  becomes a particular case of Musielak–Orlicz spaces [28]. In subsequent years several problems in  $L^{p(x)}$  were investigated in [8, 11, 24, 25, 37, 38, 40].

Variable exponent Lebesgue spaces on  $[0, 2\pi]$  (or  $[0, 1]$ ) and many fundamental results corresponding to the approximation of the function were developed by Sharapudinov [39, 41, 43–45]. Nowadays many problems for the approximation of the function are solved in these types of spaces defined on  $[0, 2\pi] \subset \mathbb{R}$  (see, e.g., [2–5, 12, 13, 19–22]). In this direction, we aim to obtain direct and inverse theorems for approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis  $\mathbb{R}$ .

Recall that studies dealing with approximation by entire function of finite degree in the real domain date back to Bernstein's works, for example [6]. After his works, Wiener and Paley [33], Ackhiezer [1], Nikolskii [30], and Ibragimov [15–17] developed this subject. Various problems related to approximation of functions on

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$\mathbb{R}$  by entire functions of exponential type in the  $L_p$  space were studied in the papers of Ackhiezer [1], Timan [48], Timan [49], Nikol'skii [30, 31], Ibragimov [15–18], Taberski [46, 47], Nasibov [29], Popov [34], Ligon [26], Vakarchuk [50], and others. Note that an entire function of finite exponential type is merely an entire function of order 1 and finite type, and in approximation theory these often play an important role similar to trigonometric polynomials in the case of approximation of periodic functions. Thus, for example, there are Bernstein-type inequalities for such functions.

In this work, we generalize the works of Ibragimov and Taberski about approximation of functions in Lebesgue spaces on the whole real axis in variable exponent settings. In what follows,  $A \lesssim B$  will mean that there exists a positive constant  $C_{u,v,\dots}$  dependent only on the parameters  $u, v, \dots$  and it can be different in different places, such that the inequality  $A \leq CB$  holds.

In Theorem 4.1, we obtain that if  $p(\cdot) \in P$  (see Definition 2.1), then there exists a positive constant depending only on  $p(\cdot)$ , such that the following Jackson–Stechkin type inequality holds:

$$A_\sigma(f)_{p(\cdot)} \lesssim \Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)}, \tag{1.1}$$

where  $f \in L^{p(\cdot)}$ ,  $h > 0$ ,  $T_h f(x) := \frac{1}{h} \int_0^h f(x+t) dt$ , ( $x \in \mathbb{R}$ ),  $\Omega(f, \delta)_{p(\cdot)} := \sup_{0 < h \leq \delta} \|(I - T_h)f\|_{p(\cdot)}$ ,  $\mathcal{G}_\sigma$  is the subspace of integral function  $f(z)$  of exponential type  $\leq \sigma$  belonging to  $L^{p(\cdot)}$  and  $A_\sigma(f)_{p(\cdot)} := \inf\{\|f - g\|_{p(\cdot)} : g \in \mathcal{G}_\sigma\}$ . Let  $W_r^{p(\cdot)}$ ,  $r \in \mathbb{N}$ , be the class of functions  $f \in L^{p(\cdot)}$  such that  $f^{(r-1)}$  is absolutely continuous and  $f^{(r)} \in L^{p(\cdot)}$ . In Theorem 6.1, for any  $f \in W_r^{p(\cdot)}$ , we show the following simultaneous approximation inequality:

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} A_\sigma(f^{(r)})_{p(\cdot)}, \quad r \in \mathbb{N}.$$

The weak inverse estimate of Theorem 4.1,

$$\Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=0}^{[\sigma]} E_\nu(f)_{p(\cdot)},$$

is obtained in Theorem 5.1, where  $[\sigma] := \max\{n \in \mathbb{Z} : n \leq \sigma\}$ . For  $0 < \beta < 1$  we define  $Lip_\beta p(\cdot) := \{f \in L^{p(\cdot)} : \Omega(f, \delta)_{p(\cdot)} \lesssim \delta^\beta, \delta > 0\}$  and  $W_{p(\cdot)}^{r,\beta} := \{f \in W_{p(\cdot)}^r : f^{(r)} \in Lip_\beta p(\cdot)\}$ , and using this notation the following constructive description of the Lipschitz class  $Lip_\beta p(\cdot)$  is proved.

Let  $0 < \beta < 1$  and  $r \in \{0\} \cup \mathbb{N}$ , and then

$$f^{(r)} \in Lip_\beta p(\cdot) \text{ iff } A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta-r}.$$

The rest of the paper is organized as follows. In Section 2 we introduce preliminaries and necessary facts. In Section 3, we give the definition of the modulus of continuity  $\Omega(f, \cdot)_{p(\cdot)}$  and obtain an equivalence between  $\Omega(f, \delta)_{p(\cdot)}$  and  $K$ -functional  $K(f, \delta, L^{p(\cdot)}, 1)_{p(\cdot)}$ . Sections 4 and 5 contain the direct and inverse theorems in variable exponent Lebesgue spaces on the real line. In Section 6 we obtain some inequalities on simultaneous approximation of functions in the corresponding Sobolev spaces  $W_r^{p(\cdot)}$  and in Section 7 we obtain some constructive characterizations of the Lipschitz class  $Lip_\beta p(\cdot)$ .

**2. Preliminaries**

Let  $p(x) : \mathbb{R} \rightarrow [1, \infty)$  be a measurable function. We suppose that

$$1 < p_- := \text{ess inf}_{x \in \mathbb{R}} p(x) \text{ and } p^+ < \infty. \tag{2.1}$$

We define  $L^{p(\cdot)} := L^{p(\cdot)}(\mathbb{R})$  as the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that

$$I_{p(\cdot)} \left( \frac{f}{\lambda} \right) := \int_{\mathbb{R}} \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy < \infty \tag{2.2}$$

for some  $\lambda > 0$ . The set of of functions  $L^{p(\cdot)}$ , with norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \eta > 0 : I_{p(\cdot)} \left( \frac{f}{\eta} \right) < 1 \right\},$$

is the Banach space.

Consider now an arbitrary, integral function  $f(z)$ ; put

$$M(r) = \max_{|z|=r} |f(z)|, \quad z = x + iy.$$

We say that  $f$  is of exponential type  $\sigma$  if the relation

$$\limsup_{r \rightarrow \infty} \frac{\ln M(r)}{r} \leq \sigma, \quad \sigma < \infty$$

is valid. Let  $\mathcal{G}_\sigma$  be the subspace of integral function  $f(z)$  of exponential type  $\sigma$  belonging to  $L^{p(\cdot)}$ . The quantity

$$A_\sigma(f)_{p(\cdot)} := \inf_g \{ \|f - g\|_{p(\cdot)} : g \in \mathcal{G}_\sigma \}$$

where  $f \in L^{p(\cdot)}$  is the deviation of the function  $f \in L^{p(\cdot)}(\mathbb{R})$  from  $\mathcal{G}_\sigma$ .

For  $f \in L^{p(\cdot)}$ , we consider Steklov’s mean operator:

$$T_h(f) = \frac{1}{h} \int_0^h f(x + t) dt.$$

**Definition 2.1** Let  $P$  be the class of measurable functions  $p(\cdot)$  satisfying the conditions (2.1),  $\exists c, C > 0$ ,  $C' \in \mathbb{R}$  such that

$$|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)}, \quad \forall x, y \in \mathbb{R}, \tag{2.3}$$

$$|p(x) - C'| \leq \frac{C}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}. \tag{2.4}$$

It was proved in [9, Theorem 4.3.8] that if  $p(\cdot) \in P$ , then for  $h > 0$ , the family of operators  $\{T_h\}$  is uniformly bounded in  $L^{p(\cdot)}$ .

### 3. Modulus of continuity and $K$ -functional

Let  $f \in L^{p(\cdot)}$  and  $h > 0$ , and then we define the Steklov mean type operator:

$$T_h f(x) := \frac{1}{h} \int_0^h f(x+t) dt, \quad x \in \mathbb{R}.$$

The modulus of continuity of  $f \in L^{p(\cdot)}$  is defined by

$$\Omega(f, \delta)_{p(\cdot)} := \sup_{0 < h \leq \delta} \|(I - T_h)f\|_{p(\cdot)}. \tag{3.1}$$

If  $f \in L^{p(\cdot)}$  and  $\delta \geq 0$ , then

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \|f\|_{p(\cdot)} \tag{3.2}$$

holds for some constant depending only on  $p(\cdot)$ .

**Theorem 3.1** For  $f, g \in L^{p(\cdot)}$  and  $\delta \geq 0$ , the modulus of continuity  $\Omega(f, \delta)_{p(\cdot)}$  has the following properties:

1.  $\Omega(f, \delta)_{p(\cdot)}$  is a nonnegative, nondecreasing function.

2. For  $f, g \in L^{p(\cdot)}$  and  $\delta > 0$ ,

$$\Omega_{p(\cdot)}(f + g, \delta) \leq \Omega_{p(\cdot)}(f, \delta) + \Omega_{p(\cdot)}(g, \delta). \tag{3.3}$$

3. For  $f \in L^{p(\cdot)}$ ,

$$\lim_{\delta \downarrow 0} \Omega_{p(\cdot)}(f, \delta) = 0. \tag{3.4}$$

**Proof** Properties (1) and (2), by definition of  $\Omega(f, \delta)_{p(\cdot)}$  and the triangle inequality of  $L^{p(\cdot)}$ , are clearly valid.

For proof of (3.4), using  $f \in L^{p(\cdot)}$  and  $\Omega(f, \delta)_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$ , we can find  $N > 1$  such that, for any fixed  $\varepsilon > 0$ ,

$$\|f - T_h(f)\|_{L^{p(\cdot)}[(-\infty, -N) \cup (N, \infty)]} \leq \frac{\varepsilon}{2}. \tag{3.5}$$

We may assume that  $\delta < 1$  and we have

$$\Omega_{p(\cdot)}(f, \delta) = \sup_{0 < h \leq \delta} \|(I - T_h)f\|_{L^{p(\cdot)}(\mathbb{R})} \leq \sup_{0 < h \leq \delta} \|(I - T_h)f\|_{L^{p(\cdot)}([-N, N])} + \frac{\varepsilon}{2}. \tag{3.6}$$

On the other hand, by [7, Corollary 2.73], there exists  $\phi \in C_c[-N, N]$  such that

$$\|f - \phi\|_{L^{p(\cdot)}[-N, N]} \leq \varepsilon. \tag{3.7}$$

Let  $N > 1$  be the same as the number found above. First we prove that in the case of  $\phi \in C_c[-N, N]$ , we have  $\|\phi - T_h\phi\|_{L^{p(\cdot)}([-N, N])} < C_0\varepsilon$ . Set

$$\mathbf{I} = I_{p(\cdot)} \left( \frac{\phi - T_h\phi}{N2^{\frac{1}{p(\cdot)} + 1}\varepsilon} \right).$$

Then we have

$$\begin{aligned} I &= \int_{[-N,N]} \left| \frac{1}{N 2^{\frac{1}{p(\cdot)}+1} \varepsilon} (\phi(y) - T_h \phi(y)) \right|^{p(y)} dy \\ &= \int_{[-N,N]} \left| \frac{1}{N 2^{\frac{1}{p(\cdot)}+1} \varepsilon} \frac{1}{\tau} \int_0^\tau (\phi(y+h) - \phi(y)) dh \right|^{p(y)} dy. \end{aligned}$$

There exists  $\delta_0 = \delta_0(\varepsilon) > 0$  such that

$$|\phi(x) - \phi(x+h)| < \varepsilon \tag{3.8}$$

for  $0 \leq h \leq \delta_0$  and  $x \in [-N, N]$ . Hence, for  $0 \leq h \leq \delta_0$ , using (3.8), we have

$$I \leq 1.$$

Then we obtain

$$\|\phi - T_h \phi\|_{L^{p(\cdot)}([-N,N])} < N 2^{\frac{1}{p(\cdot)}+1} \varepsilon \tag{3.9}$$

for  $0 \leq h < \delta_0$ .

Also, by uniform boundedness of  $T_h$  and (3.7), we have

$$\|T_h(\phi) - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \leq c(p)\varepsilon. \tag{3.10}$$

By the triangle inequality, we have

$$\begin{aligned} \|f - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \\ \leq \|f - \phi\|_{L^{p(\cdot)}([-N,N])} + \|\phi - T_h(\phi)\|_{L^{p(\cdot)}([-N,N])} + \|T_h(\phi) - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \end{aligned} \tag{3.11}$$

for any  $f \in L^{p(\cdot)}([-N, N])$ . Then, by replacing (3.7), (3.9), and (3.10) in (3.11) we have

$$\|f - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \leq c(p)\varepsilon, \quad 0 \leq h \leq \delta_0(\varepsilon). \tag{3.12}$$

Consequently, in view of (3.6), we have  $\lim_{\delta \rightarrow 0^+} \Omega(f, \delta)_{p(\cdot)} = 0$  for every  $f \in L^{p(\cdot)}$ . □

For proof of Theorem 3.4 we need the following lemma.

**Lemma 3.2** *Let  $f \in W_1^{p(\cdot)}$  be given. Then*

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \delta \|f'\|_{p(\cdot)}, \quad \delta \geq 0 \tag{3.13}$$

*holds with some constant depending only on  $p(\cdot)$ .*

**Proof** [Proof of Lemma 3.2] It is sufficient to prove the following inequality:

$$\|(I - T_h) f\|_{p(\cdot)} \lesssim h \|f'\|_{p(\cdot)}, \quad h > 0 \tag{3.14}$$

for and  $f \in L^{p(\cdot)}$ . We have

$$(I - T_h) f(x) = \frac{1}{h} \int_0^h (f(x) - f(x+t)) dt = \frac{-1}{h} \int_0^h \int_x^{x+t} f'(s) ds dt.$$

Therefore, from Minkowski’s inequality for integrals, we get

$$\begin{aligned} \|(I - T_h) f\|_{p(\cdot)} &= \left\| \frac{1}{h} \int_0^h \int_x^{x+t} f'(s) ds dt \right\|_{p(\cdot)} \\ &= \left\| \frac{1}{h} \int_0^h t \frac{1}{t} \int_0^t f'(x+s) ds dt \right\|_{p(\cdot)} = \left\| \frac{1}{h} \int_0^h t T_t(f') dt \right\|_{p(\cdot)} \\ &\leq \frac{1}{h} \int_0^h t \|T_t(f')\|_{p(\cdot)} dt \lesssim \|f'\|_{p(\cdot)} \frac{1}{h} \int_0^h t dt \lesssim h \|f'\|_{p(\cdot)} \end{aligned}$$

and (3.14) follows. Then

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \delta \|f'\|_{p(\cdot)}, \delta > 0$$

for  $f \in W_1^{p(\cdot)}$ . □

It is known that for proof of the inverse theorem we need Bernstein’s inequality. We present the following theorem corresponding to the well-known Bernstein inequality on the derivative of exponential type entire functions of finite order (integral functions) in variable exponent Lebesgue spaces that was proved by Nanobashvili and Kokilashvili in [23].

**Theorem 3.3** [23, Theorem 2] *Let  $p \in P$  and  $g_\sigma$  be an exponential type entire function of degree  $\leq \sigma$ . Assume that  $g_\sigma \in L^{p(\cdot)}$ . Then the inequality*

$$\|g'_\sigma\|_{p(\cdot)} \lesssim \sigma \|g_\sigma\|_{p(\cdot)}$$

holds with a constant, independent of  $g_\sigma$ .

Let  $f \in L^{p(\cdot)}$ . The  $K$ -functional is defined as follows:

$$K(f, t, L^{p(\cdot)}, 1)_{p(\cdot)} = \inf_{g \in W_1^{p(\cdot)}} \{ \|f - g\|_{p(\cdot)} + t \|g'\|_{p(\cdot)} \}$$

for  $t > 0$ .

In the following theorem we show that  $K$ -functional  $K(f, \delta, L^{p(\cdot)}, 1)_{p(\cdot)}$  and  $\Omega(f, \delta)_{p(\cdot)}$  are equivalent.

**Theorem 3.4** *Let  $p(\cdot) \in P$ . If  $L^{p(\cdot)}$ , then the  $K$ -functional  $K(f, t; L^{p(\cdot)}, 1)$  and the modulus  $\Omega(f, t)_{p(\cdot)}$  are equivalent; namely,*

$$\Omega(f, t)_{p(\cdot)} \lesssim K\left(f, t; L^{p(\cdot)}, 1\right)_{p(\cdot)} \lesssim \Omega(f, t)_{p(\cdot)}$$

for all  $f \in L^{p(\cdot)}$  with some constants, independent of  $f$ .

**Proof** [Proof of Theorem 3.4] Let  $t > 0$ . Then there exists  $\sigma \in \mathbb{N}$  such that  $1/\sigma \leq t < 2/\sigma$ . We define the operator

$$(U_v f)(x) := \frac{2}{v} \int_{v/2}^v \left( \frac{1}{h} \int_0^h f(x+t) dt \right) dh, \quad x \in \mathbb{R}, \quad f \in L^{p(\cdot)}, \quad v > 0.$$

On the other hand, for  $0 < v \leq 1$ , we obtain by Minkowski's inequality for integrals

$$\begin{aligned} \|U_v f\|_{p(\cdot)} &= \left\| \frac{2}{v} \int_{v/2}^v \left( \frac{1}{h} \int_0^h f(x+t) dt \right) dh \right\|_{p(\cdot)} \\ &\leq \frac{1}{v/2} \int_{v/2}^v \left\| \frac{1}{h} \int_0^h f(x+t) dt \right\|_{p(\cdot)} dh \\ &= \frac{1}{v/2} \int_{v/2}^v \|T_h f\|_{p(\cdot)} dh \lesssim \|f\|_{p(\cdot)} \frac{1}{v/2} \int_{v/2}^v dh \\ &= \|f\|_{p(\cdot)} \end{aligned}$$

and hence  $f - U_v f \in L^{p(\cdot)}$ . Also, the function  $U_v f(x)$  is absolutely continuous [43] and

$$\left| \frac{d}{dx} U_v f(x) \right| = \frac{2}{v} \left| \int_{v/2}^v \frac{1}{h} (f(x+h) - f(x)) dh \right|.$$

For  $0 < v \leq 1$  we have by Minkowski's inequality for integrals

$$\begin{aligned} \left\| \frac{d}{dx} U_v f(x) \right\|_{p(\cdot)} &\leq \frac{2}{v} \left\| \frac{1}{v} \int_0^v (f(x+t) - f(x)) dt - \frac{1}{v} \int_0^{v/2} (f(x+t) - f(x)) dt \right\|_{p(\cdot)} + \\ &\quad + \frac{2}{v} \left\| \int_{v/2}^v \frac{dh}{h^2} \left[ \int_0^h (f(x+t) - f(x)) dt - \int_0^{v/2} (f(x+t) - f(x)) dt \right] \right\|_{p(\cdot)} \\ &\leq \frac{2}{v} \left\| T_v f(x) - f(x) - \frac{1}{2} (T_{v/2} f(x) - f(x)) \right\|_{p(\cdot)} \\ &\quad + \frac{2}{v} \left\| \int_{v/2}^v \frac{1}{h} \left( T_h f(x) - f(x) - \frac{v}{2h} (T_h f(x) - f(x)) \right) dh \right\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \Omega(f, v/2)_{p(\cdot)} \\ &\quad + \frac{1}{v} \left\| \int_{v/2}^v \frac{1}{h} (|T_h f(x) - f(x)| - |T_h f(x) - f(x)|) dh \right\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \left\| \int_{v/2}^v \frac{1}{h} |T_h f(x) - f(x)| dh \right\|_{p(\cdot)} \\ &\quad + \frac{1}{v} \|T_h f(x) - f(x)\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \int_{v/2}^v \frac{1}{h} \|T_h f(x) - f(x)\|_{p(\cdot)} dh \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \Omega(f, v)_{p(\cdot)} \frac{1}{v} \int_{v/2}^v \frac{dh}{h} \lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)}. \end{aligned} \tag{3.15}$$



Hence, for a given  $v \in (0, 1]$ ,  $\frac{d}{dx}U_v f(x) \in L^{p(\cdot)}$ . Then

$$\begin{aligned} K\left(f, t, L^{p(\cdot)}, W_p^1\right) &\leq 2K\left(f, 1/\sigma, L^{p(\cdot)}, W_p^1\right) \\ &\lesssim \|f - U_{1/\sigma}f\|_{p(\cdot)} + \frac{1}{\sigma} \left\| \frac{d}{dx}U_{1/\sigma}f \right\|_{p(\cdot)} =: I_1 + I_2. \end{aligned}$$

We estimate  $I_1$ . Using Minkowski’s inequality for integrals we obtain

$$\begin{aligned} \|f - U_{1/\sigma}f\|_{p(\cdot)} &= \left\| 2\sigma \int_{1/2\sigma}^{1/\sigma} \left( \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt \right) dh \right\|_{p(\cdot)} \\ &\leq \left\| 2\sigma \int_{1/2\sigma}^{1/\sigma} |T_h f(x) - f(x)| dh \right\|_{p(\cdot)} \\ &\leq 2\sigma \int_{1/2\sigma}^{1/\sigma} \|T_h f - f\|_{p(\cdot)} dh \\ &\lesssim \sup_{0 \leq u \leq 1/\sigma} \|(I - T_u) f\|_{p(\cdot)} 2\sigma \int_{1/2\sigma}^{1/\sigma} dh = \Omega(f, 1/\sigma)_{p(\cdot)}. \end{aligned} \tag{3.16}$$

For the estimate  $I_2$ , we find from (3.15) that

$$\frac{1}{\sigma} \left\| \frac{d}{dx}U_{1/\sigma}f \right\|_{p(\cdot)} \lesssim \Omega(f, 1/\sigma)_{p(\cdot)}. \tag{3.17}$$

Now (3.16)–(3.17) give

$$K\left(f, t, L^{p(\cdot)}, 1\right) \lesssim \Omega(f, 1/\sigma)_{p(\cdot)} \leq \Omega(f, t)_{p(\cdot)}.$$

By Lemma 3.2, for  $g \in W_1^{p(\cdot)}$ ,

$$\Omega(f, t)_{p(\cdot)} \lesssim \|f - g\|_{p(\cdot)} + t \|g'\|_{p(\cdot)},$$

and taking infimum on  $g \in W_1^{p(\cdot)}$  we get

$$\Omega(f, t)_{p(\cdot)} \lesssim K\left(f, t; L^{p(\cdot)}, 1\right).$$

Now we obtain

$$\Omega(f, t)_{p(\cdot)} \approx K\left(f, t; L^{p(\cdot)}, 1\right) \tag{3.18}$$

and this is the desired result. □

As a corollary of Theorem 3.4:

**Corollary 3.5** *Let  $p(\cdot) \in P$ . If  $\delta, \lambda \in \mathbb{R}^+$ ,  $f \in L^{p(\cdot)}$  and then*

$$\Omega(f, \lambda\delta)_{p(\cdot)} \lesssim (1 + [\lambda]) \Omega(f, \delta)_{p(\cdot)} \tag{3.19}$$

*holds with some constant depending only on  $p(\cdot)$ .*

**Proof** [Proof of Corollary 3.5] Using equivalence (3.18) we have

$$\begin{aligned} \Omega(f, lt)_{p(\cdot)} &\lesssim \inf_{g \in W_1^{p(\cdot)}} \left\{ \|f - g\|_{p(\cdot)} + lt \|g'\|_{p(\cdot)} \right\} \\ &\lesssim (1 + [l]) \inf_{g \in W_1^{p(\cdot)}} \left\{ \|f - g\|_{p(\cdot)} + t \|g'\|_{p(\cdot)} \right\} \\ &\lesssim (1 + [l]) \Omega(f, t)_{p(\cdot)}, \end{aligned}$$

which gives (3.19). □

#### 4. Direct theorems

**Theorem 4.1** Let  $p(\cdot) \in P$ . If  $f \in L^{p(\cdot)}$ , then

$$A_\sigma(f)_{p(\cdot)} \lesssim \Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} \tag{4.1}$$

holds with some constant depending only on  $p(\cdot)$ .

**Proof** [Proof of Theorem 4.1] Let  $\sigma$  and  $f \in L^{p(\cdot)}$  be fixed. We consider the operator  $U_{1/\sigma}f$ . Using (3.16) and (3.17),

$$\begin{aligned} A_\sigma(f)_{p(\cdot)} &= A_\sigma(f - U_{1/\sigma}f + U_{1/\sigma}f)_{p(\cdot)} \leq A_\sigma(f - U_{1/\sigma}f)_{p(\cdot)} + A_\sigma(U_{1/\sigma}f)_{p(\cdot)} \\ &\lesssim \|f - U_{1/\sigma}f\|_{p(\cdot)} + \frac{1}{\sigma} \left\| \frac{d}{dx} U_{1/\sigma}f(x) \right\|_{p(\cdot)} \lesssim \Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)}, \end{aligned} \tag{4.2}$$

and the result follows. □

We define

$$g(x) = \left(\frac{1}{x} \sin \frac{\sigma x}{2r}\right)^{2r}$$

for  $r \geq 3/2$ . Then  $g(x) \in \mathcal{G}_\sigma$  for  $r \geq 3/2$ . Set

$$\gamma_r := \int_{\mathbb{R}} \left(\frac{1}{t} \sin \frac{\sigma t}{2r}\right)^{2r} dt.$$

In this case,

$$\gamma_r = \sigma^{2r-1}C,$$

where  $C > 0$  is dependent only on  $r$ .

Let

$$D_\sigma f(x) := \frac{1}{\gamma_r} \int_{\mathbb{R}} f(x+t)g(t)dt, \quad \sigma > 0. \tag{4.3}$$

Then  $D_\sigma f \in \mathcal{G}_\sigma$  ([17]).

**Corollary 4.2** *The subspace of integral function  $f(z)$  of exponential type  $\sigma$  belonging to  $L^{p(\cdot)}$  is dense in  $L^{p(\cdot)}$ .*

**Lemma 4.3** *Let  $p(\cdot) \in P$ . If  $f \in W_1^{p(\cdot)}$ , then*

$$\|f - D_\sigma f\|_{p(\cdot)} \lesssim \frac{1}{\sigma} \|f'\|_{p(\cdot)} \tag{4.4}$$

*holds with some constant depending only on  $p(\cdot)$ .*

**Proof** [Proof of Lemma 4.3] From (4.3), one can write

$$\begin{aligned} \|f - D_\sigma f\|_{p(\cdot)} &= \left\| \frac{1}{\gamma_r} \int_{\mathbb{R}} (f(x+t) - f(x)) g(t) dt \right\|_{p(\cdot)} \\ &= \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} (f(x+t) - f(x)) g(t) dt \right\|_{p(\cdot)} \\ &= \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} \frac{1}{t} \int_x^{x+t} f'(\tau) d\tau t g(t) dt \right\|_{p(\cdot)} = \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} T_t f'(x) t g(t) dt \right\|_{p(\cdot)} \\ &\lesssim \frac{1}{\gamma_r} \int_{\mathbb{R}} \|T_t f'\|_{p(\cdot)} |t| |g(t)| dt \lesssim \|f'\|_{p(\cdot)} \frac{2}{\gamma_r} \int_0^\infty |t| |g(t)| dt \\ &\lesssim \|f'\|_{p(\cdot)} \left\{ \frac{1}{\gamma_r} \int_{|t| \leq 1/\sigma} |t| |g(t)| dt + \frac{1}{\gamma_r} \int_{|t| \geq 1/\sigma} |t| |g(t)| dt \right\} \lesssim \frac{1}{\sigma} \|f'\|_{p(\cdot)}, \end{aligned}$$

which implies inequality (4.4). □

**5. Inverse estimate**

Now we present the inverse theorem.

**Theorem 5.1** *Let  $p(\cdot) \in P$  and  $f \in L^{p(\cdot)}$ . Then there exists a positive constant, depending only on  $p(\cdot)$ , such that*

$$\Omega \left( f, \frac{1}{\sigma} \right)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=1}^{[\sigma]} A_\nu(f)_{p(\cdot)}$$

*holds, where  $[\sigma]$  is the largest integer less than or equal to  $\sigma$ .*

**Proof** [Proof of Theorem 5.1] Let  $g_\sigma$  be an exponential type entire function of degree  $\leq \sigma$ , belonging to  $L^{p(\cdot)}$ , as the best approximation of  $f \in L^{p(\cdot)}$ . Let  $2^j \leq \sigma < 2^{j+1}$ . Thanks to the definition of  $K(f, t, L^{p(\cdot)}, 1)_{p(\cdot)}$  we have

$$\begin{aligned} K \left( f, \frac{1}{\sigma}, L^{p(\cdot)}, 1 \right)_{p(\cdot)} &= \inf_{g \in W^{p(\cdot)}} \left\{ \|f - g\|_{p(\cdot)} + \frac{1}{\sigma} \|g'\|_{p(\cdot)} \right\} \\ &\leq \|f - g_{2^{j+1}}\|_{p(\cdot)} + \frac{1}{\sigma} \|g'_{2^{j+1}}\|_{p(\cdot)}. \end{aligned}$$

Using Theorem 3.3, one can write

$$\begin{aligned} \|g'_{2^{j+1}}\|_{p(\cdot)} &= \|g'_0 - g'_1\|_{p(\cdot)} + \sum_{i=0}^j \|g'_{2^{i+1}} - g'_{2^i}\|_{p(\cdot)} \\ &\lesssim \left\{ \|g_1 - g_0\|_{p(\cdot)} + \sum_{i=0}^j 2^{i+1} \|g_{2^{i+1}} - g_{2^i}\|_{p(\cdot)} \right\} \end{aligned}$$

and then we have

$$\begin{aligned} \|g'_{2^{j+1}}\|_{p(\cdot)} &\lesssim \left\{ A_0(f)_{p(\cdot)} + A_1(f)_{p(\cdot)} + \sum_{i=0}^j 2^{i+1} (A_{2^{i+1}}(f)_{p(\cdot)} + A_{2^i}(f)_{p(\cdot)}) \right\} \\ &\lesssim \left\{ A_0(f)_{p(\cdot)} + \sum_{i=0}^j 2^{i+1} A_{2^i}(f)_{p(\cdot)} \right\} \\ &\lesssim \left\{ A_0(f)_{p(\cdot)} + 2A_1(f)_{p(\cdot)} + \sum_{i=1}^j 2^{i+1} A_{2^i}(f)_{p(\cdot)} \right\}. \end{aligned}$$

Since

$$2^{i+1} A_{2^i}(f)_{p(\cdot)} \leq 4 \sum_{\nu=2^{i-1}+1}^{2^i} A_\nu(f)_{p(\cdot)}, \tag{5.1}$$

we have

$$\|g'_{2^{j+1}}\|_{p(\cdot)} \lesssim \left\{ A_0(f)_{p(\cdot)} + 2A_1(f)_{p(\cdot)} + 4 \sum_{\nu=2}^{2^j} A_\nu(f)_{p(\cdot)} \right\}.$$

Now, using (5.1), we obtain

$$A_{2^{j+1}}(f)_{p(\cdot)} = \frac{2^{j+1} A_{2^{j+1}}(f)_{p(\cdot)}}{2^{j+1}} \leq \frac{2^{j+1} A_{2^{j+1}}(f)_{p(\cdot)}}{\sigma} \leq \frac{4}{\sigma} \sum_{\nu=2^{j-1}+1}^{2^j} A_\nu(f)_{p(\cdot)}.$$

By Theorem 3.4, one can write

$$\begin{aligned} \Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} &\lesssim K\left(f, \frac{1}{\sigma}, L^{p(\cdot)}, 1\right)_{p(\cdot)} \lesssim \left\{ \|f - g_{2^{j+1}}\|_{p(\cdot)} + \frac{1}{\sigma} \|g'_{2^{j+1}}\|_{p(\cdot)} \right\} \\ &\lesssim \frac{1}{\sigma} \sum_{\nu=2^{j-1}+1}^{2^j} A_\nu(f)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=1}^{\lfloor \sigma \rfloor} A_\nu(f)_{p(\cdot)} \end{aligned}$$

and this completes the proof. □

**Theorem 5.2** *Let  $p(\cdot) \in P$  and  $f \in L^{p(\cdot)}$ . If*

$$\sum_{\nu=0}^{\infty} \nu^{r-1} A_\nu(f)_{p(\cdot)} < \infty$$

holds for some  $r \in \mathbb{N}$ , then  $f^{(r)} \in L^{p(\cdot)}$  and

$$\Omega \left( f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=0}^{[\sigma]} (\nu+1)^r A_\nu(f)_{p(\cdot)} + \sum_{\nu=[\sigma]+1}^{\infty} \nu^{r-1} A_\nu(f)_{p(\cdot)} \tag{5.2}$$

with some constant depending only on  $p(\cdot)$ .

**Proof** [Proof of Theorem 5.2] Let  $g_\sigma$  be an exponential type entire function of degree  $\leq \sigma$ , belonging to  $L^{p(\cdot)}$ , as the best approximation of  $f \in L^{p(\cdot)}$ . For natural numbers  $p \leq r$ , we consider the series

$$g_1^{(p)} + \sum_{\nu=0}^{\infty} \{g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\}. \tag{5.3}$$

Using Bernstein’s inequality (see Theorem 3.3) we have

$$\begin{aligned} \|g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\|_{p(\cdot)} &\lesssim \sigma^p \|g_{2^{\nu+1}} - g_{2^\nu}\|_{p(\cdot)} \lesssim 2^{(\nu+1)p} \|g_{2^{\nu+1}} - g_{2^\nu}\|_{p(\cdot)} \\ &\lesssim 2^{(\nu+1)p} A_{2^\nu}(f)_{p(\cdot)}. \end{aligned}$$

Now, by the following estimation,

$$2^{(\nu+1)p} A_{2^\nu}(f)_{p(\cdot)} \leq 2^{2p} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{p-1} A_\mu(f)_{p(\cdot)},$$

we have

$$\begin{aligned} \|g_1^{(p)} + \sum_{\nu=0}^{\infty} \{g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\}\|_{p(\cdot)} &\leq \|g_1^{(p)}\|_{p(\cdot)} + \sum_{\nu=0}^{\infty} \|g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\|_{p(\cdot)} \\ &\lesssim \|g_1^{(p)}\|_{p(\cdot)} + \sum_{\nu=0}^{\infty} 2^{(\nu+1)p} A_{2^\nu}(f)_{p(\cdot)} \\ &\lesssim \|g_1^{(p)}\|_{p(\cdot)} + 2^p A_1(f)_{p(\cdot)} + \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{p-1} A_\mu(f)_{p(\cdot)} \\ &\lesssim \|g_1^{(p)}\|_{p(\cdot)} + A_1(f)_{p(\cdot)} + \sum_{\mu=2}^{\infty} \mu^{p-1} A_\mu(f)_{p(\cdot)} < \infty. \end{aligned}$$

If we denote the partial sum of the above series by  $S_n^{(p)}$ , for  $p = 0, 1, 2, \dots, r$ , then the sequence of  $S_n^{(p)}$  has convergence in the norm of  $L^{p(\cdot)}$ . For  $p = r$ , one can write

$$\Omega \left( f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} \leq \Omega \left( f^{(r)} - S_n^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} + \Omega \left( S_n^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} = I_1 + I_2.$$

Now for obtaining inequality (5.2), we must estimate  $I_1$  and  $I_2$ . First, let us deal with the first item,  $I_1$ . We choose  $2^n \leq \sigma < 2^{n+1}$ . By boundedness of the operator  $T_h$  and Bernstein's inequality, we obtain

$$\begin{aligned} \Omega \left( f^{(r)} - S_n^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} &\lesssim \|f^{(r)} - S_n^{(r)}\|_{p(\cdot)} \\ &= \left\| \sum_{\nu=n+1}^{\infty} \{g_{2^{\nu+1}}^{(r)} - g_{2^\nu}^{(r)}\} \right\|_{p(\cdot)} \lesssim \sum_{\nu=n+1}^{\infty} 2^{(\nu+1)r} A_{2^\nu}(f)_{p(\cdot)} \\ &\lesssim \sum_{\nu=n+1}^{\infty} \left\{ 2^{2r} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{r-1} A_\mu(f)_{p(\cdot)} \right\} \\ &\lesssim \sum_{\mu=2^n+1}^{\infty} \mu^{r-1} A_\mu(f)_{p(\cdot)} \lesssim \sum_{\mu=\lfloor \sigma \rfloor + 1}^{\infty} \mu^{r-1} A_\mu(f)_{p(\cdot)}. \end{aligned}$$

Next, let us estimate  $I_2$ :

$$\Omega \left( S_n^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} \leq \Omega \left( g_1^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} + \sum_{\nu=0}^n \Omega \left( g_{2^{\nu+1}}^{(r)} - g_{2^\nu}^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)}.$$

Now by inequality (3.13) and Bernstein's inequality (see Theorem 3.3), we have

$$\begin{aligned} \Omega \left( S_n^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)} &\lesssim \frac{1}{\sigma} \|g_1^{(r+1)} - g_0^{(r+1)}\|_{p(\cdot)} + \frac{1}{\sigma} \sum_{\nu=0}^n \|g_{2^{\nu+1}}^{(r+1)} - g_{2^\nu}^{(r+1)}\|_{p(\cdot)} \\ &\lesssim \frac{1}{\sigma} \|g_1 - g_0\|_{p(\cdot)} + \frac{1}{\sigma} \sum_{\nu=0}^n 2^{(\nu+1)(r+1)} A_{2^\nu}(f)_{(\cdot)} \\ &\lesssim \frac{1}{\sigma} \left\{ A_0(f)_{p(\cdot)} + A_1(f)_{p(\cdot)} + \sum_{\nu=1}^n 2^{2(r+1)} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^r A_\mu(f)_{p(\cdot)} \right\} \\ &\lesssim \frac{1}{\sigma} \left\{ \sum_{\mu=0}^{2^n} (\mu+1)^r A_\mu(f)_{p(\cdot)} \right\} \lesssim \frac{1}{\sigma} \left\{ \sum_{\mu=0}^{\lfloor \sigma \rfloor} (\mu+1)^r A_\mu(f)_{p(\cdot)} \right\}. \end{aligned}$$

The last inequality completes the proof. □

### 6. Simultaneous approximation

**Theorem 6.1** *Let  $p(\cdot) \in P$ ,  $r \in \mathbb{N}$ , and  $f \in W_r^{p(\cdot)}$ . Then*

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} A_\sigma \left( f^{(r)} \right)_{p(\cdot)} \tag{6.1}$$

*holds with some constant depending only on  $p(\cdot)$ .*

**Proof** [Proof of Theorem 6.1] Let  $r = 1$ . Suppose that  $A_\sigma(f')_{p(\cdot)} = \|f' - \Theta_n(f')\|_{p(\cdot)}$ ,  $\Theta_n(f') \in \mathcal{G}_\sigma$  and

$$F(x) := \int_0^x \Theta_n(f')(t) dt$$

for  $x > 0$ . Then  $F \in \mathcal{G}_\sigma$  ([17]) and  $F'(x) = \Theta_n(f')(x)$ . Thus,

$$\begin{aligned} A_\sigma(f)_{p(\cdot)} &= A_\sigma(f - F)_{p(\cdot)} \lesssim \frac{1}{\sigma} \|(f - F)'\|_{p(\cdot)} \\ &= \frac{1}{\sigma} \|f' - F'\|_{p(\cdot)} = \frac{1}{\sigma} \|f' - \Theta_n(f')\|_{p(\cdot)} \\ &\lesssim \frac{1}{\sigma} A_\sigma(f')_{p(\cdot)} \end{aligned}$$

(6.1) follows from the last inequality. □

**Corollary 6.2** *Let  $p(\cdot) \in P$ . Then for every  $f \in W_r^{p(\cdot)}$ ,  $r \in \{0\} \cup \mathbb{N}$ , the inequalities*

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} \Omega\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)} \tag{6.2}$$

hold with constants depending only on  $p(\cdot)$ .

**7. constructive characterization of Lipschitz classes**

**Theorem 7.1** *Under the conditions of Theorem 4.1, if the inequality*

$$A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta}$$

holds for some  $\beta > 0$ , then we have

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \begin{cases} \delta^\beta & , 1 > \beta; \\ \delta^\beta \log \frac{1}{\delta} & , 1 = \beta; \\ \delta & , 1 < \beta. \end{cases}$$

**Proof** [Proof of Theorem 7.1] Let  $f \in L^{p(\cdot)}$  and

$$A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta}$$

for some  $\beta > 0$ . We suppose that  $\delta > 0$  and  $N := \lfloor 1/\delta \rfloor$ . From Theorem 5.1 we get

$$\begin{aligned} \Omega(f, \delta)_{p(\cdot)} &\leq \Omega\left(f, \frac{1}{N}\right)_{p(\cdot)} \lesssim \frac{1}{N} \sum_{\nu=0}^N A_\nu(f)_{p(\cdot)} \\ &\lesssim \frac{1}{N} A_0(f)_{p(\cdot)} + \frac{1}{N} \sum_{\nu=1}^N A_\nu(f)_{p(\cdot)} \\ &\lesssim \frac{1}{N} \left( \|f\|_{p(\cdot)} + \sum_{\nu=1}^N \frac{1}{\nu^\beta} \right). \end{aligned}$$

If  $1 > \beta$ , then by some computations we get

$$\Omega_r(f, \delta)_{p(\cdot)} \lesssim \frac{1}{N} \left( \|f\|_{p(\cdot)} + \sum_{\nu=1}^N \frac{1}{\nu^\beta} \right) \lesssim \delta^\beta.$$

If  $1 = \beta$ , then

$$\sum_{\nu=1}^N \nu^{-\beta} = \sum_{\nu=1}^n \nu^{-1} \leq 1 + \log(1/\delta)$$

and hence

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \delta^\beta \log(1/\delta).$$

If  $1 < \beta$ , then the series  $\sum_{j=0}^\infty j^{-\beta}$  is convergent and

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \delta \left( A_0(f)_{p(\cdot)} + \sum_{j=1}^\infty j^{-\beta} \right) \lesssim \delta$$

holds. □

Using Theorem 5.2 we similarly get the following:

**Corollary 7.2** *Let  $p(\cdot) \in P$  and  $f \in L^{p(\cdot)}$ . If*

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^{r+\alpha}}, \quad \alpha > 0,$$

then  $f \in W_{p(\cdot)}^r$  and

$$\Omega(f^{(r)}, \delta)_{p(\cdot)} \lesssim \begin{cases} \delta^\alpha & , 1 > \alpha, \\ \delta^\alpha \log(1/\delta) & , 1 = \alpha, \\ \delta & , 1 < \alpha. \end{cases}$$

**Theorem 7.3** *Let  $0 < \beta < 1$  and  $r \in \mathbb{N}$ . Under the conditions of Theorem 4.1, we have:*

- (i)  $f \in Lip_\beta p(\cdot)$  iff  $A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta}$ .
- (ii)  $f \in W_{p(\cdot)}^{r,\beta}$  iff  $A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta-r}$ .

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