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r-Submodules and sr-Submodules

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Abstract: In this article, we introduce new classes of submodules called r-submodule and special r-submodule, which are two different generalizations of r-ideals. Let M be an R-module, where R is a commutative ring. We call a proper submodule N of M an r-submodule (resp., special r-submodule) if the condition $am \in N$ with $ann_M(a) = 0_M$ (resp., $ann_R(m) = 0$) implies that $m \in N$ (resp., $a \in (N :_R M)$) for each $a \in R$ and $m \in M$. We also give various results and examples concerning r-submodules and special r-submodules.

Key words: r-Ideal, prime ideal, r-submodule, special r-submodule, prime submodule

1. Introduction

Throughout, all rings will be commutative with $1 \neq 0$ and all modules will be unitary. In particular, R will always denote such a ring. The concept of r-ideals was introduced and studied by Mohamadian in [9]. Recall from [9] that a proper ideal I of R is an r-ideal if $ab \in I$ and $ann(a) = \{r \in R : ra = 0\} = 0$, and then $b \in I$ for each $a, b \in R$. In this article, we give two different generalizations of this concept to modules by r-submodules and special r-submodules.

Let us give some definitions and notations we will need throughout this study. Let M be an R-module. Then a submodule N of M is proper whenever $N \neq M$. If N is a submodule of M and K is a nonempty subset of M, then the ideal $\{r \in R : rK \subseteq N\}$ is denoted by $(N :_R K)$. In particular, we use $Ann_R(M)$ instead of $(0_M :_R M)$. Furthermore, for each element m of M, we denote $(0_M :_R \{m\})$ by $ann_R(m)$. Suppose that N is a submodule of M and S is a nonempty subset of R. Denote by $(N :_M S)$ the set of all $m \in M$ with $Sm \subseteq N$. In particular, we use $ann_M(a)$ instead of $(0_M :_M \{a\})$ for each $a \in R$. Also, the sets $\{a \in R : ann_M(a) \neq 0_M\}$ and $\{m \in M : ann_R(m) \neq 0\}$ will be designated by $Z(M)$ and $T(M)$, respectively.

The prime submodule, which is an important subject of module theory, has been widely studied by various authors. See, for example, [2, 4, 8] and [3, 5, 7]. Recall that a prime submodule is a proper submodule N of M with the property that $am \in N$ implies that $a \in (N :_R M)$ or $m \in N$ for each $a \in R, m \in M$. In that case, $(N :_R M)$ is a prime ideal of R. In Section 2, we extend the concept of r-ideals to modules by r-submodules, and we investigate some properties of r-submodules with similar prime submodules. We define a proper submodule N of M as an r-submodule if whenever $am \in N$ with $ann_M(a) = 0_M$, then $m \in N$ for each $a \in R$ and $m \in M$. Since there is no proper submodule of zero module, from now on we assume that R-module M is nonzero. Among many results in this paper, it is shown in Proposition 4 that

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a proper submodule N of M is an r-submodule if and only if $N = (N :_M a)$ for every $a \in R - Z(M)$. In Theorem 1 we show that a proper submodule N of M is an r-submodule of M if and only if whenever I is an ideal of R such that $I \cap (R - Z(M)) \neq \emptyset$ and L is a submodule of M with $IL \subseteq N$, then $L \subseteq N$. Also, it is proved in Proposition 7 that if N is a maximal r-submodule of M, then N is prime submodule. Finally, in Theorem 8, we characterize the r-submodules of Cartesian products of modules.

In Section 3, we introduce the special r-submodule, which is another generalization of r-ideals. We call a proper submodule N of M a special r-submodule (briefly sr-submodule) if for each $a \in R$ and $m \in M$, $am \in N$ with $ann_R(m) = 0$, and then $a \in (N :_R M)$. In Example 11, it is shown that r-submodules and sr-submodules are different concepts, i.e. neither implies the other. In Theorem 13, we show that an R-module M is torsion-free if and only if M is faithful and the zero submodule is the only sr-submodule of M. We characterize, in Theorem 14, all R-modules in which every proper submodule is an sr-submodule. Finally we characterize, in Theorem 15, the sr-submodules of Cartesian products of modules.

2. r-Submodules

Definition 1 *Let M be an R-module. A proper submodule N of M is said to be an r-submodule if $am \in N$ with $ann_M(a) = 0_M$ implies that $m \in N$ for each $a \in R, m \in M$.*

Note that a proper submodule N of M being an r-submodule means simply that $Z(M/N) \subseteq Z(M)$ and also the r-submodules of R-module R are precisely the r-ideals of R. Now we give some examples of r-submodules.

Example 1 *Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$ for $n \geq 2$. Let $\langle \overline{x} \rangle$ be a proper submodule of $\mathbb{Z}_n$. Then $\gcd(x, n) = d > 1$. This implies that $\langle \overline{x} \rangle = \langle \overline{d} \rangle$, and also note that $\mathbb{Z}_n / \langle \overline{x} \rangle$ is isomorphic to $\mathbb{Z}$-module $\mathbb{Z}_d$. Since $Z(\mathbb{Z}_d) \subseteq Z(\mathbb{Z}_n)$, it follows that $\langle \overline{x} \rangle$ is an r-submodule of $\mathbb{Z}_n$.*

Example 2 *Consider $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. We know that $E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^t} + \mathbb{Z}$ for $t \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}\}$ is a submodule of $\mathbb{Q}/\mathbb{Z}$, where p is a prime number. Then any proper submodule of $E(p)$ is of the form $G_{t_0} = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^{t_0}} + \mathbb{Z}$ for some $r \in \mathbb{Z}\}$ for some $t_0 \in \mathbb{N} \cup \{0\}$ [12]. $E(p)$ does not have any prime submodule. However, we show that every proper submodule of $E(p)$ is an r-submodule. First, note that $ann_{E(p)}(m) = 0_{E(p)}$ if and only if $\gcd(p, m) = 1$ for $m \in \mathbb{Z}$. Let $m \in \mathbb{Z}$, $\frac{r}{p^t} + \mathbb{Z} \in E(p)$ such that $m\left(\frac{r}{p^t} + \mathbb{Z}\right) = \frac{mr}{p^t} + \mathbb{Z} \in G_{t_0}$ and $\gcd(p, m) = 1$. If $t \leq t_0$, then we have $\frac{r}{p^t} + \mathbb{Z} \in G_{t_0}$. Now, assume that $t > t_0$. Since $\frac{mr}{p^t} + \mathbb{Z} \in G_{t_0}$, we have $\frac{mr}{p^t} + \mathbb{Z} = \frac{k}{p^{t_0}} + \mathbb{Z}$ for some $k \in \mathbb{Z}$, and so $\frac{mr}{p^t} - \frac{k}{p^{t_0}} \in \mathbb{Z}$. Then we have $mr \equiv kp^{t-t_0} \pmod{p^t}$. Since $\gcd(m, p^t) = 1$, we get $r \equiv k'kp^{t-t_0} \pmod{p^t}$ for some $k' \in \mathbb{Z}$, and so $\frac{r}{p^t} + \mathbb{Z} = \frac{k'k}{p^{t_0}} + \mathbb{Z} \in G_{t_0}$. Hence, G_{t_0} is an r-submodule of $E(p)$.*

Lemma 1 *If N is an r-submodule of M, then $(N :_R M) \subseteq Z(M)$.*

Proof It follows from the fact that $(N :_R M) = Ann(M/N) \subseteq Z(M/N) \subseteq Z(M)$. □

The converse of Lemma 1 is not always valid, i.e. if N is a submodule of M with $(N :_R M) \subseteq Z(M)$, then N need not be an r-submodule of M. We give a counter example in the following.

Example 3 *Consider the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$ and the submodule $N = 2\mathbb{Z} \times 0$ of $M = \mathbb{Z} \times \mathbb{Z}$. Note that $(N :_{\mathbb{Z}} M) = \langle 0 \rangle \subseteq Z(M)$ and also M/N is isomorphic to $\mathbb{Z}$-module $\mathbb{Z}_2 \times \mathbb{Z}$. Since $2 \in Z(\mathbb{Z}_2 \times \mathbb{Z}) - Z(M)$, we have $Z(\mathbb{Z}_2 \times \mathbb{Z}) \nsubseteq Z(M)$ and thus N is not an r-submodule of M.*

The following examples show that the concepts of prime submodule and r-submodule are different.

Example 4 *(i) Consider the $\mathbb{Z}$-module $\mathbb{Z}$. Of course, $3\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$, since $(3\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 3\mathbb{Z} \nsubseteq Z(\mathbb{Z})$, it follows that $3\mathbb{Z}$ is not an r-submodule of $\mathbb{Z}$.*

(ii) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{18}$. By Example 1, we know that $\langle \overline{9} \rangle$ is an r-submodule of $\mathbb{Z}_{18}$ but it is not a prime submodule. Since $3.\overline{3} = \overline{9} \in \langle \overline{9} \rangle$ but $3 \notin (\langle \overline{9} \rangle : \mathbb{Z}_{18}) = 9\mathbb{Z}$ and $\overline{3} \notin \langle \overline{9} \rangle$.

Note that in a vector space, any proper subspace is a prime submodule. In the following proposition, we show it is true for r-submodules and so in a vector space the prime submodule coincides with the r-submodule.

Proposition 1 *Let V be a vector space over a field F. Then every proper subspace W of V is an r-submodule.*

Proof Follows from $Z(V/W) = 0$. □

Proposition 2 *For a prime submodule N of M, N is an r-submodule if and only if $(N :_R M) \subseteq Z(M)$.*

Proof If N is prime submodule, then $Z(M/N) = (N :_R M)$ so that N is an r-submodule iff $(N :_R M) \subseteq Z(M)$. □

Proposition 3 *Let M be an R-module. Then the following hold:*

(i) The zero submodule is an r-submodule.

(ii) The intersection of an arbitrary nonempty set of r-submodules is an r-submodule.

Proof (i) It is clear that $Z(M/0_M) = Z(M)$ and so the zero submodule is an r-submodule.

(ii) Let N_i be an r-submodule of M for every $i \in \Delta$. Suppose that $am \in \bigcap_{i \in \Delta} N_i$ with $ann_M(a) = 0_M$ for $a \in R, m \in M$. Then we have $am \in N_i$ for every $i \in \Delta$. Since N_i is an r-submodule, we conclude that $m \in N_i$ for every $i \in \Delta$, and thus $m \in \bigcap_{i \in \Delta} N_i$. Hence, $\bigcap_{i \in \Delta} N_i$ is an r-submodule. □

Note that the sum of two r-submodules need not be an r-submodule. See the following example.

Example 5 *Consider the $\mathbb{Z}$-module $\mathbb{Z}_{10}$. Then $\langle \overline{2} \rangle$ and $\langle \overline{5} \rangle$ are r-submodules but $\langle \overline{2} \rangle + \langle \overline{5} \rangle = \mathbb{Z}_{10}$ is not an r-submodule of $\mathbb{Z}_{10}$.*

It is well known if N is prime submodule of M, then $(N :_R M)$ is prime ideal of R. However, the following example shows that this is not always correct for r-submodules.

Example 6 *Consider the $\mathbb{Z}$-module $\mathbb{Z}_4$. $\langle\overline{2}\rangle$ is an r-submodule but $\left(\langle\overline{2}\rangle :_{\mathbb{Z}} \mathbb{Z}_4\right) = 2\mathbb{Z}$ is not an r-ideal of $\mathbb{Z}$, since a domain has no nonzero r-ideals.*

Recall that a nonempty subset S of R is multiplicatively closed precisely when $ab \in S$ for all $a, b \in S$. For instance, $S = R - Z(M)$ is a multiplicatively closed subset of R. Suppose that S is a multiplicatively closed subset of R and M is an R-module. Then we denote the module of fraction at S by $S^{-1}M$. Note that $S^{-1}M$ is both an $S^{-1}R$-module and an R-module. Also, for every submodule N of M, $S^{-1}N$ is an $S^{-1}R$-submodule of $S^{-1}M$. Let M be an R-module. Consider $S^{-1}M$ as an R-module. The natural R-homomorphism is defined as follows:

$$\pi : M \to S^{-1}M, \text{ for all } m \in M, \ \pi(m) = \tfrac{m}{1}.$$

Proposition 4 *Let N be a proper submodule of M. Then the following are equivalent:*

(i) N is an r-submodule of M.

(ii) $aM \cap N = aN$ for every $a \in R - Z(M)$.

(iii) $(N :_M a) = N$ for every $a \in R - Z(M)$.

(iv) $N = \pi^{-1}(L)$, where $S = R - Z(M)$ and L is an $S^{-1}R$-submodule of $S^{-1}M$.

Proof $(i) \Rightarrow (ii)$: Suppose that N is an r-submodule. For every $a \in R$, the inclusion $aN \subseteq aM \cap N$ always holds. Let $a \in R$ with $ann_M(a) = 0_M$ and $x \in aM \cap N$. Then we get $x = am \in N$ for some $m \in M$. Since N is an r-submodule, $m \in N$ and thus $x = am \in aN$. Hence, we get $aM \cap N = aN$.

$(ii) \Rightarrow (iii)$: It is well known that $N \subseteq (N :_M a)$ for every $a \in R$. Let $a \in R$ such that $ann_M(a) = 0_M$ and $m \in (N :_M a)$. Then we have $am \in N$, and so $am \in aM \cap N = aN$ by (ii). Thus, we have $am = an$ for some $n \in N$. Since $ann_M(a) = 0_M$, we conclude that $m = n \in N$. Hence, we have $(N :_M a) \subseteq N$.

$(iii) \Rightarrow (iv)$: Since $N \subseteq \pi^{-1}(S^{-1}N)$, it is sufficient to show that $\pi^{-1}(S^{-1}N) \subseteq N$. Let $m \in \pi^{-1}(S^{-1}N)$. Then we have $\pi(m) = \frac{m}{1} \in S^{-1}N$ and so $am \in N$ for some $a \in S$. Thus, by (iii), we conclude that $m \in (N :_M a) = N$.

$(iv) \Rightarrow (i)$: Suppose that $N = \pi^{-1}(L)$, where $S = R - Z(M)$ and L is an $S^{-1}R$-submodule of $S^{-1}M$. Let $am \in N$ and $ann_M(a) = 0_M$. Then we have $\pi(am) = \frac{am}{1} \in L$. Since $a \in S$ and L is an $S^{-1}R$-submodule, we conclude that $\frac{1}{a}\frac{am}{1} = \frac{m}{1} = \pi(m) \in L$ and so $m \in \pi^{-1}(L) = N$, as needed. □

In [11], Ribenboim defined the pure submodule as a proper submodule N of M if $aM \cap N = aN$ for every $a \in R$. By Proposition 4, every pure submodule is also an r-submodule. However, in the following, we show that the converse is not necessarily correct.

Example 7 *Consider the $\mathbb{Z}$-module $\mathbb{Z}_{16}$ and the submodule $N = \langle\overline{2}\rangle$. Then N is an r-submodule of $\mathbb{Z}_{16}$, but N is not a pure submodule of $\mathbb{Z}_{16}$, because $2N = \langle\overline{4}\rangle \subsetneq 2\mathbb{Z}_{16} \cap N = \langle\overline{2}\rangle$.*

Proposition 5 *Suppose that N is an r-submodule of M and S is a nonempty subset of R with $S \nsubseteq (N :_R M)$. Then $(N :_M S)$ is an r-submodule of M. In particular, $(0_M :_M S)$ is always an r-submodule if $S \nsubseteq Ann_R(M)$.*

Proof Let $am \in (N :_M S)$ with $ann_M(a) = 0_M$ for $a \in R, m \in M$. Then we have $asm \in N$ for every $s \in S$. Since N is an r-submodule, we get $sm \in N$ for every $s \in S$ and this yields $m \in (N :_M S)$, as is needed. The rest follows easily. □

Corollary 1 *If $a \notin Ann_R(M)$, then $ann_M(a)$ is an r-submodule of M.*

Proposition 6 *For any R-module M, the following hold if the zero submodule is the only r-submodule:*

(i) The zero submodule is a prime submodule of M.

(ii) $Ann_R(M)$ is a prime ideal of R.

Proof (i) Let $am = 0_M$ and $a \notin Ann_R(M)$, where $a \in R$, $m \in M$. Then by previous corollary, $ann_M(a)$ is an r-submodule and thus $ann_M(a) = 0_M$. Hence, we have $m = 0_M$, as needed.

(ii) It follows from (i). □

Remember that a proper submodule N of M is prime if and only if for every ideal I of R and submodule L of M with $IL \subseteq N$, then either $I \subseteq (N :_R M)$ or $L \subseteq N$. Now we present a similar result for r-submodules as follows.

Theorem 1 *For a proper submodule N of M, the following hold:*

(i) N is an r-submodule of M if and only if whenever I is an ideal of R such that $I \cap (R - Z(M)) \neq \emptyset$ and L is a submodule of M with $IL \subseteq N$, then $L \subseteq N$.

(ii) If $(N :_R M) \subseteq Z(M)$ and N is not an r-submodule of M, then there exist an ideal I of R and a submodule L of M such that $I \cap (R - Z(M)) \neq \emptyset$, $N \subsetneqq L$, $(N :_R M) \subsetneqq I$, and $IL \subseteq N$.

Proof (i) Suppose that N is an r-submodule and $IL \subseteq N$ for some ideal I of R with $I \cap (R - Z(M)) \neq \emptyset$ and submodule L of M. Then there exist $a \in I$ such that $ann_M(a) = 0_M$. Since $al \in N$ for every $l \in L$ and N is an r-submodule, we conclude that $l \in N$, and thus $L \subseteq N$. For the converse, let $am \in N$ and $ann_M(a) = 0_M$ for $a \in R, m \in M$. We take $I = aR$ and $L = Rm$. Note that $I \cap (R - Z(M)) \neq \emptyset$ and $IL \subseteq N$. Then by assumption we have $Rm \subseteq N$, and so $m \in N$. Hence, N is an r-submodule.

(ii) Since N is not an r-submodule, there exist $a \in R, m \in M$ such that $am \in N$ with $ann_M(a) = 0_M$ and $m \notin N$. We take $I = (N :_R m)$. Note that $a \in I$ and $a \notin (N :_R M)$ since $ann_M(a) = 0_M$. Thus, $(N :_R M) \subsetneqq I$. Now we take $L = (N :_M I)$. Since $m \notin N$ and $m \in L$, $N \subsetneqq L$. Hence, we get $N \subsetneqq L$, $(N :_R M) \subsetneqq I$ and $IL = I(N :_M I) \subseteq N$. □

Theorem 2 *Suppose that K_1, K_2, L are submodules of M and I is an ideal of R with $I \cap (R - Z(M)) \neq \emptyset$. Then the following hold:*

(i) If K_1, K_2 are r-submodules of M with $IK_1 = IK_2$, then $K_1 = K_2$.

(ii) If IL is an r-submodule, then $IL = L$. In particular, L is an r-submodule.

Proof (i) Since $IK_1 \subseteq K_2$ and K_2 is an r-submodule, we have $K_1 \subseteq K_2$ by Theorem 1(i). Similarly, we have $K_2 \subseteq K_1$, and so $K_1 = K_2$.

(ii) Since IL is an r-submodule and $IL \subseteq IL$, we have $L \subseteq IL \subseteq L$ by Theorem 1(i), and so $IL = L$. □

Theorem 3 *Suppose that $N_1, N_2, ..., N_n$ are prime submodules of M such that $(N_i :_R M)$s are not comparable. If $\bigcap_{i=1}^{n} N_i$ is an r-submodule, then N_i is an r-submodule for each $i \in \{1, 2, ..., n\}$.*

Proof Let $am \in N_k$ with $ann_M(a) = 0_M$ for $a \in R, m \in M$. Since $(N_i :_R M)$s are not comparable, we have $r \in \left(\bigcap_{\substack{i=1 \\ i\neq k}}^{n} (N_i :_R M)\right) - (N_k :_R M)$ for some $r \in R$. Then we have $ram \in \bigcap_{i=1}^{n} N_i$. Since $\bigcap_{i=1}^{n} N_i$ is an r-submodule, we conclude that $rm \in \bigcap_{i=1}^{n} N_i \subseteq N_k$. Thus, we have $m \in N_k$, because N_k is a prime submodule and $r \notin (N_k :_R M)$. Hence, N_k is an r-submodule. □

Proposition 7 *If N is a maximal r-submodule of M, then N is prime submodule.*

Proof Let $am \in N$ and $m \notin N$; we show that $a \in (N :_R M)$. Assume that $a \notin (N :_R M)$. Then $(N :_M a)$ is an r-submodule by Proposition 5. Since N is a maximal r-submodule, we conclude that $m \in (N :_M a) = N$, a contradiction. Thus, we have $a \in (N :_R M)$, as needed. □

Let recall the following well-known theorem of the prime avoidance lemma: suppose that $N \subseteq \bigcup_{j=1}^{n} N_j$ and at most two of N_j are not prime submodules. Then $N \subseteq N_i$ for some $1 \leq i \leq n$ if the condition $(N_i :_R M) \nsubseteq (N_j :_R M)$ holds for every $i \neq j$ [4, 7]. Now we present a result with a similar prime avoidance lemma for r-submodules.

Proposition 8 *Let $N \subseteq \bigcup_{j=1}^{n} N_j$ for submodules $N, N_1, N_2, ..., N_n$ of M. Suppose that N_k is an r-submodule and $(N_j :_R M) \cap (R - Z(M)) \neq \emptyset$ for every $j \neq k$. If $N \nsubseteq \bigcup_{j\neq k} N_j$, then $N \subseteq N_k$.*

Proof We may asume that $k = 1$. Since $N \nsubseteq \bigcup_{j=2}^{n} N_j$, there exists $m \in N$ such that $m \notin \bigcup_{j=2}^{n} N_j$, namely $m \in N_1$. Let $n \in N \cap N_2 \cap N_3 \cap ... \cap N_n$. Then it is clear that $m + n \in N - \bigcup_{j=2}^{n} N_j$, and thus $m + n \in N_1$. This gives $n \in N_1$, and so $N \cap N_2 \cap N_3 \cap ... \cap N_n \subseteq N_1$. Since $(N_j :_R M) \cap (R - Z(M)) \neq \emptyset$, there exists $a_j \in (N_j :_R M)$ such that $ann_M(a_j) = 0_M$ for $j = 2, 3, ..., n$. Then note that $ann_M(a_2 a_3 ... a_n) = 0_M$. Now we take $I = \bigcap_{j=2}^{n} (N_j :_R M)$. Then we have $a_2 a_3 ... a_n \in I \cap (R - Z(M))$. Since $IN \subseteq N \cap N_2 \cap N_3 \cap ... \cap N_n \subseteq N_1$ and $I \cap (R - Z(M)) \neq \emptyset$, by Theorem 1, we get $N \subseteq N_1$. □

Definition 2 *A nonempty subset S of R is said to be an r-multiplicatively closed subset precisely when $R - Z(M) \subseteq S$ and $ab \in S$, for all $a \in R - Z(M)$ and $b \in S$.*

Example 8 *For every r-submodule N of M, $R-(N:_R M)$ is an r-multiplicatively closed subset of R. We know that if N is an r-submodule, then $(N:_R M)\subseteq Z(M)$ and so $R-Z(M)\subseteq R-(N:_R M)$. Let $a\in R-Z(M)$ and $b\in R-(N:_R M)$. Suppose that $ab\in(N:_R M)$. Then we have $abm\in N$ for every $m\in M$ and $ann_M(a)=0_M$. Since N is an r-submodule, it follows that $bm\in N$ and thus $b\in(N:_R M)$, a contradiction. Hence, $R-(N:_R M)$ is an r-multiplicatively closed subset.*

Definition 3 *Let S be an r-multiplicatively closed subset of R and S^* be a nonempty subset of M. Then S^* is called an S-closed subset of M if $am\in S^*$ for each $a\in S$ and $m\in S^*$.*

Theorem 4 *Let S^* be an S-closed subset of M, where S is an r-multiplicatively closed subset of R. Suppose that N is a submodule of M with $N\cap S^*=\emptyset$. Then there exists an r-submodule L of M with $N\subseteq L$ and $L\cap S^*=\emptyset$.*

Proof Let $\Omega=\{L' :\ L'$ be a submodule of M with $N\subseteq L'$ and $L'\cap S^*=\emptyset\}$. Since $N\in\Omega$, we have $\Omega\neq\emptyset$. By Zorn's lemma, Ω has a maximal element L with $N\subseteq L$ and $L\cap S^*=\emptyset$. Assume that L is not an r-submodule of M. Then there exist $a\in R, m\in M$ such that $am\in L$, $ann_M(a)=0_M$ and $m\notin L$. Since $m\notin L$ and $m\in(L:_M a)$, $L\subsetneqq(L:_M a)$. By the maximality of L, we get $m'\in(L:_M a)\cap S^*$. Since $a\in S$, we get the result that $am'\in L\cap S^*$, a contradiction. Hence, L is an r-submodule. □

Theorem 5 *Let M be an R-module. Then every proper submodule of M is an r-submodule if and only if for every submodule N of M, $aN=N$ for every $a\in R-Z(M)$.*

Proof Suppose that every proper submodule of M is an r-submodule. Let N be a submodule and $a\in R-Z(M)$. Assume that $N=M$. If $aM\neq M$, then aM is an r-submodule of M. Since $am\in aM$ for every $m\in M$ and $ann_M(a)=0_M$, we conclude that $m\in aM$, and thus $aM=M$, a contradiction. Thus, we have $aM=M$. Now assume that N is a proper submodule of M. Then $aN\subseteq N\neq M$ and so aN is an r-submodule of M. Since $an\in aN$ for every $n\in N$, similarly we get the result that $aN=N$. Conversely, suppose that $aN=N$ for every submodule N of M and every $a\in R-Z(M)$. Let N be a proper submodule of M and $a\in R-Z(M)$. Then we have $aM\cap N=aN$, and so by Proposition 4, N is an r-submodule of M. □

Let M be an R-module. Recall that the idealization of M in R, which is denoted by $R(+)M=\{(a,m): a\in R,\ m\in M\}$, is a commutative ring with component-wise addition and multiplication $(a_1,m_1)(a_2,m_2)=(a_1a_2,a_1m_2+a_2m_1)$ [10]. In [1,6], the zero divisor set of $R(+)M$ was characterized as follows:

$$Z(R(+)M)=\{(a,m): a\in Z(R)\cup Z(M),\ m\in M\},$$

where $Z(R)=\{a\in R: ann(a)\neq 0\}$.

Corollary 2 *For every $a\in R$ and $m\in M$, $ann_{R(+)M}(a,m)=0$ if and only if $ann(a)=0$ and $ann_M(a)=0_M$.*

Suppose that N is a submodule of M and J is an ideal of R. Then it is clear that $J(+)N$ is an ideal of $R(+)M$ if and only if $JM\subseteq N$. In that case $J(+)N$ is called a homogeneous ideal.

Proposition 9 *Suppose that J is an r-ideal of R. Then $J(+)M$ is an r-ideal of $R(+)M$.*

Proof Let J be an r-ideal of R. Suppose that $(a_1,m_1)(a_2,m_2) = (a_1a_2, a_1m_2 + a_2m_1) \in J(+)M$ and $ann_{R(+)M}(a_1,m_1) = 0$. Since $ann_{R(+)M}(a_1,m_1) = 0$, we have $ann(a_1) = 0$. Then we get the result that $a_2 \in J$, because J is an r-ideal and $a_1a_2 \in J$. Thus, we have $(a_2,m_2) \in J(+)M$. Consequently, $J(+)M$ is an r-ideal. □

The converse of the previous proposition is not always true. We have a counterexample as follows.

Example 9 *Consider the $\mathbb{Z}(+)\mathbb{Z}_2$ and the ideal $2\mathbb{Z}(+)\mathbb{Z}_2$ of $\mathbb{Z}(+)\mathbb{Z}_2$. We know that $2\mathbb{Z}$ is not an r-ideal of $\mathbb{Z}$ but $2\mathbb{Z}(+)\mathbb{Z}_2$ is an r-ideal of $\mathbb{Z}(+)\mathbb{Z}_2$.*

Theorem 6 *Suppose that J is an r-ideal of R and N is an r-submodule of M with $JM \subseteq N$. Then $J(+)N$ is an r-ideal of $R(+)M$.*

Proof Let $(a_1,m_1)(a_2,m_2) \in J(+)N$ with $ann_{R(+)M}(a_1,m_1) = 0$. Then we have $ann(a_1) = 0$ and $ann_M(a_1) = 0_M$. Since J is an r-ideal and $a_1a_2 \in J$, we have $a_2 \in J$. Thus, we have $a_2m_1 \in N$ and so $a_1m_2 \in N$. As N is an r-submodule, it follows that $m_2 \in N$ and so $(a_2,m_2) \in J(+)N$. Hence, $J(+)N$ is an r-ideal. □

Example 9 also serves as a counterexample of the previous theorem, but we prove that the converse of Theorem 6 is valid when $Z(R) = Z(M)$ as follows.

Theorem 7 *Let M be an R-module and $Z(R) = Z(M)$. If $J(+)N$ is an r-ideal of $R(+)M$ with $N \neq M$, then J is an r-ideal of R and N is an r-submodule of M.*

Proof Suppose that $J(+)N$ is an r-ideal. Since $Z(R) = Z(M)$, $ann_{R(+)M}(a_1,m_1) = 0$ if and only if $ann(a_1) = 0$. Let $a,b \in R$ with $ab \in J$ and $ann(a) = 0$. Then we have $ann_{R(+)M}(a,0_M) = 0$ and so $(a,0_M)(b,0_M) = (ab,0_M) \in J(+)N$. Since $J(+)N$ is an r-ideal, we get the result that $(b,0_M) \in J(+)N$ and thus $b \in J$. Hence, J is an r-ideal of R. Suppose that $am \in N$ with $ann_M(a) = 0_M$ for $a \in R$, $m \in M$. Then $ann_{R(+)M}(a,0_M) = 0$, so we get $(a,0_M)(0,m) = (0,am) \in J(+)N$. As $J(+)N$ is an r-ideal, we conclude that $(0,m) \in J(+)N$ and so $m \in N$. Hence, N is an r-submodule. □

Let M_1 be an R_1-module and M_2 an R_2-module, where R_1 and R_2 are commutative rings with identity. Suppose that $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M becomes an R-module with coordinate-wise addition and the scalar multiplication $(a_1,a_2)(m_1,m_2) = (a_1m_1, a_2m_2)$ for every $a_1 \in R_1, a_2 \in R_2$; $m_1 \in M_1$ and $m_2 \in M_2$. Also, every submodule N of M has the form $N = N_1 \times N_2$, where N_1 is a submodule of M_1 and N_2 is a submodule of M_2. The following theorem characterizes the r-submodule of Cartesian product of modules.

Lemma 2 *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a submodule of M. Then the following are equivalent:*

(i) N is an r-submodule of M.

(ii) $N_1 = M_1$ and N_2 is an r-submodule of M_2 or N_1 is an r-submodule of M_1 and $N_2 = M_2$ or N_1, N_2 are r-submodules of M_1 and M_2, respectively.

Proof $(i) \Rightarrow (i)$: First note that M/N is isomorphic to $(M_1/N_1) \times (M_2/N_2)$ and $Z(M/N) = (Z(M_1/N_1) \times R_2) \cup (R_1 \times Z(M_2/N_2))$. Suppose that N is an r-submodule of M and assume that $N_1 = M_1$. Since N is a proper submodule of M, $N_2 \neq M_2$. Then $Z(M/N) = R_1 \times Z(M_2/N_2) \subseteq Z(M) = (Z(M_1) \times R_2) \cup (R_1 \times Z(M_2))$ and so $Z(M_2/N_2) \subseteq Z(M_2)$. This implies that N_2 is an r-submodule of M_2. In other cases, a similar argument shows that (i) implies (ii).

$(ii) \Rightarrow (i)$: Conversely, suppose that (ii) holds. Assume that N_1, N_2 are r-submodules of M_1 and M_2, respectively. Then $Z(M_1/N_1) \subseteq Z(M_1)$ and $Z(M_2/N_2) \subseteq Z(M_2)$. This implies that $Z(M/N) = (Z(M_1/N_1) \times R_2) \cup (R_1 \times Z(M_2/N_2)) \subseteq (Z(M_1) \times R_2) \cup (R_1 \times Z(M_2)) = Z(M)$, i.e. N is an r-submodule of M. In other cases, one can similarly prove that N is an r-submodule. □

Theorem 8 *Suppose that $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$, where M_i is an R_i-module for $n \geq 1$ and $1 \leq i \leq n$. Let $N = N_1 \times N_2 \times ... \times N_n$ be a submodule of M. Then the following are equivalent:*

(i) N is an r-submodule of M.

(ii) $N_i = M_i$ for $i \in \{t_1, t_{2,}..., t_k : k < n\}$ and N_i is an r-submodule of M_i for $i \in \{1, 2, ..., n\} \setminus \{t_1, t_{2,}..., t_k\}$.

Proof To prove the claim, we use induction on n. If $n = 1$, then it is clear that $(i) \Leftrightarrow (ii)$. If $n = 2$, by Lemma 2, (i) and (ii) are equal. Assume that $n \geq 3$ and the claim is valid when $K = M_1 \times M_2 \times ... \times M_{n-1}$. We prove that the claim is true when $M = K \times M_n$. Then by Lemma 2 we get the result that N is an r-submodule if and only if $N = K \times N_n$ for some r-submodule N_n of M_n or $N = L \times M_n$ for some r-submodule L of K or $N = L \times N_n$ for some r-submodule L of K and some r-submodule N_n of M_n. By induction hypothesis, the result is valid in three cases. □

3. Special r-submodules

In this section, we give another type of generalization of r-ideals to modules.

Definition 4 *Let M be an R-module. Then a submodule N of M is said to be a special r-submodule (briefly sr-submodule) if $N \neq M$, for each $a \in R, m \in M$ with $am \in N$ and $ann_R(m) = 0$, then $a \in (N :_R M)$.*

If we consider R-module R, the sr-submodules and r-submodules coincide. Now we give some examples of sr-submodules in the following.

Example 10 *By Example 1, we know that all proper submodules of $\mathbb{Z}$-module $\mathbb{Z}_n$ are r-submodules. One can easily see that all proper submodules of $\mathbb{Z}_n$ are also sr-submodules. Now consider the $\mathbb{Z}$-module $E(p)$. By Example 2, all proper submodules of $E(p)$ are r-submodules. Since $ann_{\mathbb{Z}}\left(\frac{r}{p^t} + \mathbb{Z}\right) \neq 0$ for each $\frac{r}{p^t} + \mathbb{Z} \in E(p)$, we conclude that all proper submodules of $E(p)$ are also sr-submodules.*

In the previous example, r-submodules and sr-submodules are equal, but these concepts are different. See the following examples.

Example 11 *(i) By Proposition 1, the subspace $N = \{(x, 0) : x \in \mathbb{R}\}$ of $M = \mathbb{R}^2$ is an r-submodule, but $2(1, 0) = (2, 0) \in N$, $ann_{\mathbb{R}}(1, 0) = 0$, and $2 \notin (N :_{\mathbb{R}} M)$; thus, we get the result that N is not an sr-submodule.*

(ii) Consider the $R = \mathbb{Z} \times \mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_2$ and the submodule $N = 2\mathbb{Z} \times \overline{0}$. Since $ann_R(m) \neq 0$ for every $m \in M$, it follows that N is an sr-submodule of M. However, it is not an r-submodule since $(2, 1)\left(1, \overline{0}\right) = \left(2, \overline{0}\right) \in N$, $ann_M(2, 1) = 0_M$, and $\left(1, \overline{0}\right) \notin N$.

Lemma 3 *If N is an sr-submodule of M, then $N \subseteq T(M)$.*

Proof Assume that $N \nsubseteq T(M)$. There exists $m \in N$ with $ann_R(m) = 0$. Since $1.m = m \in N$ and N is an sr-submodule, we get the result that $1 \in (N :_R M)$, i.e. $N = M$, a contradiction. Hence, we have $N \subseteq T(M)$. □

The converse of the previous lemma is not always true. See the following example.

Example 12 *Consider the $R = \mathbb{R} \times \mathbb{Z}$-module $M = \mathbb{C} \times \mathbb{Z}$ and the submodule $N = \mathbb{R} \times 0$ of M. Note that $T(M) = (0_{\mathbb{C}} \times \mathbb{Z}) \cup (\mathbb{C} \times 0)$ and $(N :_R M) = 0_R$. Thus, we have $N \subseteq T(M)$. Since $(2,0)(2+0i,1) = (4,0) \in N$, $ann_R(2+0i,1) = 0_R$, and $(2,0) \notin (N :_R M)$, we get the result that N is not an sr-submodule.*

Example 13 *(i) Every nonzero prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$ is not an sr-submodule.*

(ii) $\langle \overline{4} \rangle$ is an sr-submodule of $\mathbb{Z}$-module $\mathbb{Z}_{12}$ but it is not prime.

Now we give a condition for a prime submodule to be an sr-submodule in the following proposition.

Proposition 10 *For a prime submodule N of M, N is an sr-submodule if and only if $N \subseteq T(M)$.*

Proof Assume that N is a prime submodule. If N is an sr-submodule, then $N \subseteq T(M)$ by Lemma 3. Now, suppose $N \subseteq T(M)$. Let $am \in N$ and $ann_R(m) = 0$ for $a \in R$ and $m \in M$. Since $ann_R(m) = 0$, $m \notin T(M)$ and so $m \notin N$. Since N is prime submodule, we have $a \in (N :_R M)$ and hence N is an sr-submodule. □

Proposition 11 *Let M be an R-module. Then the following hold:*

(i) The zero submodule is an sr-submodule of M.

(ii) The intersection of an arbitrary nonempty set of sr-submodules is an sr-submodule.

Proof (i) Let $a \in R, m \in M$ with $am = 0_M$ and $ann_R(m) = 0$. Then we have $a = 0 \in (0_M :_R M)$. Hence, we get the result that the zero submodule is an sr-submodule.

(ii) Suppose that $\{N_i\}_{i \in \Delta}$ is an arbitrary nonempty set of sr-submodules of M. Let $am \in \bigcap_{i \in \Delta} N_i$ and $ann_R(m) = 0$. Since N_i is an sr-submodule and $am \in N_i$, we get $a \in (N_i :_R M)$ for every $i \in \Delta$. Hence, we get $a \in \bigcap_{i \in \Delta} (N_i :_R M) = \left(\left(\bigcap_{i \in \Delta} N_i \right) :_R M \right)$ and so $\bigcap_{i \in \Delta} N_i$ is an sr-submodule. □

The following example shows that $(N :_R M)$ need not be an r-ideal even if N is an sr-submodule of M.

Example 14 *Consider the $\mathbb{Z}$-module $\mathbb{Z}_6[x]$ and the submodule $N = \{p(x) \in \mathbb{Z}_6[x] : p(\overline{0}) \in \langle \overline{2} \rangle\}$. Then N is an sr-submodule but $(N :_{\mathbb{Z}} \mathbb{Z}_6[x]) = 2\mathbb{Z}$ is not an r-ideal of $\mathbb{Z}$.*

Proposition 12 *Let N be a proper submodule of M. Then the following are equivalent:*

(i) N is an sr-submodule of M.

(ii) $Rm \cap N = (N :_R M)m$ for every $m \in M - T(M)$.

(iii) $(N :_R M) = (N :_R m)$ for every $m \in M - T(M)$.

Proof $(i) \Rightarrow (ii)$: Suppose that N is an sr-submodule. The inclusion $(N :_R M)\, m \subseteq Rm \cap N$ always holds for each $m \in M$. Let $m \in M - T(M)$ and $x \in Rm \cap N$. Then we have $x = am \in N$ for some $a \in R$. As N is an sr-submodule of M and $ann_R(m) = 0$, $a \in (N :_R M)$ and so $x = am \in (N :_R M)\, m$, as desired.

$(ii) \Rightarrow (iii)$: It is easy to see that $(N :_R M) \subseteq (N :_R m)$ for every $m \in M$. Suppose that $m \in M - T(M)$ and $a \in (N :_R m)$. Then we have $am \in N$. Thus, we have $am \in Rm \cap N = (N :_R M)\, m$ by assumption. Then $am = rm$ for some $r \in (N :_R M)$. Since $ann_R(m) = 0$ and $(a - r)\, m = 0_M$, we conclude that $a \in (N :_R M)$. Hence, we have $(N :_R M) = (N :_R m)$.

$(iii) \Rightarrow (i)$: Let $am \in N$ and $ann_R(m) = 0$. Then we get $m \in M - T(M)$ and so $a \in (N :_R m) = (N :_R M)$ by the assumption. Consequently, N is an sr-submodule of M. □

Theorem 9 *Let $f : M_1 \to M_2$ be an R-module homomorphism. Then the following hold:*

(i) If f is a monomorphism and L is an sr-submodule of M_2 with $f^{-1}(L) \neq M_1$, then $f^{-1}(L)$ is an sr-submodule of M_1.

(ii) If f is an epimorphism and K is an sr-submodule of M_1 containing $Ker(f)$, then $f(K)$ is an sr-submodule of M_2.

Proof (i) Let $am \in f^{-1}(L)$ with $ann_R(m) = 0$ for $a \in R$, $m \in M_1$. Then $f(am) = af(m) \in L$ and $ann_R(f(m)) = 0$. Since L is an sr-submodule of M_2, we conclude that $a \in (L :_R M_2) \subseteq (f^{-1}(L) :_R M_1)$. Hence, $f^{-1}(L)$ is an sr-submodule of M_1.

(ii) Let $am' \in f(K)$ and $ann_R(m') = 0$ for $a \in R, m' \in M_2$. Since f is epimorphism, there exists $m \in M_1$ such that $f(m) = m'$. Then we have $am' = af(m) = f(am) \in f(K)$. As $Ker(f) \subseteq K$, we have $am \in K$. Since $ann_R(m) = 0$, we conclude that $a \in (K :_R M_1) \subseteq (f(K) :_R M_2)$. Consequently, $f(K)$ is an sr-submodule. □

Corollary 3 *Let K be a submodule of M. Then the following hold:*

(i) For every sr-submodule N of M with $K \nsubseteq N$, $N \cap K$ is an sr-submodule of K.

(ii) For every sr-submodule N of M with $K \subseteq N$, N/K is an sr-submodule of M/K.

Proof (i) Consider the injection $i : K \to M$ and note that $i^{-1}(N) = K \cap N$. Thus, $N \cap K$ is an sr-submodule of K by Theorem 9(i).

(ii) Assume $\pi : M \longrightarrow M/K$ to be the natural homomorphism and note that $Ker(\pi) = K \subseteq N$. Thus, N/K is an sr-submodule of M/K by Theorem 9(ii). □

Remark 1 *For any nonempty subset S of R and submodule N of M, $((N :_M S) :_R M) = ((N :_R M) :_R S)$ always holds.*

Proposition 13 *Let M be an R-module. Then the following hold:*

(i) For every sr-submodule N of M and every subset S of R with $S \nsubseteq (N :_R M)$, $(N :_M S)$ is an sr-submodule of M. In particular, $(0_M :_M S)$ is always an sr-submodule if $S \nsubseteq Ann_R(M)$.

(ii) $ann_M(a)$ is an sr-submodule of M for every $a \notin Ann_R(M)$.

Proof (i) Let $am \in (N :_M S)$ with $ann_R(m) = 0$ for $a \in R, m \in M$. Then $asm \in N$ for every $s \in S$. Since N is an sr-submodule, we get the result that $as \in (N :_R M)$ for every $s \in S$ and so $a \in ((N :_R M) :_R S)$. By Remark 1, $a \in ((N :_M S) :_R M)$, and thus $(N :_M S)$ is an sr-submodule.

(ii) Follows from (i) and Proposition 11. □

Theorem 10 *For a proper submodule N of M, the following hold:*

(i) N is an sr-submodule of M if and only if whenever L is a submodule of M with $L \cap (M - T(M)) \neq \emptyset$ and J is an ideal of R with $JL \subseteq N$, then $J \subseteq (N :_R M)$.

(ii) If N is not an sr-submodule with $N \subseteq T(M)$, then there is an ideal J of R and submodule L of M with $L \cap (M - T(M)) \neq \emptyset$, $N \subsetneqq L$, $(N :_R M) \subsetneqq J$, and $JL \subseteq N$.

Proof (i) Suppose N is an sr-submodule. For submodule L of M with $L \cap (M - T(M)) \neq \emptyset$ and ideal J of R, assume that $JL \subseteq N$. Since $L \cap (M - T(M)) \neq \emptyset$, $ann_R(m) = 0$ for some $m \in L$. By assumption, $am \in N$ for every $a \in J$, and thus $a \in (N :_R M)$. We get the result that $J \subseteq (N :_R M)$. Conversely, let $am \in N$ and $ann_R(m) = 0$ for $a \in R, m \in M$. Now we take $J = aR$ and $L = Rm$. Then we have $JL \subseteq N$ for submodule L of M with $L \cap (M - T(M)) \neq \emptyset$ and ideal J of R. By assumption, $J = aR \subseteq (N :_R M)$ so that $a \in (N :_R M)$. Consequently, N is an sr-submodule.

(ii) If N is not an sr-submodule, then $am \in N$ with $ann_R(m) = 0$ but $a \notin (N :_R M)$ for some $a \in R, m \in M$. Now we take $L = (N :_M a)$. Since $m \in L - N$, $N \subsetneqq L$. Also, we take $J = (N :_R L)$. Since $a \in J - (N :_R M)$, we get $(N :_R M) \subsetneqq J$. Then we get $JL = (N :_R L) L \subseteq N$, as desired. □

As a consequence of Theorem 10, we have the following result.

Theorem 11 *Let L be a submodule of M with $L \cap (M - T(M)) \neq \emptyset$. Then the following hold:*

(i) If N_1, N_2 are sr-submodules of M with $(N_1 :_R M)L = (N_2 :_R M)L$, then $(N_1 :_R M) = (N_2 :_R M)$.

(ii) If JL is an sr-submodule for an ideal J of R, then $JL = JM$. Particularly, JM is an sr-submodule of M.

Theorem 12 *Suppose that $N_1, N_2, ..., N_n$ are prime submodules of M with $(N_i :_R M)s$ not comparable. If $\bigcap_{i=1}^{n} N_i$ is an sr-submodule, then N_i is an sr-submodule for each $i \in \{1, 2, ..., n\}$.*

Proof The proof is similar to Theorem 3. □

The following theorem characterizes the torsion-free modules by sr-submodule.

Theorem 13 *For any R-module M, the following are equivalent:*

(i) M is torsion-free.

(ii) M is faithful and the zero submodule is the only sr-submodule.

Proof $(i) \Rightarrow (ii)$: It is obvious that M is faithful. For every sr-submodule N of M, $N \subseteq T(M) = 0_M$ and so $N = 0_M$ by Lemma 3. However, the zero submodule is always an sr-submodule.

$(ii) \Rightarrow (i)$: Let $m \in T(M)$. Then we have $0 \neq r \in R$ such that $rm = 0_M$. We know that $ann_M(r)$ is an sr-submodule by Proposition 13(ii), and we have $m \in ann_M(r) = 0_M$ by assumption. Hence, we have $T(M) = 0_M$. □

Proposition 14 *If N is a maximal sr-submodule of M, then N is prime submodule.*

Proof Let $am \in N$ and $a \notin (N :_R M)$; we show that $m \in N$. Then $(N :_M a)$ is an sr-submodule by Proposition 13(i). Since N is maximal sr-submodule, $m \in (N :_M a) = N$. Consequently, N is prime submodule. □

Theorem 14 *Let M be an R-module. Then every proper submodule is an sr-submodule of M if and only if $T(M) = M$ or $Rm = M$ for every $m \in M - T(M)$.*

Proof Suppose every proper submodule of M is an sr-submodule and $T(M) \neq M$. Let $m \in M - T(M)$. If $Rm \neq M$, then we get the result that Rm is an sr-submodule. Since $rm \in Rm$ for every $r \in R$ and $ann_R(m) = 0$, $(Rm :_R M) = R$. Thus, we have $Rm = RM = M$, which contradicts the assumption. Hence, we have $Rm = M$ for all $m \in M - T(M)$. Conversely, if $T(M) = M$, then every proper submodule is an sr-submodule. Now assume that $Rm = M$ for all $m \in M - T(M)$. Suppose N is a proper submodule of M. Let $am \in N$ and $ann_R(m) = 0$ for $a \in R, m \in M$. Then we get the result that $Rm = M$, because $m \in M - T(M)$. Thus, $a \in (N :_R m) = (N :_R M)$. Consequently, N is an sr-submodule. □

Lemma 4 *For every R_1-module M_1 and R_2-module M_2, $T(M_1 \times M_2) = (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$ always holds.*

Proof Let $(m_1, m_2) \in T(M_1 \times M_2)$. Then there exists $(0_{R_1}, 0_{R_2}) \neq (a_1, a_2) \in R_1 \times R_2$ such that $(a_1, a_2)(m_1, m_2) = (0_{M_1}, 0_{M_2})$ and so $a_1m_1 = 0_{M_1}$, $a_2m_2 = 0_{M_2}$. Since $a_1 \neq 0_{R_1}$ or $a_2 \neq 0_{R_2}$, we conclude that $m_1 \in T(M_1)$ or $m_2 \in T(M_2)$. Hence, we have $(m_1, m_2) \in (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$. Conversely, let $(m_1, m_2) \in (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$. Without loss of generality, we may assume that $(m_1, m_2) \in T(M_1) \times M_2$. There exists $0_{R_1} \neq a_1 \in R_1$ such that $a_1m_1 = 0_{M_1}$ since $m_1 \in T(M_1)$. Thus, we have $(0_{R_1}, 0_{R_2}) \neq (a_1, 0_{R_2}) \in R_1 \times R_2$ such that $(a_1, 0_{R_2})(m_1, m_2) = (0_{M_1}, 0_{M_2})$ and so $(m_1, m_2) \in T(M_1 \times M_2)$. Hence, we have $T(M_1 \times M_2) = (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$. □

Corollary 4 *If $T(M_1) = M_1$ or $T(M_2) = M_2$, then we have $T(M_1 \times M_2) = M_1 \times M_2$ and so every proper submodule of $M_1 \times M_2$ is an sr-submodule of $M_1 \times M_2$.*

Now we characterize the sr-submodules of Cartesian products of modules in case $T(M_1) \neq M_1$ and $T(M_2) \neq M_2$.

Lemma 5 *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$, where M_i is an R_i-module with $T(M_i) \neq M_i$ for $i = 1, 2$. Suppose that $N = N_1 \times N_2$ is a submodule of M. Then the following are equivalent:*

(i) N is an sr-submodule.

(ii) $N_1 = M_1$ and N_2 is an sr-submodule of M_2 or N_1 is an sr-submodule of M_1 and $N_2 = M_2$ or N_1, N_2 are sr-submodules of M_1 and M_2, respectively.

Proof $(i) \Rightarrow (ii)$: Assume that $N = N_1 \times N_2$ is an sr-submodule and $N_1 = M_1$. Since N is proper, we conclude that $N_2 \neq M_2$. Now we show that N_2 is an sr-submodule of M_2. Suppose not. Then there exist $a_2 \in R_2, m_2 \in M_2$ such that $a_2m_2 \in N_2$ with $ann_{R_2}(m_2) = 0_{R_2}$ but $a_2 \notin (N_2 :_{R_2} M_2)$. Since $T(M_1) \neq M_1$, we get $ann_{R_1}(m_1) = 0_{R_1}$ for some $m_1 \in M_1$. Thus, we have $ann_R(m_1, m_2) = 0_R$ and

$(0_{R_1}, a_2)(m_1, m_2) = (0_{M_1}, a_2 m_2) \in N$ but $(0_{R_1}, a_2) \notin (N :_R M)$, which contradicts N being an sr-submodule of M. Hence, we have that N_2 is an sr-submodule of M_2. If $N_2 = M_2$, in a similar way we can see that N_1 is an sr-submodule of M_2. If $N_1 \neq M_1$ and $N_2 \neq M_2$, it can be proved that N_1, N_2 are sr-submodules of M_1 and M_2, respectively.

$(ii) \Rightarrow (i)$: Assume N_1, N_2 are sr-submodules of M_1 and M_2, respectively. Let $(a_1, a_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(a_1, a_2)(m_1, m_2) = (a_1 m_1, a_2 m_2) \in N$ with $ann_R(m_1, m_2) = (0_{R_1}, 0_{R_2})$. Then we have $ann_{R_i}(m_i) = 0_{R_i}$ and $a_i m_i \in N_i$ for $i = 1, 2$. Since N_i is an sr-submodule of M_i, we conclude that $a_i \in (N_i :_{R_i} M_i)$ and so $(a_1, a_2) \in (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2) = (N :_R M)$. Hence, we get the result that N is an sr-submodule. In other cases, one can easily prove the result. □

Theorem 15 *Suppose that $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$, where M_i is an R_i-module with $T(M_i) \neq M_i$ for $n \geq 1$ and $1 \leq i \leq n$. For a submodule $N = N_1 \times N_2 \times ... \times N_n$ of M, the following are equivalent:*

(i) N is an sr-submodule.

(ii) $N_i = M_i$ for $i \in \{t_1, t_{2,...,} t_k : k < n\}$ and N_i is an sr-submodule of M_i for $i \in \{1, 2, ..., n\} \setminus \{t_1, t_{2,...,} t_k\}$.

Proof We use induction on n. If $n = 1$, of course $(i) \Leftrightarrow (ii)$. If $n = 2$, by Lemma 5, (i) and (ii) are equal. Assume $n \geq 3$ and $(i) \Leftrightarrow (ii)$ holds when $K = M_1 \times M_2 \times ... \times M_{n-1}$. Now we prove that (i) and (ii) are equal when $M = K \times M_n$. Then, by Lemma 5, N is an sr-submodule of M if and only if $N = K \times N_n$ for some sr-submodule N_n of M_n or $N = L \times M_n$ for some sr-submodule L of K or $N = L \times N_n$ for some sr-submodule L of K and some sr-submodule N_n of M_n. By induction hypothesis, the result is true in three cases. □

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