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The natural brackets on couples of vector fields and 1-forms

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Abstract: All natural bilinear operators transforming pairs of couples of vector fields and 1-forms into couples of vector fields and 1-forms are found. All natural bilinear operators as above satisfying the Leibniz rule are extracted. All natural Lie algebra brackets on couples of vector fields and 1-forms are collected.

Key words: Natural operator, vector field, 1-form, Leibniz rule

1. Introduction

Let $\mathcal{M}f_m$ be the category of m -dimensional C^∞ manifolds and their embeddings.

The "doubled" tangent bundle $T \oplus T^*$ over $\mathcal{M}f_m$ is of great interest because of the seminal papers, where it is proved that it has the natural inner product, and the Courant bracket, see, e.g., [1, 4, 5].

If $m \geq 2$, we classify all $\mathcal{M}f_m$ -natural bilinear operators

$$A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$$

transforming pairs of couples $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ ($i = 1, 2$) of vector fields and 1-forms on m -manifolds M into couples $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \mathcal{X}(M) \oplus \Omega^1(M)$ of vector fields and 1-forms on M .

In particular, we get that if $m \geq 2$ then any $\mathcal{M}f_m$ -natural skew-symmetric bilinear operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ coincides with the Courant bracket up to three real constants; see Corollary 3.3.

If $m \geq 2$, we find all $\mathcal{M}f_m$ -natural bilinear operators $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ satisfying the Leibniz rule

$$A(X, A(Y, Z)) = A(A(X, Y), Z) + A(Y, A(X, Z))$$

for any $X, Y, Z \in \mathcal{X}(M) \oplus \Omega^1(M)$ and $M \in \text{obj}(\mathcal{M}f_m)$.

If $m \geq 2$, we also find all $\mathcal{M}f_m$ -natural Lie algebra brackets $[-, -]$ on $\mathcal{X}(M) \oplus \Omega^1(M)$, i.e. all $\mathcal{M}f_m$ -natural skew-symmetric bilinear operators $A = [-, -]$ as above satisfying the Leibniz rule.

Some linear natural operators on vector fields, forms, and some other tensor fields have been studied in many papers; see [2, 3, 7, 8], etc.

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From now on, (x^i) ($i = 1, \dots, m$) denote the usual coordinates on \mathbb{R}^m and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on \mathbb{R}^m .

All manifolds are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class C^∞). Maps between manifolds are assumed to be C^∞ .

2. The basic notions

The notion of natural operators is rather well known. In the present note we need the following particular definitions of natural operators.

Definition 2.1 A bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ is a $\mathcal{M}f_m$ -invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M)$$

for m -dimensional manifolds M , where $\mathcal{X}(M)$ is the space of vector fields on M and $\Omega^1(M)$ is the space of 1-forms on M . The $\mathcal{M}f_m$ -invariance of A means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M))$ and $(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2) \in (\mathcal{X}(\bar{M}) \oplus \Omega^1(\bar{M})) \times (\mathcal{X}(\bar{M}) \oplus \Omega^1(\bar{M}))$ are φ -related by an $\mathcal{M}f_m$ -map $\varphi : M \rightarrow \bar{M}$ (i.e. $\bar{X}^i \circ \varphi = T\varphi \circ X^i$ and $\bar{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2)$.

Definition 2.2 A bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is a $\mathcal{M}f_m$ -invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \mathcal{X}(M)$$

for m -manifolds M .

Definition 2.3 A bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T^*$ is a $\mathcal{M}f_m$ -invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \Omega^1(M)$$

for m -manifolds M .

Remark 2.4 By the multilinear Peetre theorem, see [6], any $\mathcal{M}f_m$ -natural bilinear operator A (as above) is of finite order. It means that there is a finite number r such that we have the following implication

$$(j_x^r X_i = j_x^r \bar{X}_i, j_x^r \omega_i = j_x^r \bar{\omega}_i, i = 1, 2) \Rightarrow A(X_1 \oplus \omega_1, X_2 \oplus \omega_2)|_x = A(\bar{X}_1 \oplus \bar{\omega}_1, \bar{X}_2 \oplus \bar{\omega}_2)|_x$$

Remark 2.5 We say that an operator A is regular if it transforms smoothly parametrized families of objects into smoothly parametrized families. One can show that bilinear $\mathcal{M}f_m$ -natural operators are regular because of the Peetre theorem.

Definition 2.6 A $\mathcal{M}f_m$ -natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ is a $\mathcal{M}f_m$ -invariant family of regular (not necessarily bilinear) operators

$$B : \mathcal{X}(M) \oplus C^\infty(M) \rightarrow \Omega^1(M)$$

for m -manifolds M , where $C^\infty(M)$ is the space of smooth maps $M \rightarrow \mathbb{R}$.

The most interesting example of a bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ is the famous Courant bracket $[-, -]_C$ presented below.

Example 2.7 *On the vector bundle $TM \oplus T^*M$ there exist canonical symmetric and skew-symmetric pairings*

$$\langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_{\pm} = \frac{1}{2}(\langle X^2, \omega^1 \rangle \pm \langle X^1, \omega^2 \rangle)$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $\langle -, - \rangle : TM \times_M T^*M \rightarrow \mathbb{R}$ is the usual canonical pairing. Further, a bracket (Courant bracket) is given by

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_C = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + d \langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_-)$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where \mathcal{L} denotes the usual Lie derivative, d denotes the usual differentiation, and $[-, -]$ denotes the usual bracket on vector fields.

Definition 2.8 *A $\mathcal{M}f_m$ -natural bilinear operator A in the sense of Definition 2.1 satisfies the Leibniz rule if*

$$A(X, A(Y, Z)) = A(A(X, Y), Z) + A(Y, A(X, Z))$$

for any $X, Y, Z \in \mathcal{X}(M) \oplus \Omega^1(M)$.

The Courant bracket is skew-symmetric bilinear but does not satisfy the Jacobi identity.

3. The main results

The main results of the present note are the following classification theorems.

Theorem 3.1 *If $m \geq 2$, any bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ is of the form*

$$A(\rho^1, \rho^2) = a[X^1, X^2] \oplus (b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3d \langle \rho^1, \rho^2 \rangle_+ + b_4d \langle \rho^1, \rho^2 \rangle_-)$$

for (uniquely determined by A) real numbers a, b_1, b_2, b_3, b_4 , where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$ and where $\langle -, - \rangle_+$ and $\langle -, - \rangle_-$ are as in Example 2.7.

Theorem 3.2 *If $m \geq 2$, any bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ satisfying the Leibniz rule is the constant multiple of the one of the following four operators:*

$$\begin{aligned} A_1(\rho^1, \rho^2) &= [X^1, X^2] \oplus 0 \\ A_2(\rho^1, \rho^2) &= [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) \\ A_3(\rho^1, \rho^2)_3 &= [X^1, X^2] \oplus \mathcal{L}_{X^1}\omega^2 \\ A_4(\rho^1, \rho^2) &= [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + d \langle X^2, \omega^1 \rangle) \end{aligned}$$

where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

From Theorem 3.1 we obtain immediately

Corollary 3.3 *If $m \geq 2$, any skew-symmetric bilinear $\mathcal{M}f_m$ -natural operator $A : (T \otimes T^*) \oplus (T \otimes T^*) \rightsquigarrow T \oplus T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd \langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_-)$$

for (uniquely determined by A) real numbers a, b, c .

Roughly speaking, Corollary 3.3 says that any skew-symmetric bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ coincides with the Courant bracket up to three real constants.

From Theorem 3.2 and Corollary 3.3 it follows immediately

Corollary 3.4 *If $\dim(M) \geq 2$, any $\mathcal{M}f_m$ -natural Lie algebra bracket on $\mathcal{X}(M) \oplus \Omega^1(M)$ is the constant multiple of the one of the following two Lie algebra brackets:*

$$\begin{aligned} [X^1 \oplus \omega^1, X^2 \oplus \omega^2]_1 &= [X^1, X^2] \oplus 0, \\ [X^1 \oplus \omega^1, X^2 \oplus \omega^2]_2 &= [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1). \end{aligned}$$

The rest of the paper is dedicated to proving the results mentioned above.

4. The natural operators in the sense of Definition 2.2

In this section we prove the following:

Proposition 4.1 *If $m \geq 2$, any bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2]$$

for a (uniquely determined by A) real number a .

Proof Consider a bilinear $\mathcal{M}f_m$ -natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$. Clearly, A is determined by the values

$$\langle A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, \eta \rangle \in \mathbb{R}$$

for all $X^i \oplus \omega^i \in \mathcal{X}(\mathbb{R}^m) \oplus \Omega^1(\mathbb{R}^m)$, $\eta \in T_0^*\mathbb{R}^m$, $i = 1, 2$. Moreover, by the invariance and the regularity of A and the Frobenius theorem we may additionally assume that $X^1 = \partial_1$ and $\eta = d_0x^1$. In other words, A is determined by the values

$$\langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)|_0, d_0x^1 \rangle \in \mathbb{R}$$

for all $X \in \mathcal{X}(\mathbb{R}^m)$, $\omega^i \in \Omega^1(\mathbb{R}^m)$, $i = 1, 2$. Using the invariance of A with respect to the homotheties and the bilinearity of A we have the homogeneity condition

$$\langle A(\partial_1 \oplus t(\frac{1}{t}id)_*\omega^1, t(\frac{1}{t}id)_*X \oplus t(\frac{1}{t}id)_*\omega^2)|_0, d_0x^1 \rangle = t \langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)|_0, d_0x^1 \rangle .$$

Thus, by the homogeneous function theorem, since A is of finite order and regular, the value $\langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)|_0, d_0x^1 \rangle$ depends on j_0^1X only. Then A is determined by the values

$$\langle A(\partial_1 \oplus 0, (\sum_{k=1}^m a^k \partial_k + \sum_{i,j=1}^m b_i^j x^i \partial_j) \oplus 0)|_0, d_0x^1 \rangle$$

for all $a^k, b_i^j \in \mathbb{R}$, $i, j, k = 1, \dots, m$. Then, by the invariance of A with respect to the diffeomorphisms $(t_1x^1, t_2x^2, \dots, t_mx^m)$, $t_l \in \mathbb{R}_+$, $l = 1, \dots, m$, and by the bilinearity of A , we may assume that $a^k = 0$ for $k = 1, \dots, m$ and $b_i^j = 0$ for $i, j = 1, \dots, m$ with $i \neq j$, that is, A is determined by the values $\langle A(\partial_1 \oplus 0, x^i \partial_i \oplus 0)|_0, d_0x^1 \rangle \in \mathbb{R}$, $i = 1, \dots, m$, and then A is determined by the values

$$\langle A(\partial_1 \oplus 0, x^1 \partial_1 \oplus 0)|_0, d_0x^1 \rangle \in \mathbb{R} \text{ and } A(\partial_1 \oplus 0, X \oplus 0)|_0 \in T_0\mathbb{R}^m$$

for all $X \in \mathcal{X}(\mathbb{R}^{m-1})$ (depending on x^2, \dots, x^m). Further by the regularity of A we may assume that $X|_0 \neq 0$, and then (by the invariance of A with respect to local diffeomorphisms of the form $id_{\mathbb{R}} \times \psi(x^2, \dots, x^m)$ and the Frobenius theorem) we may assume $X = \partial_2$. Using the bilinearity and the invariance of A with respect to the homotheties one can easily see that $A(\partial_1 \oplus 0, \partial_2 \oplus 0)|_0 = 0$. Consequently, A is determined by the value

$$\langle A(\partial_1 \oplus 0, x^1 \partial_1 \oplus 0)|_0, d_0x^1 \rangle \in \mathbb{R},$$

i.e. the vector space of all bilinear $\mathcal{M}f_m$ -natural operators $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is not more than 1-dimensional. On the other hand, we have the bilinear $\mathcal{M}f_m$ -natural operator A_o (in question) given by $A_o(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2]$. The proof of Proposition 4.1 is complete. \square

5. On natural operators in the sense of Definition 2.6

In this section we prove the following:

Lemma 5.1 *Let $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ be a $\mathcal{M}f_m$ -natural operator satisfying*

$$\begin{aligned} B(tX \oplus f) &= t^2 B(X \oplus f) = B(X \oplus t^2 f), \\ B(X \oplus (f + f_1)) &= B(X \oplus f) + B(X \oplus f_1). \end{aligned}$$

If $m \geq 2$, then B is of the form

$$B(X \oplus f) = \lambda d(XXf),$$

for a (uniquely determined by B) real number λ , where d is the usual differentiation.

Proof By the classical Petree theorem (since B is linear in f), B is of finite order in f , i.e. for any m -manifold M , any point $x \in M$ and any vector field $X \in \mathcal{X}(M)$ there is a natural number r such that for any $f, \bar{f} \in \mathcal{C}^\infty(M)$ from $j_x^r f = j_x^r \bar{f}$ it follows $B(X, f)|_x = B(X, \bar{f})|_x$. Clearly, B is determined by the values $\langle B(X \oplus f)|_0, v \rangle \in \mathbb{R}$ for $X \in \mathcal{X}(\mathbb{R}^m)$, $f \in \mathcal{C}^\infty(M)$, $v \in T_0\mathbb{R}^m$. By the regularity of B and $m \geq 2$, we may assume $X|_0$ and v are linearly independent, and then by the invariance of B and the Frobenius theorem, we may assume $X = \partial_1$, and $v = \partial_2|_0$, i.e. B is determined by the values

$$\langle B(\partial_1 \oplus f)|_0, \partial_2|_0 \rangle \in \mathbb{R}$$

for $f \in \mathcal{C}^\infty(\mathbb{R}^m)$. Since B is of finite order in f , we may assume, f is polynomial. Now, by the invariance of B with respect to the diffeomorphisms (t_1x^1, \dots, t_mx^m) , $t_l \in \mathbb{R}_+$, $l = 1, \dots, m$ and the conditions of B , we derive that $\langle B(\partial_1 \oplus f)|_0, \partial_2|_0 \rangle$ is determined by $\langle B(\partial_1 \oplus (x^1)^2 x^2)|_0, \partial_2|_0 \rangle$. Consequently, the vector space of all such operators B is of dimension not more than 1. On the other hand, we have an $\mathcal{M}f_m$ -operator B_o in question given by $B_o(X \oplus f) = d(XXf)$. Lemma 5.1 is complete. \square

6. The natural operators in the sense of Definition 2.3

In this section we prove the following:

Proposition 6.1 *Let $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T^*$ be a bilinear $\mathcal{M}f_m$ -natural operator. If $m \geq 2$, then A is the linear combination with real coefficients of the bilinear $\mathcal{M}f_m$ -natural operators $A^{<j>} : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T^*$ given by*

$$\begin{aligned} A^{<1>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= \mathcal{L}_{X^2} \omega^1 \\ A^{<2>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= \mathcal{L}_{X^1} \omega^2 \\ A^{<3>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= d \langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_+ \\ A^{<4>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= d \langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_- . \end{aligned}$$

Proof Clearly, A is determined by the values

$$\langle A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, v \rangle \in \mathbb{R}$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^1(\mathbb{R}^m)$, $v \in T_0\mathbb{R}^m$. Consequently, using the bilinearity of A , A is determined by the values

$$\begin{aligned} &\langle A(0 \oplus \omega^1, 0 \oplus \omega^2)|_0, v \rangle, \langle A(0 \oplus \omega^1, X^2 \oplus 0)|_0, v \rangle, \\ &\langle A(X^1 \oplus 0, 0 \oplus \omega^2)|_0, v \rangle, \langle A(X^1 \oplus 0, X^2 \oplus 0)|_0, v \rangle \end{aligned}$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^1(\mathbb{R}^m)$, $v \in T_0\mathbb{R}^m$. Using the invariance of A with respect to the homotheties and the bilinearity of A and then applying the homogeneous function theorem, we easily deduce that

$$\langle A(0 \oplus \omega^1, 0 \oplus \omega^2)|_0, v \rangle = 0.$$

By the same argument, $\langle A(0 \oplus \omega^1, X^2 \oplus 0)|_0, v \rangle$ depends on $j_0^1 \omega^1$ and $j_0^1 X^2$ only, and (symmetrically) $\langle A(X^1 \oplus 0, 0 \oplus \omega^2)|_0, v \rangle$ depends on $j_0^1 \omega^2$ and $j_0^1 X^1$ only, and (similarly) $\langle A(X^1 \oplus 0, X^2 \oplus 0)|_0, v \rangle$ depends on $j_0^3 X^1$ and $j_0^3 X^2$ only. Next, by the regularity of A we may assume X_0^1 and v are linearly independent, and then by the Frobenius theorem we may assume that $X^1 = \partial_1$ and $v = \partial_{2|_0}$. Then, using the invariance of A with respect to $(t_1 x^1, \dots, t_m x^m)$ for $t_l \in \mathbb{R}_+$, $l = 1, \dots, m$, we may assume $\langle A(X^1 \oplus 0, 0 \oplus \omega^2)|_0, v \rangle$ is determined by

$$c_1 := \langle A(\partial_1 \oplus 0, 0 \oplus x^1 dx^2)|_0, \partial_{2|_0} \rangle \text{ and } c_3 := \langle A(\partial_1 \oplus 0, 0 \oplus x^2 dx^1)|_0, \partial_{2|_0} \rangle .$$

By similar arguments, we may assume $\langle A(0 \oplus \omega^1, X^2 \oplus 0)|_0, v \rangle$ is determined by

$$c_2 := \langle A(0 \oplus x^1 dx^2, \partial_1 \oplus 0)|_0, \partial_{2|_0} \rangle \text{ and } c_4 := \langle A(0 \oplus x^2 dx^1, \partial_1 \oplus 0)|_0, \partial_{2|_0} \rangle ,$$

and (similarly) we may assume $\langle A(X^1 \oplus 0, X^2 \oplus 0)|_0, v \rangle$ is determined by

$$d_k := \langle A(\partial_1 \oplus 0, x^1 x^2 x^k \partial_k \oplus 0)|_0, \partial_{2|_0} \rangle, \quad k = 1, \dots, m .$$

The above facts imply that A is determined by the real numbers d_k and

$$b_1 := c_2, \quad b_2 := c_1, \quad b_3 := c_3 + c_4, \quad b_4 := c_4 - c_3 .$$

We prove that $A = \sum_{j=1}^4 b_j A^{<j>}$.

Replacing A by $A - \sum_{j=1}^4 b_j A^{<j>}$, we may assume $b_1 = b_2 = b_3 = b_4 = 0$, i.e. we may assume that A is determined by the values d_k , i.e. we may assume that A is determined by the value

$$\langle A(\partial_1 \oplus 0, (x^1)^2 x^2 \partial_1 \oplus 0)|_0, \partial_2|_0 \rangle \in \mathbb{R}$$

together with the values

$$A(\partial_1 \oplus 0, x^1 Y \oplus 0)|_0 \in T_0^* \mathbb{R}^m$$

for all vector fields $Y \in \mathcal{X}(\mathbb{R}^{m-1})$ (depending on x^2, \dots, x^m). Next, by the regularity of A , we may assume $Y|_0 \neq 0$, and then, by the invariance of A with respect to local diffeomorphisms of the form $id_{\mathbb{R}} \times \psi(x^2, \dots, x^m)$ and the Frobenius theorem, we may assume $Y = \partial_2$. However,

$$A(\partial_1 \oplus 0, x^1 \partial_2 \oplus 0)|_0 = 0$$

because of the invariance of A with respect to the homotheties. Consequently, A is determined by the $\mathcal{M}f_m$ -natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ given by

$$B(X \oplus f) := A(X \oplus 0, fX \oplus 0) ,$$

$M \in \text{obj}(\mathcal{M}f_m)$, $X \in \mathcal{X}(M)$, $f \in \mathcal{C}^\infty(M)$. Clearly, B satisfies the assumptions of Lemma 5.1. Then we have $\lambda \in \mathbb{R}$ such that

$$B(X \oplus f) = \lambda d(XXf)$$

for any $M \in \text{obj}(\mathcal{M}f_m)$, $X \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. In particular,

$$A(x^1 \partial_1 \oplus 0, \partial_1 \oplus 0) = B(x^1 \partial_1 \oplus \frac{1}{x^1}) = \lambda d[(x^1 \partial_1) \circ (x^1 \partial_1)(\frac{1}{x^1})] = \lambda \frac{1}{x^1} dx^1$$

over $\mathbb{R}^m \setminus \{0\}$. Then $\lambda = 0$ because $A(x^1 \partial_1 \oplus 0, \partial_1 \oplus 0)$ is a smooth form on all \mathbb{R}^m and $\frac{1}{x^1} dx^1$ is not extendable to a smooth form on \mathbb{R}^m . Then $B = 0$, and then $A = 0$ (under the additional assumption). It means $A = \sum_{j=1}^4 b_j A^{<j>}$, where the numbers b_j are defined above. The proof of Proposition 6.1 is complete. \square

7. Proof of Theorem 3.1

Proof Theorem 3.1 is an immediate consequence of Propositions 4.1 and 6.1. \square

8. Proof of Theorem 3.2

In this section we prove Theorem 3.2 as follows:

Proof Let $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ be a bilinear $\mathcal{M}f_m$ -natural operator satisfying the Leibniz rule. By Theorem 3.1, A is of the form

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + c_1 d \langle X^2, \omega^1 \rangle + c_2 d \langle X^1, \omega^2 \rangle)$$

for (uniquely determined by A) real numbers a, b_1, b_2, c_1, c_2 , where $\langle -, - \rangle$ is as in Example 2.7. Then for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $\omega^1, \omega^2, \omega^3 \in \Omega^1(M)$ we have

$$A(X^1 \oplus \omega^1, A(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) = a^2[X^1, [X^2, X^3]] \oplus \Omega,$$

$$A(A(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) = a^2[[X^1, X^2], X^3] \oplus \Theta,$$

$$A(X^2 \oplus \omega^2, A(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) = a^2[X^2, [X^1, X^3]] \oplus \mathcal{T},$$

where

$$\begin{aligned} \Omega = & b_1 \mathcal{L}_{a[X^2, X^3]} \omega^1 + c_1 d \langle a[X^2, X^3], \omega^1 \rangle \\ & + b_2 \mathcal{L}_{X^1} (b_1 \mathcal{L}_{X^3} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^3 + c_1 d \langle X^3, \omega^2 \rangle + c_2 d \langle X^2, \omega^3 \rangle) \\ & + c_2 d \langle X^1, b_1 \mathcal{L}_{X^3} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^3 + c_1 d \langle X^3, \omega^2 \rangle + c_2 d \langle X^2, \omega^3 \rangle \rangle, \end{aligned}$$

$$\begin{aligned} \Theta = & b_2 \mathcal{L}_{a[X^1, X^2]} \omega^3 + c_2 d \langle a[X^1, X^2], \omega^3 \rangle \\ & + b_1 \mathcal{L}_{X^3} (b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + c_1 d \langle X^2, \omega^1 \rangle + c_2 d \langle X^1, \omega^2 \rangle) \\ & + c_1 d \langle X^3, b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + c_1 d \langle X^2, \omega^1 \rangle + c_2 d \langle X^1, \omega^2 \rangle \rangle, \end{aligned}$$

$$\begin{aligned} \mathcal{T} = & b_1 \mathcal{L}_{a[X^1, X^3]} \omega^2 + c_1 d \langle a[X^1, X^3], \omega^2 \rangle \\ & + b_2 \mathcal{L}_{X^2} (b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^3 + c_1 d \langle X^3, \omega^1 \rangle + c_2 d \langle X^1, \omega^3 \rangle) \\ & + c_2 d \langle X^2, b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^3 + c_1 d \langle X^3, \omega^1 \rangle + c_2 d \langle X^1, \omega^3 \rangle \rangle. \end{aligned}$$

The Leibniz rule of A is equivalent to

$$\Omega = \Theta + \mathcal{T}.$$

Applying the differentiation d to both sides of the last equality and using the well-known formula $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$ we get

$$\begin{aligned} & b_1 a \mathcal{L}_{[X^2, X^3]} d\omega^1 + b_2 b_1 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d\omega^2 + b_2^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d\omega^3 \\ & = (b_2 a \mathcal{L}_{[X^1, X^2]} d\omega^3 + b_1^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d\omega^1 + b_1 b_2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d\omega^2) \\ & + (b_1 a \mathcal{L}_{[X^1, X^3]} d\omega^2 + b_2 b_1 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d\omega^1 + b_2^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d\omega^3). \end{aligned}$$

If we put $X^1 = \partial_1$, $X^2 = x^1 \partial_1$, $X^3 = 0$ and $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = (x^1)^2 dx^2$, we get

$$4b_2^2 dx^1 \wedge dx^2 = 2b_2 a dx^1 \wedge dx^2 + 2b_2^2 dx^1 \wedge dx^2.$$

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = x^1\partial_1$ and $\omega^1 = (x^1)^2dx^2$, $\omega^2 = 0$, $\omega^3 = 0$, we get

$$2b_1adx^1 \wedge dx^2 = 2b_1^2dx^1 \wedge dx^2 + 4b_2b_1dx^1 \wedge dx^2 .$$

If we put $X^1 = \partial_1$, $X^2 = 0$, $X^3 = x^1\partial_1$ and $\omega^1 = 0$, $\omega^2 = (x^1)^2dx^2$, $\omega^3 = 0$, we get

$$4b_2b_1dx^1 \wedge dx^2 = 2b_1b_2dx^1 \wedge dx^2 + 2b_1adx^1 \wedge dx^2 .$$

Thus,

$$b_2a = b_2^2, \quad b_1a = b_1^2 + 2b_1b_2, \quad b_1b_2 = b_1a .$$

From the first equality we get $b_2 = 0$ or $b_2 = a$. From the third one we get $b_1 = 0$ or $b_2 = a$. Adding the first two equalities we get $(b_2 + b_1)a = (b_2 + b_1)^2$, i.e. $b_2 + b_1 = 0$ or $b_2 + b_1 = a$. Consequently,

$$(b_1, b_2) = (0, 0) \text{ or } (b_1, b_2) = (0, a) \text{ or } (b_1, b_2) = (-a, a) . \quad (1)$$

Then, using the formula $\mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = \mathcal{L}_{[X,Y]}\omega$ and (1), we get

$$\begin{aligned} & b_1a\mathcal{L}_{[X^2, X^3]}\omega^1 + b_2b_1\mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 + b_2^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 \\ &= (b_2a\mathcal{L}_{[X^1, X^2]}\omega^3 + b_1^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1b_2\mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2) \\ &+ (b_1a\mathcal{L}_{[X^1, X^3]}\omega^2 + b_2b_1\mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 + b_2^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3) . \end{aligned}$$

Consequently, the Leibniz rule of A is equivalent to the system of (1) and

$$\begin{aligned} & c_1ad \langle [X^2, X^3], \omega^1 \rangle + b_2\mathcal{L}_{X^1}(c_1d \langle X^3, \omega^2 \rangle + c_2d \langle X^2, \omega^3 \rangle) \\ &+ c_2d \langle X^1, b_1\mathcal{L}_{X^3}\omega^2 + b_2\mathcal{L}_{X^2}\omega^3 + c_1d \langle X^3, \omega^2 \rangle + c_2d \langle X^2, \omega^3 \rangle \rangle \\ &= c_2ad \langle [X^1, X^2], \omega^3 \rangle + b_1\mathcal{L}_{X^3}(c_1d \langle X^2, \omega^1 \rangle + c_2d \langle X^1, \omega^2 \rangle) \\ &+ c_1d \langle X^3, b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1d \langle X^2, \omega^1 \rangle + c_2d \langle X^1, \omega^2 \rangle \rangle \\ &+ c_1ad \langle [X^1, X^3], \omega^2 \rangle + b_2\mathcal{L}_{X^2}(c_1d \langle X^3, \omega^1 \rangle + c_2d \langle X^1, \omega^3 \rangle) \\ &+ c_2d \langle X^2, b_1\mathcal{L}_{X^3}\omega^1 + b_2\mathcal{L}_{X^1}\omega^3 + c_1d \langle X^3, \omega^1 \rangle + c_2d \langle X^1, \omega^3 \rangle \rangle . \end{aligned} \quad (2)$$

If we put $X^1 = \partial_1$, $X^2 = \partial_2$, $X^3 = 0$ and $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = (x^2)^2dx^1$, we get

$$2c_2b_2dx^2 = 2c_2b_2dx^2 + 2c_2^2dx^2 .$$

Then $c_2 = 0$.

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = \partial_2$ and $\omega^1 = (x^2)^2dx^1$, $\omega^2 = 0$, $\omega^3 = 0$ we get

$$0 = 2b_1c_1dx^2 + 2c_1^2dx^2 + 2c_2b_1dx^2 .$$

Then (as $c_2 = 0$) we get $c_1 = 0$ or $c_1 = -b_1$.

Consequently, we get $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$. On the other hand one can directly verify that the operators A_1, \dots, A_4 from Theorem 3.2 satisfy the Leibniz rule. Theorem 3.2 is complete. \square

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