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Harmonic functions associated with some polynomials in several variables

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Abstract: The aim of this paper is to give various properties of homogeneous operators associated with Chan–Chyan–Srivastava polynomials and, by using these results, to obtain harmonic functions by applying Laplace and ultrahyperbolic operators to the Chan–Chyan–Srivastava polynomials.

Key words: Chan–Chyan–Srivastava polynomials, Lagrange polynomials, homogeneous function, ultrahyperbolic operator, Laplace operator, harmonic function

1. Introduction

The functions satisfying Laplace's equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$$

are called harmonic functions where Δ is the Laplace operator. Spherical harmonics are the angular portion of a set of solutions of Laplace's equation. The spherical harmonics are orthogonal functions on the sphere and they are also a frequency-space basis for indicating functions defined on the surface of a sphere (see [7],[8],[10],[11]).

Spherical harmonics play important roles in several theoretical and practical applications such as the group of rotations in three dimensions, representation of gravitational fields, geoids and the magnetic fields of planetary bodies, and the representation of electromagnetic fields. There are many applications of spherical harmonics to 3D computer graphics in areas such as global illumination, precomputed radiance transfer, and modeling of 3D shapes ([2],[11]). They are also used to solve partial differential equations, which are encountered in mathematics and physical science. In particular, they arise in the solutions of Scrödinger's equation in spherical coordinates [9, p. 193].

In this paper, we derive harmonic functions by using the Chan–Chyan–Srivastava polynomials and Lagrange polynomials. First, we recall that the Chan–Chyan–Srivastava polynomials of $p+q$ variables, which are the multivariable extension of the classical Lagrange polynomials [4, p.267], are generated by [3] (see also [1] [5], [6]):

$$\prod_{i=1}^{p}\left\{(1-x_i t)^{-\alpha_i}\right\}\prod_{j=1}^{q}\left\{(1-y_j t)^{-\beta_j}\right\} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})\, t^n, \tag{1.1}$$

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$$\left(\begin{array}{c}\alpha_i,\beta_j\in\mathbb{C}\ \ (i=1,2,...,p\ ;\ j=1,2,...,q)\ ;\\ |t|<min\left\{|x_1|^{-1},...,|x_p|^{-1},|y_1|^{-1},...,|y_q|^{-1}\right\}\end{array}\right),$$

where $\mathbf{x}=(x_1,x_2,...,x_p)$, $\mathbf{y}=(y_1,y_2,...,y_q)$, $\alpha=(\alpha_1,...,\alpha_p)$, and $\beta=(\beta_1,...,\beta_q)$. From this generating function, its explicit representation is as below:

$$g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})=\sum_{\substack{k_1+...+k_p+l_1+...+l_q=n\\(k_i,l_j\in\mathbb{N}_0:=\{0\}\cup\mathbb{N})}}\frac{(\alpha_1)_{k_1}...(\alpha_p)_{k_p}(\beta_1)_{l_1}...(\beta_q)_{l_q}}{k_1!...k_p!l_1!...l_q!}x_1^{k_1}...x_p^{k_p}y_1^{l_1}...y_q^{l_q}, \tag{1.2}$$

where Chan–Chyan–Srivastava polynomials are of degree k $(k=k_1+...+k_p)$ and l $(l=l_1+...+l_q)$ with respect to the variables $\mathbf{x}=(x_1,x_2,...,x_p)$ and $\mathbf{y}=(y_1,y_2,...,y_q)$, respectively. It is clear that it is a homogeneous function of total degree $k+l=n$.

Let us consider the ultrahyperbolic operator and Laplace operator, respectively:

$$L=\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{k=1}^{q}\frac{\partial^2}{\partial y_k^2}\quad\text{and}\quad\Delta=\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}. \tag{1.3}$$

We organize the paper as follows. In Section 2, we investigate some properties of homogeneous operators associated with Chan–Chyan–Srivastava polynomials of $p+q$ variables and then by applying the ultrahyperbolic operator to the Chan–Chyan–Srivastava polynomials by means of these properties we obtain the ultraspherical harmonic function. In the next section, by using the results obtained in the previous section, we give the ultraspherical harmonic function in terms of the Laplace operator and Chan–Chyan–Srivastava polynomials of p $(p\geq 3)$ variables. Finally, for $p=2$, a harmonic function is obtained via Lagrange polynomials and the Laplace operator.

2. Some properties of homogeneous operators

Lemma 2.1 *Let $f_n(\mathbf{x},\mathbf{y})$ be a homogeneous polynomial of total degree n $(n=k+l)$ and let k and l denote the degree of the polynomial with respect to the variables $\mathbf{x}=(x_1,x_2,...,x_p)$ and $\mathbf{y}=(y_1,y_2,...,y_q)$, respectively. Then*

$$f_n\left(\frac{\partial}{\partial\mathbf{x}},\frac{\partial}{\partial\mathbf{y}}\right)g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})=g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial\mathbf{x}},\frac{\partial}{\partial\mathbf{y}}\right)f_n(\mathbf{x},\mathbf{y}) \tag{2.1}$$

holds where $\frac{\partial}{\partial\mathbf{x}}=\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_p}\right)$ *and* $\frac{\partial}{\partial\mathbf{y}}=\left(\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_q}\right)$.

Proof A homogeneous function $f_n(\mathbf{x},\mathbf{y})$ has representation in the form of

$$f_n(\mathbf{x},\mathbf{y})=\sum_{\substack{k_1+...+k_p+l_1+...+l_q=n\\(k_i,l_j\in\mathbb{N}_0:=\{0\}\cup\mathbb{N})}}A_{k_1,...,k_p,l_1,...,l_q}x_1^{k_1}...x_p^{k_p}y_1^{l_1}...y_q^{l_q},$$

which is of degree k $(k=k_1+...+k_p)$ and l $(l=l_1+...+l_q)$ with respect to the variables $\mathbf{x}=(x_1,x_2,...,x_p)$ and $\mathbf{y}=(y_1,y_2,...,y_q)$, respectively, and also is a homogeneous function of total degree n $(n=k+l)$. The left side of equation (2.1) can be written as

$$\begin{aligned} f_n\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right) g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) &= \sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=n} A_{k_1,\ldots,k_p,l_1,\ldots,l_q}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\ldots\left(\frac{\partial}{\partial x_p}\right)^{k_p}\left(\frac{\partial}{\partial y_1}\right)^{l_1}\ldots\left(\frac{\partial}{\partial y_q}\right)^{l_q} \\ &\times \sum_{m_1+\ldots+m_p+r_1+\ldots+r_q=n} \frac{(\alpha_1)_{m_1}\ldots(\alpha_p)_{m_p}(\beta_1)_{r_1}\ldots(\beta_q)_{r_q}}{m_1!\ldots m_p! r_1!\ldots r_q!} x_1^{m_1}\ldots x_p^{m_p} y_1^{r_1}\ldots y_q^{r_q} \\ &= \sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=n} A_{k_1,\ldots,k_p,l_1,\ldots,l_q}(\alpha_1)_{k_1}\ldots(\alpha_p)_{k_p}(\beta_1)_{l_1}\ldots(\beta_q)_{l_q}. \end{aligned}$$

Similarly, the right side of (2.1) is equal to

$$\begin{aligned} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right) f_n(\mathbf{x},\mathbf{y}) &= \sum_{m_1+\ldots+m_p+r_1+\ldots+r_q=n} \frac{(\alpha_1)_{m_1}\ldots(\alpha_p)_{m_p}(\beta_1)_{r_1}\ldots(\beta_q)_{r_q}}{m_1!\ldots m_p! r_1!\ldots r_q!}\left(\frac{\partial}{\partial x_1}\right)^{m_1}\ldots\left(\frac{\partial}{\partial x_p}\right)^{m_p} \\ &\times \left(\frac{\partial}{\partial y_1}\right)^{r_1}\ldots\left(\frac{\partial}{\partial y_q}\right)^{r_q} \sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=n} A_{k_1,\ldots,k_p,l_1,\ldots,l_q} x_1^{k_1}\ldots x_p^{k_p} y_1^{l_1}\ldots y_q^{l_q} \\ &= \sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=n} A_{k_1,\ldots,k_p,l_1,\ldots,l_q}(\alpha_1)_{k_1}\ldots(\alpha_p)_{k_p}(\beta_1)_{l_1}\ldots(\beta_q)_{l_q}, \end{aligned}$$

which proves the lemma. □

Lemma 2.2 *Chan–Chyan–Srivastava polynomials* $g_s^{(\alpha,\beta)}(\mathbf{u},\mathbf{v})$ *satisfy the following relation:*

$$\left(x_1\frac{\partial}{\partial u_1}+x_2\frac{\partial}{\partial u_2}+\ldots+x_p\frac{\partial}{\partial u_p}-y_1\frac{\partial}{\partial v_1}-y_2\frac{\partial}{\partial v_2}-\ldots-y_q\frac{\partial}{\partial v_q}\right)^s g_s^{(\alpha,\beta)}(\mathbf{u},\mathbf{v}) = (-1)^l s! g_s^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}),$$

where $\mathbf{u}=(u_1,u_2,\ldots,u_p)$, $\mathbf{v}=(v_1,v_2,\ldots,v_q)$, *and* l $(l=l_1+\ldots+l_q)$ *denotes the degree of Chan–Chyan–Srivastava polynomials* $g_s^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ *with respect to the variable* $\mathbf{y}=(y_1,y_2,\ldots,y_q)$ *and* s $(s=k+l)$ *denotes the total degree of* $g_s^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ *with respect to the variables* $\mathbf{x}=(x_1,x_2,\ldots,x_p)$ *and* $\mathbf{y}=(y_1,y_2,\ldots,y_q)$.

Proof From the binomial theorem, we can write

$$\begin{aligned} &\left(x_1\frac{\partial}{\partial u_1}+x_2\frac{\partial}{\partial u_2}+\ldots+x_p\frac{\partial}{\partial u_p}-y_1\frac{\partial}{\partial v_1}-y_2\frac{\partial}{\partial v_2}-\ldots-y_q\frac{\partial}{\partial v_q}\right)^s \\ &= \sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=s}\frac{s!}{k_1!\ldots k_p! l_1!\ldots l_q!}\left(x_1\frac{\partial}{\partial u_1}\right)^{k_1}\ldots\left(x_p\frac{\partial}{\partial u_p}\right)^{k_p}\left(-y_1\frac{\partial}{\partial v_1}\right)^{l_1}\ldots\left(-y_q\frac{\partial}{\partial v_q}\right)^{l_q} \\ &= (-1)^l\sum_{k_1+\ldots+k_p+l_1+\ldots+l_q=s}\frac{s!}{k_1!\ldots k_p! l_1!\ldots l_q!}\left(x_1\frac{\partial}{\partial u_1}\right)^{k_1}\ldots\left(x_p\frac{\partial}{\partial u_p}\right)^{k_p}\left(y_1\frac{\partial}{\partial v_1}\right)^{l_1}\ldots\left(y_q\frac{\partial}{\partial v_q}\right)^{l_q}, \end{aligned}$$

where $l=l_1+\ldots+l_q$. Applying this operator to the Chan–Chyan–Srivastava polynomials $g_s^{(\alpha,\beta)}(\mathbf{u},\mathbf{v})$ and then using equality (1.2) again, we obtain the result. □

Theorem 2.3 *Let $g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ be Chan–Chyan–Srivastava polynomials. Let $w=\phi(\mathbf{x},\mathbf{y})$ and $F=F(w)$ be functions that have continuous derivatives of nth order in the domain $D\subset\mathbb{R}^{p+q}$. Then we have*

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial\mathbf{x}},\frac{\partial}{\partial\mathbf{y}}\right)F(w)=\chi_0(\mathbf{x},\mathbf{y})\frac{d^nF}{dw^n}+\chi_1(\mathbf{x},\mathbf{y})\frac{d^{n-1}F}{dw^{n-1}}+...+\chi_{n-1}(\mathbf{x},\mathbf{y})\frac{dF}{dw}. \tag{2.2}$$

Proof In view of equality (1.2), one gets

$$\begin{aligned}g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial\mathbf{x}},\frac{\partial}{\partial\mathbf{y}}\right)F(w) &= \sum_{k_1+...+k_p+l_1+...+l_q=n}\frac{(\alpha_1)_{k_1}...(\alpha_p)_{k_p}(\beta_1)_{l_1}...(\beta_q)_{l_q}}{k_1!...k_p!l_1!...l_q!}\left(\frac{\partial}{\partial x_1}\right)^{k_1}...\left(\frac{\partial}{\partial x_p}\right)^{k_p}\\ &\quad\times\left(\frac{\partial}{\partial y_1}\right)^{l_1}...\left(\frac{\partial}{\partial y_q}\right)^{l_q}F(w).\end{aligned}$$

It is obvious that

$$\frac{\partial^{k_1}}{\partial x_1^{k_1}}F(w)=B_0\frac{d^{k_1}}{dw^{k_1}}F(w)+B_1\frac{d^{k_1-1}}{dw^{k_1-1}}F(w)+...+B_{k_1-1}\frac{d}{dw}F(w), \tag{2.3}$$

where $B_0,B_1,...,B_{k_1-1}$ are functions of the variables $x_1,x_2,...,x_p,\ y_1,y_2,...,y_q$. If we take the derivative of equality (2.3) k_2 times with respect to the variable x_2, we find

$$\frac{\partial^{k_1+k_2}}{\partial x_1^{k_1}\partial x_2^{k_2}}F(w)=C_0\frac{d^{k_1+k_2}}{dw^{k_1+k_2}}F(w)+C_1\frac{d^{k_1+k_2-1}}{dw^{k_1+k_2-1}}F(w)+...+C_{k_1+k_2-1}\frac{d}{dw}F(w), \tag{2.4}$$

where $C_0,C_1,...,C_{k_1+k_2-1}$ are functions that depend on the variables $x_1,x_2,...,x_p,\ y_1,y_2,...,y_q$. Similarly, differentiating consecutively equation (2.4) with respect to the variables $x_3,...,x_p,\ y_1,y_2,...,y_q$, we obtain

$$\begin{aligned}\frac{\partial^n}{\partial x_1^{k_1}...\partial x_p^{k_p}\partial y_1^{l_1}...\partial y_q^{l_q}}F(w) &= D_0\frac{d^n}{dw^n}F(w)+D_1\frac{d^{n-1}}{dw^{n-1}}F(w)+...+D_{n-1}\frac{d}{dw}F(w)\\ &= \sum_{j=0}^{n-1}D_j\frac{d^{n-j}}{dw^{n-j}}F(w),\end{aligned}$$

where $n=k_1+...+k_p+l_1+...+l_q$ and $D_j\ (j=0,1,...,n-1)$ are functions of $x_1,x_2,...,x_p,\ y_1,y_2,...,y_q$. Taking into consideration the last equality in the first relation we conclude that

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial\mathbf{x}},\frac{\partial}{\partial\mathbf{y}}\right)F(w)=\sum_{j=0}^{n-1}\left(\sum_{k_1+...+k_p+l_1+...+l_q=n}D_j\frac{(\alpha_1)_{k_1}...(\alpha_p)_{k_p}(\beta_1)_{l_1}...(\beta_q)_{l_q}}{k_1!...k_p!l_1!...l_q!}\right)\frac{d^{n-j}}{dw^{n-j}}F(w),$$

and by denoting the function in the bracket by $\chi_j,$ this implies the desired equality. □

Now let us give some theorems to obtain the coefficients χ_j in Theorem 2.3.

Theorem 2.4 *Assume that* $w = \phi(\mathbf{x}, \mathbf{y})$ *is a function which has* n*th order continuous derivatives in the domain* $D \subset \mathbb{R}^{p+q}$*. If we choose* $F(w) = w^n = (\phi(\mathbf{x}, \mathbf{y}))^n$ *in Theorem 2.3, equality (2.2) reduces to*

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}\right)(\phi(\mathbf{x}, \mathbf{y}))^n = n!\left\{\chi_0(\mathbf{x}, \mathbf{y}) + \chi_1(\mathbf{x}, \mathbf{y})\ \phi(\mathbf{x}, \mathbf{y}) + \ldots + \frac{1}{m!}\chi_m(\mathbf{x}, \mathbf{y})\ (\phi(\mathbf{x}, \mathbf{y}))^m + \ldots + \frac{1}{(n-1)!}\chi_{n-1}(\mathbf{x}, \mathbf{y})\ (\phi(\mathbf{x}, \mathbf{y}))^{n-1}\right\}.$$

Theorem 2.5 *In view of Theorem 2.4, it follows that*

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}\right)(\phi(\mathbf{x}, \mathbf{y}))^n = \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}}, \frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^n,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_p)$, $\mathbf{v} = (v_1, v_2, \ldots, v_q)$, $\frac{\partial}{\partial \mathbf{u}} = \left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_p}\right)$, *and* $\frac{\partial}{\partial \mathbf{v}} = \left(\frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_q}\right)$.

Proof Setting $x_i + u_i,\ y_j + v_j\ (i = 1, 2, \ldots, p\ ;\ j = 1, 2, \ldots, q)$ instead of $x_i,\ y_j (i = 1, 2, \ldots, p\ ;\ j = 1, 2, \ldots, q)$, respectively, in Theorem 2.4, and then using

$$\frac{\partial}{\partial(x_i + u_i)} = \frac{\partial}{\partial u_i} \quad \text{and} \quad \frac{\partial}{\partial(y_j + v_j)} = \frac{\partial}{\partial v_j}$$
$$(i = 1, 2, \ldots, p\ ;\ j = 1, 2, \ldots, q)$$

we have

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}}, \frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^n = n!\{\chi_0(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}) + \chi_1(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})\ \phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}) + \ldots + \frac{1}{m!}\chi_m(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})\ (\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^m + \ldots + \frac{1}{(n-1)!}\chi_{n-1}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})\ (\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^{n-1}\right\}.$$

If we take the limit as $(\mathbf{u}, \mathbf{v}) \to (\mathbf{0}, \mathbf{0})$ in the last equation and use Theorem 2.4, we complete the proof. □

Now let us obtain the explicit forms of the coefficients $\chi_j,\ (j = 0, 1, \ldots, n-1)$.

Theorem 2.6 *The coefficients* $\chi_j\ (j = 0, 1, \ldots, n-1)$ *in Theorem 2.4 are given by*

$$\chi_j(\mathbf{x}, \mathbf{y}) = \frac{1}{(n-j)!}\lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}}, \frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}) - \phi(\mathbf{x}, \mathbf{y}))^{n-j}.$$

Proof If we apply the binomial theorem to the right side of the equality $(\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^n = (\phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}) - \phi(\mathbf{x}, \mathbf{y}))^n$, we can write

$$(\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}))^n = \sum_{t=0}^{n} \frac{n!}{(n-t)!t!}(\phi(\mathbf{x}, \mathbf{y}))^t (\phi(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}) - \phi(\mathbf{x}, \mathbf{y}))^{n-t}.$$

Substituting this in Theorem 2.5, it follows that

$$\begin{aligned} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right)(\phi(\mathbf{x},\mathbf{y}))^n &= \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)\sum_{t=0}^{n}\frac{n!}{(n-t)!t!}(\phi(\mathbf{x},\mathbf{y}))^t(\phi(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\phi(\mathbf{x},\mathbf{y}))^{n-t} \\ &= \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\phi(\mathbf{x},\mathbf{y}))^n \\ &\quad + \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)\frac{n!}{(n-1)!}(\phi(\mathbf{x},\mathbf{y}))(\phi(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\phi(\mathbf{x},\mathbf{y}))^{n-1} \\ &\quad + \ldots + \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x},\mathbf{y}))^n . \end{aligned}$$

When we compare this equality and Theorem 2.4, we obtain

$$\chi_{n-j}(\mathbf{x},\mathbf{y}) = \frac{1}{j!}\lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)(\phi(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\phi(\mathbf{x},\mathbf{y}))^j \quad , \quad j=1,2,...,n,$$

from which, by replacing j by $n-j$, the proof is completed. □

Corollary 2.7 *Choosing $\phi(\mathbf{x},\mathbf{y}) = |\mathbf{x}|^2 - |\mathbf{y}|^2 = r^2$ in Theorem 2.6 where r is the Lorentz distance, we can obtain the coefficients $\chi_j(\mathbf{x},\mathbf{y})$ in terms of the ultrahyperbolic operator L as*

$$\chi_j(\mathbf{x},\mathbf{y}) = \frac{(-1)^l\, 2^{n-2j}}{j!} L^j g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) \quad \text{for} \quad j=0,1,...,[|n/2|], \tag{2.5}$$

and $\chi_j(\mathbf{x},\mathbf{y}) = 0$ for $j = [|n/2|]+1,...,(n-1)$ where $n = 0,1,2,...$. Here l $(l = l_1 + ... + l_q)$ denotes the degree of Chan–Chyan–Srivastava polynomials $g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ with respect to the variable $\mathbf{y} = (y_1, y_2, ..., y_q)$ and L is ultrahyperbolic operator given by (1.3).

Proof In the case of $\phi(\mathbf{x},\mathbf{y}) = |\mathbf{x}|^2 - |\mathbf{y}|^2 = (x_1^2 + ... + x_p^2) - (y_1^2 + ... + y_q^2)$, Theorem 2.6 reduces to

$$\begin{aligned} \chi_j(\mathbf{x},\mathbf{y}) &= \frac{1}{(n-j)!}\lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right) \\ &\quad \times \left[\sum_{i=1}^{p}(2x_iu_i + u_i^2) - \sum_{k=1}^{q}(2y_kv_k + v_k^2)\right]^{n-j}. \end{aligned}$$

From the binomial theorem, we may write

$$\begin{aligned} \chi_j(\mathbf{x},\mathbf{y}) &= \frac{1}{(n-j)!}\lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right) \\ &\quad \times \sum_{\eta=0}^{n-j}\binom{n-j}{\eta}\left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)^{\eta} 2^{n-j-\eta}(\langle \mathbf{x},\mathbf{u}\rangle - \langle \mathbf{y},\mathbf{v}\rangle)^{n-j-\eta}, \end{aligned}$$

where $\langle \mathbf{x},\mathbf{u}\rangle$ denotes the inner product defined by $\langle \mathbf{x},\mathbf{u}\rangle = \sum\limits_{i=1}^{p} x_i u_i$, and $|\mathbf{u}|^2 = u_1^2 + ... + u_p^2$ and $|\mathbf{v}|^2 = v_1^2 + ... + v_q^2$. Since the other terms are zero except for $\eta = j$ in the right side of the equation above, we have

$$\chi_j(\mathbf{x},\mathbf{y}) = \frac{2^{n-2j}}{j!\,(n-2j)!} \lim_{(\mathbf{u},\mathbf{v})\to(\mathbf{0},\mathbf{0})} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)\left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)^j \left(\langle \mathbf{x},\mathbf{u}\rangle - \langle \mathbf{y},\mathbf{v}\rangle\right)^{n-2j}, \tag{2.6}$$

for $j = 0,1,...,[|n/2|]$. On the other side, since the function $h(\mathbf{u},\mathbf{v}) = \left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)^j \left(\langle \mathbf{x},\mathbf{u}\rangle - \langle \mathbf{y},\mathbf{v}\rangle\right)^{n-2j}$ is a homogeneous function of degree n, from Lemma 2.1,

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{u}},\frac{\partial}{\partial \mathbf{v}}\right)\left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)^j \left(\langle \mathbf{x},\mathbf{u}\rangle - \langle \mathbf{y},\mathbf{v}\rangle\right)^{n-2j} = \left(\left\langle \mathbf{x},\frac{\partial}{\partial \mathbf{u}}\right\rangle - \left\langle \mathbf{y},\frac{\partial}{\partial \mathbf{v}}\right\rangle\right)^{n-2j} L^j g_n^{(\alpha,\beta)}(\mathbf{u},\mathbf{v})$$

holds where $L = \sum\limits_{i=1}^{p} \frac{\partial^2}{\partial u_i^2} - \sum\limits_{k=1}^{q} \frac{\partial^2}{\partial v_k^2}$, $\frac{\partial}{\partial \mathbf{u}} = \left(\frac{\partial}{\partial u_1},...,\frac{\partial}{\partial u_p}\right)$, and $\frac{\partial}{\partial \mathbf{v}} = \left(\frac{\partial}{\partial v_1},...,\frac{\partial}{\partial v_q}\right)$. Also, because of the fact that the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha,\beta)}(\mathbf{u},\mathbf{v})$ are homogeneous, it follows that

$$L^j g_n^{(\alpha,\beta)}(\mathbf{u},\mathbf{v}) = S_{n-2j}(\mathbf{u},\mathbf{v}),$$

where $S_{n-2j}(\mathbf{u},\mathbf{v})$ is homogeneous polynomial of degree $n-2j$. From this equality, in view of Lemma 2.2, we conclude that

$$\begin{aligned}\left(\left\langle \mathbf{x},\frac{\partial}{\partial \mathbf{u}}\right\rangle - \left\langle \mathbf{y},\frac{\partial}{\partial \mathbf{v}}\right\rangle\right)^{n-2j} S_{n-2j}(\mathbf{u},\mathbf{v}) &= (-1)^l (n-2j)! S_{n-2j}(\mathbf{x},\mathbf{y}) \\ &= (-1)^l (n-2j)! L^j g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}).\end{aligned}$$

By taking into account this equality in (2.6), we get

$$\chi_j(\mathbf{x},\mathbf{y}) = \frac{(-1)^l\, 2^{n-2j}}{j!} L^j g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}),$$

for $j = 0,1,...,[|n/2|]$ where $L = \sum\limits_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum\limits_{k=1}^{q} \frac{\partial^2}{\partial y_k^2}$. Also, $\chi_j(\mathbf{x},\mathbf{y}) = 0$ for $j = [|n/2|]+1,...,(n-1)$. □

Theorem 2.8 *If we get $w = \phi(\mathbf{x},\mathbf{y}) = |\mathbf{x}|^2 - |\mathbf{y}|^2 = r^2$ in Theorem 2.3 and then use the coefficients given by (2.5), we find*

$$\begin{aligned}g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right) F(w) = (-1)^l \Big\{ & 2^n g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) \frac{d^n F}{dw^n} + \frac{2^{n-2}}{1!} L g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) \frac{d^{n-1}F}{dw^{n-1}} \\ & + ... + \frac{2^{n-2j}}{j!} L^j g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) \frac{d^{n-j}F}{dw^{n-j}} \Big\},\end{aligned}$$

where $j = [|n/2|]$, $(n = 0,1,...)$ and l $(l = l_1 + ... + l_q)$ denotes the degree of $g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ with respect to the variable $\mathbf{y} = (y_1, y_2, ..., y_q)$.

Theorem 2.9 *For the Chan–Chyan–Srivastava polynomials, we have*

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right) r^{2-p-q} = (-1)^k \frac{(p+q-2)(p+q)(p+q+2)\dots(p+q+2n-4)}{r^{p+q+2n-2}}$$
$$\left\{1-\frac{r^2L}{2(p+q+2n-4)}+\frac{r^4L^2}{2.4.(p+q+2n-4)(p+q+2n-6)}-\dots\right.$$
$$\left.+(-1)^j\frac{r^{2j}L^j}{j!2^j(p+q+2n-4)\dots(p+q+2(n-j))(p+q+2(n-j-1))}\right\} g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}),$$

where $j=[|n/2|]$, $(n=0,1,\dots)$ *and* k $(k=k_1+\dots+k_p)$ *denotes the degree of Chan–Chyan–Srivastava polynomials* $g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ *with respect to the variable* $\mathbf{x}=(x_1,x_2,\dots,x_p)$.

Proof If we write $F(w)=F(r^2)=\psi(r)$ in Theorem 2.8, it follows that

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right)\psi(r) = (-1)^l\left\{2^n g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})\frac{d^n\psi(r)}{d(r^2)^n}+\frac{2^{n-2}}{1!}Lg_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})\frac{d^{n-1}\psi(r)}{d(r^2)^{n-1}}\right.$$
$$\left.+\dots+\frac{2^{n-2j}}{j!}L^j g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})\frac{d^{n-j}\psi(r)}{d(r^2)^{n-j}}\right\}.$$

By choosing $\psi(r)=r^{2-p-q}$ in this equation and then calculating the derivatives, we complete the proof. □

Corollary 2.10 *Let* $p=q$ *and* $F(w)=w=|\mathbf{x}|^2-|\mathbf{y}|^2=\left(x_1^2+\dots+x_p^2\right)-\left(y_1^2+\dots+y_p^2\right)$ *in Theorem 2.8. Then, it holds that*

$$g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial \mathbf{y}}\right) w = \begin{cases} 2(-1)^l g_1^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) & \text{for } n=1,\\ (-1)^l Lg_2^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) & \text{for } n=2,\\ 0, & \text{for } n=3,4,\dots. \end{cases} \tag{2.7}$$

Theorem 2.11 *The function on the right side of equation (2.7),*

$$u(\mathbf{x},\mathbf{y}) = \begin{cases} 2(-1)^l g_1^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) & \text{for } n=1,\\ (-1)^l Lg_2^{(\alpha,\beta)}(\mathbf{x},\mathbf{y}) & \text{for } n=2,\\ 0, & \text{for } n=3,4,\dots, \end{cases}$$

satisfies the Laplace equation

$$\Delta u = u_{x_1x_1}+\dots+u_{x_px_p}+u_{y_1y_1}+\dots+u_{y_py_p};$$

that is, it is an ultraspherical harmonic function.

Proof Applying the Laplace operator to both sides of equality (2.7), the proof is completed with the fact that the function $F(w)=w=\left(x_1^2+\dots+x_p^2\right)-\left(y_1^2+\dots+y_p^2\right)$ is an ultraspherical harmonic function. □

3. Ultraspherical harmonic functions associated with Chan–Chyan–Srivastava polynomials

Now we consider the Chan–Chyan–Srivastava polynomials of p variables generated by [3]

$$\prod_{i=1}^{p}(1-x_i t)^{-\alpha_i}=\sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})\, t^n \tag{3.1}$$

$$\alpha_i\in\mathbb{C}\ \ (i=1,2,\ldots,p\ )\ \ ;\ \ |t|<min\left\{|x_1|^{-1},\ldots,|x_p|^{-1}\right\};\quad \mathbf{x}=(x_1,x_2,\ldots,x_p),$$

from which its explicit form is as follows:

$$g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})=\sum_{\substack{k_1+\ldots+k_p=n\\ k_i\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}}}\frac{(\alpha_1)_{k_1}\ldots(\alpha_p)_{k_p}}{k_1!\ldots k_p!}x_1^{k_1}\ldots x_p^{k_p}.$$

If we get the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})$ of p variables and the Laplace operator instead of the polynomials $g_n^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})$ of $p+q$ variables and ultrahyperbolic operator L, respectively, in the results obtained in the previous section, we can give the following results.

Corollary 3.1 *From Theorem 2.8, we have the next result for the Chan–Chyan–Srivastava polynomials of p variables:*

$$\begin{aligned} g_n^{(\alpha_1,\ldots,\alpha_p)}\left(\frac{\partial}{\partial\mathbf{x}}\right)F(w) &= 2^n g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})\frac{d^nF}{dw^n}+\frac{2^{n-2}}{1!}\frac{d^{n-1}F}{dw^{n-1}}\ \Delta g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x}) \\ &\quad +\ldots+\frac{2^{n-2j}}{j!}\frac{d^{n-j}F}{dw^{n-j}}\Delta^j g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x}), \end{aligned}$$

where $j=[|n/2|]$, $(n=0,1,\ldots)$ and $w=\phi(\mathbf{x})=|\mathbf{x}|^2=x_1^2+\ldots+x_p^2=r^2$, r is Euclidean distance, and $\frac{\partial}{\partial\mathbf{x}}=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_p}\right)$. If we denote $F(w)=F(r^2)$ by $\psi(r)$, we can write

$$\begin{aligned} g_n^{(\alpha_1,\ldots,\alpha_p)}\left(\frac{\partial}{\partial\mathbf{x}}\right)\psi(r) &= \left\{2^n\frac{d^n\psi(r)}{d(r^2)^n}+\frac{2^{n-2}}{1!}\frac{d^{n-1}\psi(r)}{d(r^2)^{n-1}}\ \Delta\right. \\ &\quad \left. +\ldots+\frac{2^{n-2j}}{j!}\frac{d^{n-j}\psi(r)}{d(r^2)^{n-j}}\Delta^j\right\} g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x}). \end{aligned}$$

Theorem 3.2 *For the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})$ of p variables, it holds that*

$$\begin{aligned} g_n^{(\alpha_1,\ldots,\alpha_p)}\left(\frac{\partial}{\partial\mathbf{x}}\right)r^{2-p} &= (-1)^n\frac{(p-2)(p)(p+2)\ldots(p+2n-4)}{r^{p+2n-2}} \\ &\quad\times\left\{1-\frac{r^2\Delta}{2(p+2n-4)}+\frac{r^4\Delta^2}{2.4.(p+2n-6)(p+2n-4)}\right. \\ &\quad\left. -\ldots+(-1)^j\frac{r^{2j}\Delta^j}{j!2^j(p+2n-4)\ldots(p+2(n-j))(p+2(n-j-1))}\right\} g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x}), \end{aligned}$$

where $j=[|n/2|]$, $(n=0,1,\ldots)$, and $r^2=x_1^2+\ldots+x_p^2$.

Proof In Corollary 3.1, setting $\psi(r) = r^{2-p}$, we have

$$\begin{aligned}
\frac{d\psi(r)}{d(r^2)} &= \frac{dr^{2-p}}{d(r^2)} = \frac{d\left(w^{\frac{2-p}{2}}\right)}{dw} = (-1)\frac{p-2}{2}r^{-p}, \\
\frac{d^2\psi(r)}{d(r^2)^2} &= \frac{d^2r^{2-p}}{d(r^2)^2} = \frac{d^2\left(w^{\frac{2-p}{2}}\right)}{dw^2} = (-1)^2\frac{(p-2)(p)}{2^2}r^{-p-2}, \\
&\ldots \\
\frac{d^n\psi(r)}{d(r^2)^n} &= \frac{d^nr^{2-p}}{d(r^2)^n} = \frac{d^n\left(w^{\frac{2-p}{2}}\right)}{dw^n} \\
&= (-1)^n\frac{(p-2)(p)\ldots(p+2n-4)}{2^n}r^{-p-2n+2}.
\end{aligned}$$

Taking into account these derivatives in Corollary 3.1, we obtain the result. □

Theorem 3.3 *The function*

$$\begin{aligned}
u(\mathbf{x}) &= \frac{1}{r^{p+2n-2}}\left\{1 - \frac{r^2\Delta}{2(p+2n-4)} + \frac{r^4\Delta^2}{2.4.(p+2n-6)(p+2n-4)}\right. \\
&\quad \left. - \ldots + (-1)^j\frac{r^{2j}\Delta^j}{j!2^j(p+2n-4)\ldots(p+2(n-j))(p+2(n-j-1))}\right\} g_n^{(\alpha_1,\ldots,\alpha_p)}(\mathbf{x})
\end{aligned}$$

where $j = [|n/2|]$, $(n = 0, 1, \ldots)$, *satisfies the Laplace equation*

$$\Delta u = u_{x_1x_1} + \ldots + u_{x_px_p};$$

that is, it is an ultraspherical harmonic function.

Proof Since the function r^{2-p} , $p \geq 3$, is a harmonic function, $\Delta\left(r^{2-p}\right) = 0$ holds. If we apply the Laplace operator to the both sides of the equality in Theorem 3.2 by considering the fact that the operator $g_n^{(\alpha_1,\ldots,\alpha_p)}\left(\frac{\partial}{\partial \mathbf{x}}\right)$ is an operator with constant coefficient and the function r^{2-p} , $p \geq 3$, is a harmonic function, we have the desired result. □

Remark 3.4 *We note that Theorem 3.2 and Theorem 3.3 were given for any homogeneous function in [7].*

4. Harmonic functions associated with Lagrange polynomials

In the case of $p = 2$, the Chan–Chyan–Srivastava polynomials given by (3.1) reduce to the classical Lagrange polynomials, which are seen in certain problems in statistics [4, p. 267] (see also [12, p. 441]) defined by

$$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\,t^n,$$

from which we get

$$g_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_{n-k}}{k!(n-k)!}x^ky^{n-k}.$$

In Corollary 3.1, setting $\psi(r) = \log \frac{1}{r}$ where $r^2 = x^2 + y^2$, we can give the next theorem without the proof.

Theorem 4.1 *For the Lagrange polynomials* $g_n^{(\alpha,\beta)}(x,y)$, *it follows that*

$$\begin{aligned} g_n^{(\alpha,\beta)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \log \frac{1}{r} &= (-1)^n \frac{(n-1)!2^{n-1} \log e}{r^{2n}} \left\{ 1 - \frac{r^2 \Delta}{1!(n-1)2^2} + \frac{r^4 \Delta^2}{2!(n-1)(n-2)2^4} \right. \\ &\left. - \frac{r^6 \Delta^3}{3!(n-1)(n-2)(n-3)2^6} + \ldots + (-1)^j \frac{(n-j-1)! r^{2j} \Delta^j}{j!(n-1)! 2^{2j}} \right\} g_n^{(\alpha,\beta)}(x,y), \end{aligned}$$

where $j = [|n/2|]$, $(n = 1, 2, \ldots)$, $r^2 = x^2 + y^2$, *and* $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *is two-dimensional Laplacian.*

Now we can give a harmonic function associated with Lagrange polynomials.

Theorem 4.2 *The function*

$$\begin{aligned} u(x,y) &= \frac{1}{r^{2n}} \left\{ 1 - \frac{r^2 \Delta}{1!(n-1)2^2} + \frac{r^4 \Delta^2}{2!(n-1)(n-2)2^4} - \frac{r^6 \Delta^3}{3!(n-1)(n-2)(n-3)2^6} \right. \\ &\left. + \ldots + (-1)^j \frac{(n-j-1)! r^{2j} \Delta^j}{j!(n-1)! 2^{2j}} \right\} g_n^{(\alpha,\beta)}(x,y) \end{aligned}$$

is a harmonic function where $j = [|n/2|]$, $(n = 1, 2, \ldots)$.

Proof If we apply the Laplace operator to both sides of the equality in Theorem 4.1 and then use the fact that $\log \frac{1}{r}$ is a harmonic function, we complete the proof. □

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