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A nonexistence result for blowing up sign-changing solutions of the Brezis–Nirenberg-type problem

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Abstract: We consider the Brezis–Nirenberg problem: $-\triangle u = |u|^{p-1}u \pm \varepsilon u$ in Ω, with $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, $p+1 = 2n/(n-2)$ is the critical Sobolev exponent, and $\varepsilon > 0$ is a positive parameter. The main result of this paper shows that if $n \geq 4$ there are no sign-changing solutions u_ε of $(P_{-\varepsilon})$ with two positive and one negative blow up points.

Key words: Blow-up analysis, sign-changing solutions, lack of compactness, critical exponent

1. Introduction

In this paper, we study the following semilinear elliptic problem:

$$(P_{\pm\varepsilon}) \qquad \begin{cases} -\triangle u = |u|^{p-1}u \pm \varepsilon u & \text{in } \Omega, \qquad (1) \\ u = 0 & \text{on } \partial\Omega, \qquad (2) \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, $p+1 = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, and ε is a real positive parameter.

The problem is known as the Brezis–Nirenberg problem because the first fundamental results about the existence of positive solutions were obtained by Brezis and Nirenberg in 1983. The authors explain in [8] that dimension plays a crucial role in the study of $(P_{+\varepsilon})$. They proved that if $n \geq 4$ there exists a positive solution of $(P_{+\varepsilon})$ for every $\varepsilon \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\triangle$ in Ω with Dirichlet boundary conditions.

Moreover, for $n \geq 4$, by using Pohozaev's identity, it is easy to check that if Ω is a star-shaped domain, the problem $(P_{-\varepsilon})$ has no nontrivial solutions. Finally, in [15], Musso and Pistoia show that if ε is close to 0, there exists a family of solutions that blow up and concentrate in two points if Ω is a domain with a small "hole".

Concerning the case of sign-changing solutions of $(P_{+\varepsilon})$, the existence results hold for $n \geq 4$ for both $\varepsilon \in (0, \lambda_1(\Omega))$ and $\varepsilon > \lambda_1(\Omega)$ as shown in [1, 9, 10]. Note that the small dimensions $n = 4, 5, 6$ are specific to this problem. Indeed, Atkinson et al. show in [2] that if Ω is a ball, then there exists $\widetilde{\lambda} := \widetilde{\lambda}(n)$ so there are no radial sign-changing solutions of $(P_{+\varepsilon})$ for $\varepsilon \in (0, \widetilde{\lambda})$. However, for $n \geq 7$, Schechter and Zou have shown in [17] that in any bounded smooth domain, there is an infinity of sign-changing solutions of $(P_{+\varepsilon})$ for any $\varepsilon > 0$.

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Concerning the low energy sign-changing solutions of $(P_{+\varepsilon})$, a study has been carried out in [6] by Ben Ayed et al. concerning the solutions u_ε satisfying

$$\frac{1}{c_1} \leq -\frac{\max u_\varepsilon}{\min u_\varepsilon} \leq c_1.$$

The authors proved an axial symmetry result for the same kinds of solutions in a ball. Next, Iacopetti and Vaira built in [14] solutions of $(P_{+\varepsilon})$ in the form of $u_\varepsilon = \delta_{a,\lambda_1} - \delta_{a,\lambda_2} + \widetilde{v}_\varepsilon$ with $\|\widetilde{v}_\varepsilon\| \to 0$ as $\varepsilon \to 0$, where

$$\delta(x) := \delta_{a,\lambda}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(1+\lambda^2|x-a|^2)^{(n-2)/2}}, \qquad \lambda > 0, \quad a \in \mathbb{R}^n,$$

$c_0 := (n(n-2))^{\frac{n-2}{4}}$, describe all regular positive solutions of the Yamabe problem

$$-\triangle u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n.$$

This result has been proved only for large dimensions $n \geq 7$. Note that the bound $n \geq 7$ is optimal, since Iacopetti and Pacella [13] and Dammak [11] show that in dimension $n = 4, 5, 6$ the low energy sign-changing solutions u_ε of $(P_{+\varepsilon})$ (with $\max u_\varepsilon / \min u_\varepsilon \to 0$ or $-\infty$) do not exist in any bounded smooth domain. However, in the 3-dimensional case a nonexistence result was already proved in [7]. Indeed, in studying the asymptotic behavior of low-energy nodal solutions it was shown in [7] that their positive and negative parts cannot concentrate at the same point, as ε tends to a limit value $\overline{\lambda} > 0$.

Concerning the case of sign-changing solutions of $(P_{-\varepsilon})$, in [12], Hammami and Ismail have shown in Theorem 1.1.1 that for $n \geq 4$ there exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem $(P_{-\varepsilon})$ has no solutions u_ε in the form of

$$u_\varepsilon = P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}} - P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}} + \widetilde{v}_\varepsilon, \tag{3}$$

with $\|\widetilde{v}_\varepsilon\|$ small, where $P\delta := P\delta_{a,\lambda}$ denotes the projection of $\delta_{a,\lambda}$ on $H_0^1(\Omega)$, i.e.

$$-\triangle P\delta = \delta^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \qquad P\delta = 0 \quad \text{on } \partial\Omega, \tag{4}$$

and in Theorem 1.1.2, assuming that there exists a family of solutions under the form

$$u_\varepsilon := P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}} - P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}} + P\delta_{a_{3,\varepsilon},\lambda_{3,\varepsilon}} + \widetilde{v}_\varepsilon,$$

with $\|\widetilde{v}_\varepsilon\|_{H_0^1} \to 0$, then the authors have been able to characterize the points and the rates of concentration of $\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}}$ and $\delta_{a_{3,\varepsilon},\lambda_{3,\varepsilon}}$. More precisely, they have shown that there exist constants c_1, c_2, and c_3 such that

$$\begin{cases} c_1^{-1} \leq \lambda_{1,\varepsilon}/\lambda_{3,\varepsilon} \leq c_1, \quad c_2\lambda_{i,\varepsilon} \leq \lambda_{2,\varepsilon}, \forall i \in \{1,3\}, \\ \varepsilon \simeq \frac{1}{\ln(\lambda_{1,\varepsilon})} \quad \text{if } n = 4, \quad \varepsilon \simeq \lambda_{1,\varepsilon}^{4-n} \quad \text{if } n \geq 5, \\ d(a_{i,\varepsilon}, \partial\Omega) \geq c_3, \forall i \in \{1,3\}, \quad |a_{1,\varepsilon} - a_{3,\varepsilon}| \geq c_3. \end{cases} \tag{5}$$

Note that in [12] the authors could not extract information concerning the point $a_{2,\varepsilon}$. This defect flow is due to the poor control of $\|\widetilde{v}_\varepsilon\|_{H_0^1}$.

Now we state our result, which can be considered the continuation of a part of [12].

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$. There exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem $(P_{-\varepsilon})$ has no sign-changing solutions u_ε in the form of*

$$u_\varepsilon = P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}} - P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}} + P\delta_{a_{3,\varepsilon},\lambda_{3,\varepsilon}} + \widetilde{v}_\varepsilon, \text{ such that} \tag{6}$$

$$\begin{cases} a_{i,\varepsilon} \in \Omega, \quad \lambda_{i,\varepsilon} d(a_{i,\varepsilon}, \partial\Omega) \to +\infty \quad \forall i \in \{1,2,3\}, \\ \forall i \neq j, \langle P\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}, P\delta_{a_{j,\varepsilon},\lambda_{j,\varepsilon}} \rangle \to 0, \text{ and } \|\widetilde{v}_\varepsilon\| \to 0 \text{ as } \varepsilon \to 0. \end{cases}$$

Our argument is carried out by contradiction. It is based on analysis of the Euler functional related to this problem $(P_{-\varepsilon})$. Note that, assuming the existence of a family of solutions of the form (6), then, according to Bahri, the solution u_ε can be written as

$$u_\varepsilon := \alpha_{1,\varepsilon} P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}} - \alpha_{2,\varepsilon} P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}} + \alpha_{3,\varepsilon} P\delta_{a_{3,\varepsilon},\lambda_{3,\varepsilon}} + v_\varepsilon, \tag{7}$$

where $\alpha_{i,\varepsilon}$ is close to 1 and v_ε satisfies: $\|v_\varepsilon\|_{H_0^1} \to 0$, $v_\varepsilon \in F^\perp$, where F is the space spanned by the $3n+6$ functions $P\delta_i, \partial P\delta_i/\partial\lambda_i, \partial P\delta_i/\partial a_i^k$, $k \in \{1, ..., n\}$ and $i \in \{1,2,3\}$. The main difficulty of our proof comes from the v_ε. Following the ideas introduced by Bahri and Xu in [4], we managed to decompose this function v_ε into two parts $v_\varepsilon := v_1^\varepsilon + v_2^\varepsilon$. Some accurate estimates proven on these two functions allowed us to improve the remaining of certain formulas. This improvement led to a contradiction justifying the nonexistence of such a family of solutions of $(P_{-\varepsilon})$ in the form (6).

The remainder of this paper is organized as follows. In Section 2, we collect estimates of some integrals needed in our work. The main result of Section 3 is devoted to studying the v-part. We decompose it into two functions v_1 and v_2. We find a punctual estimate of v_1 and then we deduce the estimate of $\|v_2\|$. Section 4 is devoted to the already acquired results to improve the remaining of Proposition 3.3. Finally, in Section 5 we prove the main theorem.

Note that, throughout this paper, for the sake of simplicity, we will omit the index ε of our variables and we denote by $\delta_i := \delta_{a_i,\lambda_i}$.

2. Some a priori estimates

To show the main theorem, we argue by contradiction. We assume that there exists a family of solutions under the form (6).

To facilitate the recourse to the reference, in this section, we collect estimates of some integrals needed in our work.

Lemma 2.1 *[16, p. 29-30] Let $n \geq 4$. We have the following estimates:*

(a) $$\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d)^n}\right), \qquad \text{where } S = \int_{\mathbb{R}^n} \delta_{0,1}^{\frac{2n}{n-2}} \quad \text{and } d := d(a, \partial\Omega),$$

(b) $$\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \theta_{a,\lambda} = O\left(\frac{1}{(\lambda d)^{n-2}}\right), \qquad \text{where } \theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda},$$

(c) $$\int_\Omega P\delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d)^{n-2}}\right).$$

Lemma 2.2 *[3, Estimate 1,2. p. 4], [11, Lemma 2.2. p. 3] Let* $n \geq 4$ *and let* δ_{a_1,λ_1} *and* δ_{a_2,λ_2} *be such that* $\max(\lambda_1/\lambda_2, \lambda_2/\lambda_1, \lambda_1\lambda_2|a_1 - a_2|^2)$ *is very large. We have the following estimates:*

(a) $\int_{\mathbb{R}^n} \delta_1^{\frac{n+2}{n-2}}\delta_2 = \int_{\mathbb{R}^n} \delta_2^{\frac{n+2}{n-2}}\delta_1 = c\varepsilon_{12} + O(\varepsilon_{12}^{\frac{n}{n-2}}),$ *where* $\varepsilon_{12} := \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_1\lambda_2|a_1 - a_2|^2\right)^{\frac{2-n}{2}}$

(b) $\int_{\mathbb{R}^n} (\delta_1\delta_2)^{\frac{n}{n-2}} = O\left(\varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1})\right),$

(c) $\int_\Omega \delta_1\delta_2 = O(\varepsilon_{12}),$ *(d)* $\int_\Omega \delta_1^\alpha\delta_2^\beta = O\left(\varepsilon_{12}^{min(\alpha,\beta)}\right),$

for all positive real numbers α *and* β *checking* $\alpha \neq \beta$ *and* $\alpha + \beta = 2n/(n-2)$.

Lemma 2.3 *We assume that* ε_{12} *is very small. For all* $h \in H_0^1(\Omega)$, *we have*

(a) $\int_\Omega \delta_1^{\frac{4}{n-2}}\delta_2|h| = \begin{cases} O\left(\|h\|\varepsilon_{12}(\ln\varepsilon_{12}^{-1})^{2/3}\right) & \text{if } n = 6, \\ O\left(\|h\|\varepsilon_{12}^{\min(1,4/(n-2))}\right) & \text{if } n \neq 6. \end{cases}$

(b) $\int_\Omega \delta_1\delta_2^{\frac{6-n}{n-2}}h^2 = \begin{cases} O\left(\|h\|^2\varepsilon_{12}(\ln\varepsilon_{12}^{-1})^{1/2}\right) & \text{if } n = 4 \\ O\left(\|h\|^2\varepsilon_{12}^{1/3}\right) & \text{if } n = 5. \end{cases}$

(c) $\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}|h| = \begin{cases} O\left(\frac{\|h\|}{(\lambda d)^4}(\ln(\lambda d))^{2/3}\right) & \text{if } n = 6, \\ O\left(\frac{\|h\|}{(\lambda d)^{\min(n-2,(n+2)/2)}}\right) & \text{if } n \neq 6. \end{cases}$

Proof (a) and (b) follows from Lemma 2.2 by using Hölder's inequality. Concerning claim (c), we decompose the integral on the ball $B := B_{(a,d)}$ and on $\Omega \setminus B$.

On B, we use the fact that $\|\theta_{a,\lambda}\|_\infty \leq c\lambda^{(2-n)/2}d^{(2-n)}$ and then we apply Hölder's inequality.

On $\Omega \setminus B$, we increase $\theta_{a,\lambda}$ by $\delta_{a,\lambda}$ and then we apply Hölder's inequality. □

Lemma 2.4 *[16, p. 34] For all* $v \in \langle P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a^1}, ..., \frac{1}{\lambda}\frac{\partial P\delta}{\partial a^n}\rangle^\perp$ *we have*

$$\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}}v = \begin{cases} O\left(\frac{\|v\|}{(\lambda d)^4}(\ln(\lambda d))^{2/3}\right) & \text{if } n = 6, \\ O\left(\frac{\|v\|}{(\lambda d)^{\min(n-2,(n+2)/2)}}\right) & \text{if } n \neq 6. \end{cases}$$

Lemma 2.5 *[3, Proposition 3.1 p. 64] Let* $N \geq 1$, *there exists* $\beta_0 > 0$ *such that for each* $v \in F^\perp$ *we have:*

$$\|v\|_{H_0^1}^2 - \frac{n+2}{n-2}\sum_{i=1}^N \int_\Omega P\delta_i^{4/n-2}v^2 \geq \beta_0\|v\|_{H_0^1}^2,$$

where F *is the space spanned by the* $N(n+2)$ *functions* $P\delta_i, \lambda_i\partial P\delta_i/\partial\lambda_i, (1/\lambda_i)(\partial P\delta_i/\partial a_i^k),$ $k \in \{1, ..., n\}$ *and* $i \in \{1, ..., N\}$.

Lemma 2.6 *[6, Lemma 3.3 p. 778] The function v defined in (7) satisfies the following estimate:*

$$\|v\| \leq c \begin{cases} \sum_{k=1}^{3} \frac{1}{(\lambda_k d_k)^{n-2}} + \sum_{k=1}^{3} \frac{\varepsilon}{(\lambda_k)^{(n-2)/2}} + \sum_{i\neq j} \varepsilon_{ij} (\ln \varepsilon_{ij}^{-1})^{(n-2)/2} & \text{if } n = 4, 5, \\ \sum_{k=1}^{3} \frac{(\ln \lambda_k d_k)^{2/3}}{(\lambda_k d_k)^4} + \sum_{k=1}^{3} \varepsilon \frac{(\ln \lambda_k)^{2/3}}{(\lambda_k)^2} + \sum_{i\neq j} \varepsilon_{ij} (\ln \varepsilon_{ij}^{-1})^{2/3} & \text{if } n = 6, \\ \sum_{k=1}^{3} \frac{1}{(\lambda_k d_k)^{(n+2)/2}} + \sum_{k=1}^{3} \frac{\varepsilon}{\lambda_k^2} + \sum_{i\neq j} \varepsilon_{ij}^{(n+2)/2(n-2)} (\ln \varepsilon_{ij}^{-1})^{(n+2)/2n} & \text{if } n > 6, \end{cases}$$

where $\varepsilon_{ij} := \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{2-n}{2}}$ *and* $d_i := d(a_i, \partial\Omega)$.

We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part i.e. for $x \in \Omega$,

$$-\triangle G(x,.) = c_n \delta_x \quad \text{in } \Omega \qquad G(x,.) = 0 \qquad \text{on } \partial\Omega,$$

$$H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2), \quad \forall (x_1, x_2) \in \Omega^2.$$

To simplify the presentation, we denote by $H_{ij} := H(a_i, a_j)$, and $G_{ij} := G(a_i, a_j)$; then we have

Lemma 2.7 *[11, Lemma 2.8-2.10. p. 6-7] For $n \geq 4$, and for all $y \in \Omega$ we have*

(a) $\int_\Omega \delta_{a,\lambda}(x) G(x,y) dx \leq c \delta_{a,\lambda}(y)$,

(b) for all $\alpha \in \left(\frac{n-4}{n-2}, 1 \right)$, *there exists a positive constant* c_α *such that*

$$\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}(x) \theta_{a,\lambda}(x) G(x,y) dx \leq \frac{c_\alpha}{(\lambda d)^{(n-2)(1-\alpha)}} \delta_{a,\lambda}(y),$$

(c) $\int_\Omega \delta_{a,\lambda}^{\frac{n}{n-2}}(x) |h(x)|^{\frac{2}{n-2}} G(x,y) dx \leq c \|h\|_{L^{2n/(n-2)}}^{2/(n-2)} \delta_{a,\lambda}(y), \quad \forall h \in L^{\frac{2n}{n-2}}(\Omega)$.

We remark that in [11], Claim (c) is written with the function v. However, the proof is also true with a general function $h \in L^{2n/(n-2)}$.

Lemma 2.8 *[5, Lemmas A5 and A6. p. 571-572] For $n \geq 4$, we have:*

(a) $\int_\Omega P\delta_i^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = 2 \langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle + O\left(\frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^n} \right)$,

(b) $\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle = \frac{n-2}{2} \widetilde{c} \frac{H_{ii}}{\lambda_i^{n-2}} + O\left(\frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^n} \right)$,

where $\widetilde{c} := c_0^{2n/n-2} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}$ *and* $d_i := d(a_i, \partial\Omega)$.

Lemma 2.9 *[11, Lemma 2.13. p. 8] For $n \geq 4$, and $i \neq j$, we have*

$$\int_\Omega P\delta_i P\delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} = \frac{n-2}{n+2}\langle P\delta_i, \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}\rangle + O\left(\varepsilon_{ij}^{n/n-2}\ln(\varepsilon_{ij}^{-1}) + \frac{\ln(\lambda_j d_j)}{(\lambda_j d_j)^n}\right).$$

Lemma 2.10 *[11, Lemma 2.14. p. 9] For $n \geq 4$, if $d_1 \nrightarrow 0$, then we have*

$$\langle P\delta_1, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2}\rangle = \widetilde{c}\left(\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) + O\left(\varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) + \frac{\ln(\lambda_2 d_2)}{(\lambda_2 d_2)^2}\varepsilon_{12} + \frac{\varepsilon_{12}}{\lambda_1^2}\right).$$

Proposition 2.11 *[12, Eqs (1.19) and (1.20) p. 36] Let u_ε be a solution of $(P_{-\varepsilon})$ under the form (7); then for $i = 1, 2, 3$ and $n \geq 4$ we have*

$$\widetilde{c}\frac{n-2}{2}\frac{H_{ii}}{\lambda_i^{n-2}} + \gamma_i \sum_{j\neq i}\gamma_j \widetilde{c}\left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2}\frac{H_{ij}}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\right) + \frac{C_2\varepsilon}{\lambda_i^2}(\text{if } n \geq 5) + C_3\frac{\varepsilon\ln(\lambda_i d_i)}{\lambda_i^2}(\text{if } n = 4)$$
$$= o\left(\sum_{j=1}^{3}\left(\frac{1}{(\lambda_j d_j)^{n-2}} + \frac{\varepsilon}{\lambda_j^2}\right) + \sum_{j\neq k}\varepsilon_{jk}\right),$$

where $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = -1$, $\widetilde{c}$ is defined in Lemma 2.8, ε_{ij} and d_i are defined in Lemma 2.6, $C_2 = \frac{n-2}{2}c_0^2\int_{\mathbb{R}^n}\frac{|x|^2-1}{(1+|x|^2)^{n-1}} > 0$ and $C_3 = \frac{1}{2}c_0^2 mes(S^3)$ with $mes(S^3)$ denoting the area of the unit sphere of $\mathbb{R}^4$.

The proof of the following Lemma is immediate by applying the maximum principle and some proprieties of the harmonic functions.

Lemma 2.12 *We have*

(a) $\left|\lambda\frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right| \leq cP\delta_{a,\lambda}$,

(b) $\theta_{a,\lambda}(x) = \frac{c_0}{\lambda^{\frac{n-2}{2}}}H(a,x) + O\left(\frac{H(a,x)}{\lambda^{\frac{n+2}{2}}d^2}\right)$ *(where $\theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda}$),*

(c) let τ be a positive constant. For all $x \in \Omega\backslash B_{(a,\tau)}$ we have

$$P\delta_{a,\lambda}(x) = O\left(\frac{d}{\lambda^{\frac{n-2}{2}}} + \frac{1}{\lambda^{\frac{n-2}{2}}(\lambda d)^2}\right).$$

Lemma 2.13 *Assume that $|a_1 - a_2| \geq c > 0$ and $d_1 \geq c > 0$. Then there holds $G_{12} \sim cd_2$.*

Proof This relation follows immediately if $d_2 \nrightarrow 0$ since $d(a_1, \partial\Omega) \geq c > 0$. In the other case, that is $d_2 \to 0$, there exists $t_0 \in (0,1)$ and $a_0 = t_0 a_2 + (1-t_0)\overline{a}_2$ where $\overline{a}_2$ denotes the orthogonal projection of a_2 on $\partial\Omega$ such that

$$G(a_1, a_2) = \frac{\partial G}{\partial b}(a_1, a_0)(a_2 - \overline{a}_2) = -\frac{\partial G}{\partial b}(a_1, a_0)\cdot \nu_{\overline{a}_2} d_2.$$

Note that $(\partial G/\partial b)(a_1, a_0) = (\partial G/\partial b)(a_1, \overline{a}_2) + O(|a_0 - \overline{a}_2|)$, which implies that

$$G_{12} = -\frac{\partial G}{\partial \nu}(a_1, \overline{a}_2)d_2 + O(d_2^2).$$

The proof follows from Hopf's Lemma and the fact that $\partial\Omega$ is a compact set and $|a_1 - a_2| \geq c$. □

Lemma 2.14 *[11, Lemma 6.2. p. 22] We assume that $d_1 \geq c > 0$ and $|a_1 - a_2| \geq c > 0$. For $n \geq 4$, we have*

$$\int_\Omega \delta_1^\alpha P\delta_2 \leq \left(d_2 + \frac{1}{(\lambda_2 d_2)^2}\right)\frac{c}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}, \ \forall\ 1 \leq \alpha \leq \frac{n+2}{n-2}.$$

3. Study of the v-part

Before starting this section, we are going to explain the reasons why we are conducting an elaborate study of the function v. Let u_ε be a solution of $(P_{-\varepsilon})$ under the form (7); then we have: $u_\varepsilon := \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v$. Taking into account Eq. (5), two cases may occur.

The first case: (if $\lambda_2 \simeq \lambda_1 \simeq \lambda_3$), in this case Proposition 2.11 allows us to find a contradiction justifying Theorem 1.1, but in the second case: (if $\lambda_1 \simeq \lambda_3 \ll \lambda_2$), the remaining of Proposition 2.11 (for $i = 2$) is inadequate for our situation; in fact the term $\frac{1}{(\lambda_i d_i)^{n-2}}$, $i = 1, 3$ of the same proposition is not small compared to the main terms of the proposition. Thus this proposition, for $i = 2$, as written, is not very important. This term comes mainly from $\|v\|$ (see Lemma 2.6); therefore, we have thought about decomposing the function v in two functions v_1 and v_2 such that the function v_1 contains all the disturbing terms and v_2 has a good estimate.

The main objective of this section is to find a punctual estimate of v_1; then we deduce an estimate of $\|v_2\|$. Thus, throughout this section, we assume that $\lambda_1 \simeq \lambda_3 \ll \lambda_2$.

Let u_ε be a solution of $(P_{-\varepsilon})$ under the form (7); then we have

$$-\triangle v = f := -\alpha_1\delta_1^{\frac{n+2}{n-2}} + \alpha_2\delta_2^{\frac{n+2}{n-2}} - \alpha_3\delta_3^{\frac{n+2}{n-2}} + |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon - \varepsilon u_\varepsilon. \tag{8}$$

Let

$$\begin{aligned} f_1(w) :=& -\alpha_1\delta_1^{\frac{n+2}{n-2}} - \alpha_3\delta_3^{\frac{n+2}{n-2}} + |\alpha_1 P\delta_1 + \alpha_3 P\delta_3 + w|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 + \alpha_3 P\delta_3 + w) \\ & - \varepsilon\alpha_1 P\delta_1 - \varepsilon\alpha_3 P\delta_3 - \varepsilon w, \end{aligned} \tag{9}$$

and let $(\varphi_1, ..., \varphi_{2n+4})$ be an orthonormal basis of E_1, where

$$E_1 := \langle P\delta_1, \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}, \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1^1}, ..., \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1^n}, P\delta_3, \lambda_3\frac{\partial P\delta_3}{\partial\lambda_3}, \frac{1}{\lambda_3}\frac{\partial P\delta_3}{\partial a_3^1}, ..., \frac{1}{\lambda_3}\frac{\partial P\delta_3}{\partial a_3^n}\rangle. \tag{10}$$

We define the function v_1 by

$$\begin{cases} -\triangle v_1 = f_1\left((\inf(|v_1|, \mu\delta_1 + \mu\delta_3))sign(v_1)\right) \\ \qquad\qquad - \displaystyle\sum_{k=1}^{2n+4}\left[\int_\Omega f_1\left((\inf(|v_1|, \mu\delta_1 + \mu\delta_3))sign(v_1)\right)\varphi_k\right](-\triangle\varphi_k) \quad \text{in } \Omega \\ v_1 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{11}$$

where μ is a small positive constant.

The idea of introducing v_1 in this form is inspired by [4], where the authors use equations of type (11) in order to get punctual estimates of some functions similar to our function v.

We start with the following result.

Proposition 3.1 *For $n \geq 4$, the function v_1 satisfies*

(a) $v_1 \in E_1^{\perp}$

(b) $\|v_1\| \leq c\varepsilon_{13}^{\frac{n+2}{2(n-2)}} \left(\ln \varepsilon_{13}^{-1}\right)^{\frac{n+2}{2n}} + \dfrac{c\varepsilon}{\lambda_1^{\frac{n-2}{2}}}$ *(if* $n = 4, 5$*)* $+ \dfrac{c(\ln \lambda_1)^{2/3}}{\lambda_1^4}$ *(if* $n = 6$*)* $+ \dfrac{c}{\lambda_1^{\frac{n+2}{2}}}$ *(if* $n > 6$*).*

Proof Claim (a) follows immediately by computing $\langle v_1, \varphi_j \rangle_{H_0^1}$ and using the fact that $(\varphi_1, ..., \varphi_{2n+4})$ is an orthonormal basis of E_1. Concerning Claim (b), to simplify the presentation, we denote by $v_1^* := \inf(|v_1|, \mu\delta_1 + \mu\delta_3) sign(v_1)$. Multiplying (11) by v_1, and integrating on Ω, we obtain

$$\begin{aligned}
\|v_1\|^2_{H_0^1} = & \int_\Omega |\alpha_1 P\delta_1 + \alpha_3 P\delta_3 + v_1^*|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 + \alpha_3 P\delta_3 + v_1^*) v_1 \\
& - \varepsilon \int_\Omega \alpha_1 P\delta_1 v_1 - \varepsilon \int_\Omega \alpha_3 P\delta_3 v_1 - \varepsilon \int_\Omega v_1^* v_1 \\
\leq & \int_\Omega (\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{n+2}{n-2}} v_1 + \frac{n+2}{n-2} \int_\Omega (\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{4}{n-2}} v_1^* v_1 \\
& + O_{(\text{if } n=4,5)} \left(\int_\Omega (\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{6-n}{n-2}} (v_1^*)^2 |v_1| \right) + O\left(\int_\Omega |v_1^*|^{\frac{n+2}{n-2}} |v_1| \right) \\
& + c\varepsilon \int_\Omega \delta_1 |v_1| + c\varepsilon \int_\Omega \delta_3 |v_1| \\
\leq & \int_\Omega (\alpha_1 P\delta_1)^{\frac{n+2}{n-2}} v_1 + \int_\Omega (\alpha_3 P\delta_3)^{\frac{n+2}{n-2}} v_1 + \frac{n+2}{n-2} \int_\Omega (\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{4}{n-2}} v_1^2 \\
& + c \int_\Omega (\sup(\alpha_1 P\delta_1, \alpha_3 P\delta_3))^{\frac{4}{n-2}} \inf(\alpha_1 P\delta_1, \alpha_3 P\delta_3) |v_1| \\
& + (c\mu + c\mu^{\frac{4}{n-2}}) \int_\Omega (\delta_1 + \delta_3)^{\frac{4}{n-2}} |v_1|^2 + c\varepsilon \int_\Omega \delta_1 |v_1| + c\varepsilon \int_\Omega \delta_3 |v_1|. \qquad (12)
\end{aligned}$$

From (12), Lemmas 2.3, 2.4, Hölder's inequality, and the fact that $\varepsilon \simeq \lambda_1^{4-n}, \forall n \geq 5$, (see (5)), it follows that

$$\begin{aligned}
\|v_1\|^2_{H_0^1} \leq & \frac{n+2}{n-2} \int_\Omega (\alpha_1 P\delta_1)^{\frac{4}{n-2}} v_1^2 + \frac{n+2}{n-2} \int_\Omega (\alpha_3 P\delta_3)^{\frac{4}{n-2}} v_1^2 + \|v_1\| \varepsilon_{13}^{\frac{n+2}{2(n-2)}} \left(\ln \varepsilon_{13}^{-1}\right)^{\frac{n+2}{2n}} \\
& + c \int_\Omega \underbrace{(\sup(\alpha_1 P\delta_1, \alpha_3 P\delta_3))^{\frac{6-n}{n-2}} \inf(\alpha_1 P\delta_1, \alpha_3 P\delta_3) v_1^2}_{\text{if } n \neq 6} + (c\mu + c\mu^{\frac{4}{n-2}}) \|v_1\|^2 \\
& + \frac{c\varepsilon \|v_1\|}{\lambda_1^{\frac{n-2}{2}}} (\text{if } n = 4, 5) + \frac{c\|v_1\|}{\lambda_1^4} (\ln \lambda_1)^{2/3} (\text{if } n = 6) + \frac{c\|v_1\|}{\lambda_1^{\frac{n+2}{2}}} (\text{if } n > 6).
\end{aligned}$$

Note that $v_1 \in E_1^{\perp}$, and Lemma 2.5 ($N = 2$) implies that there exists $\beta_0 > 0$ such that

$$\left(\beta_0 - c\mu - c\mu^{\frac{4}{n-2}} - c|A_1| - c|A_3| - c\varepsilon_{13}^{\frac{2}{(n-2)}}\left(\ln \varepsilon_{13}^{-1}\right)^{\frac{2}{n}}\right)\|v_1\|^2$$

$$\leq \left[c\varepsilon_{13}^{\frac{n+2}{2(n-2)}}\left(\ln \varepsilon_{13}^{-1}\right)^{\frac{n+2}{2n}} + \frac{c\varepsilon}{\lambda_1^{\frac{n-2}{2}}} \text{ (if } n = 4, 5) + \frac{c\left(\ln \lambda_1\right)^{2/3}}{\lambda_1^4} \text{ (if } n = 6) + \frac{c}{\lambda_1^{\frac{n+2}{2}}} \text{ (if } n > 6)\right]\|v_1\|,$$

where $A_i := 1 - \alpha_i^{4/n-2}$.

The proof follows by choosing μ such that $c\mu + c\mu^{\frac{4}{n-2}} + c|A_1| + c|A_3| + c\varepsilon_{13}^{\frac{2}{(n-2)}}\left(\ln \varepsilon_{13}^{-1}\right)^{\frac{2}{n}} \leq \frac{\beta_0}{2}$. □

Proposition 3.2 *Let $n \geq 4$. For all y in Ω, we have*

$$|v_1(y)| \leq C\Lambda(\delta_1(y) + \delta_3(y)),$$

where $\Lambda := |A_1| + |A_3| + \varepsilon + \frac{1}{\lambda_1} + \frac{1}{\lambda_3} + \varepsilon_{13}^{\frac{1}{(n-2)}}\left(\ln \varepsilon_{13}^{-1}\right)^{\frac{1}{n}}$.

Proof Note that v_1 satisfies (11); then

$$|v_1(y)| \leq \left|\int_\Omega f_1(v_1^*)G(x,y)dx\right| + \left|\sum_{k=1}^{2n+4}\left[\left(\int_\Omega f_1(v_1^*)\varphi_k(x)dx\right)(-\varphi_k(y))\right]\right| := |I_1| + |I_2|. \tag{13}$$

We now estimate I_1; we have

$$\begin{aligned}|I_1| \leq &\sum_{i=1,3}\int_\Omega \alpha_i|A_i|\delta_i^{\frac{n+2}{n-2}}(x)G(x,y)dx + c\sum_{i=1,3}\int_\Omega \theta_i(x)\delta_i^{\frac{4}{n-2}}(x)G(x,y)dx\\ &+c\int_\Omega \delta_1(x)\delta_3(x)^{\frac{4}{n-2}}G(x,y)dx + c\int_\Omega \delta_1(x)^{\frac{4}{n-2}}\delta_3(x)G(x,y)dx\\ &+c\mu^{\frac{n-4}{n-2}}\sum_{i=1,3}\int_\Omega \delta_i^{\frac{n}{n-2}}(x)|v_1(x)|^{\frac{2}{n-2}}G(x,y)dx + c\sum_{i=1,3}\varepsilon\int_\Omega \delta_i(x)G(x,y)dx\\ &+c\mu^{\frac{n-4}{n-2}}\int_\Omega\left(\delta_1^{\frac{4}{n-2}}(x)\delta_3^{\frac{n-4}{n-2}}(x) + \delta_1^{\frac{n-4}{n-2}}(x)\delta_3^{\frac{4}{n-2}}(x)\right)|v_1(x)|^{\frac{2}{n-2}}G(x,y)dx\end{aligned} \tag{14}$$

Note that by using Claim (c) of Lemma 2.7 and Lemma 2.2 we deduce that for $i \neq j$

$$\begin{aligned}\int_\Omega \delta_i(x)\delta_j^{\frac{4}{n-2}}(x)G(x,y)dx &= \int_\Omega \delta_i^{\frac{n-3}{n-2}}\delta_j^{\frac{3}{n-2}}(\delta_i\delta_j)^{\frac{1}{n-2}}G\\ &\leq \int_\Omega \delta_i^{\frac{n}{n-2}}(\delta_i\delta_j)^{\frac{1}{n-2}}G + \int_\Omega \delta_j^{\frac{n}{n-2}}(\delta_i\delta_j)^{\frac{1}{n-2}}G\\ &\leq \varepsilon_{ij}^{1/(n-2)}\left(\ln \varepsilon_{ij}^{-1}\right)^{1/n}(\delta_i(y) + \delta_j(y)).\end{aligned} \tag{15}$$

Now (14) and (15) imply that

$$\begin{aligned}|I_1| \leq &c|A_1|P\delta_1(y) + c|A_3|P\delta_3(y) + \frac{c}{\lambda_1}\delta_1(y) + \frac{c}{\lambda_3}\delta_3(y) + \varepsilon_{13}^{1/(n-2)}\left(\ln \varepsilon_{13}^{-1}\right)^{1/n}(\delta_1(y) + \delta_3(y))\\ &+ c\mu^{\frac{n-4}{n-2}}\|v_1\|^{2/(n-2)}(\delta_1(y) + \delta_3(y)) + c\varepsilon(\delta_1(y) + \delta_3(y)).\end{aligned} \tag{16}$$

For I_2, note that, for all $k \in \{1,...,2n+4\}$, $\varphi_k = \sum_{j=1}^{2n+4} C_j \overline{\varphi_j}$, where $\overline{\varphi_j}$ is one of the functions $P\delta_i, \lambda_i(\partial P\delta_i/\partial\lambda_i), (1/\lambda_i)(\partial P\delta_i/$ with $i = 1,3$. Since $\|\varphi_k\| = 1$ and $\|\overline{\varphi_j}\| \leq C$ (independent of ε), we derive that $|C_j| \leq c$ uniformly with respect to ε. Hence easy computation implies that $|\varphi_k(y)| \leq c(\delta_1(y)+\delta_3(y))$. Thus

$$|I_2| \leq cK(\delta_1(y)+\delta_3(y)) \quad \text{with } K := \left|\sum_{k=1}^{2n+4}\left(\int_\Omega f_1(v_1^*)\varphi_k(x)dx\right)\right|. \tag{17}$$

We also have

$$\begin{aligned}|f_1(w)| \leq &\alpha_1|A_1|\delta_1^{\frac{n+2}{n-2}} + c\theta_1\delta_1^{\frac{4}{n-2}} + \alpha_3|A_3|\delta_3^{\frac{n+2}{n-2}} + c\theta_3\delta_3^{\frac{4}{n-2}} + c\delta_1\delta_3^{\frac{4}{n-2}} + c\delta_1^{\frac{4}{n-2}}\delta_3\\ &+ c\delta_1^{\frac{4}{n-2}}|w| + c\delta_3^{\frac{4}{n-2}}|w| + c|w|^{\frac{n+2}{n-2}} + c\varepsilon\delta_1 + c\varepsilon\delta_3 + c\varepsilon|w|. \end{aligned}\tag{18}$$

(17) and (18) imply that

$$\begin{aligned}K \leq &c|A_1| + c|A_1|\int_\Omega \delta_1^{\frac{n+2}{n-2}}\delta_3 + c\int_\Omega \theta_1\delta_1^{\frac{n+2}{n-2}} + c|A_3| + c|A_3|\int_\Omega \delta_1\delta_3^{\frac{n+2}{n-2}}\\ &+ c\int_\Omega \theta_3\delta_3^{\frac{n+2}{n-2}} + c\int_\Omega \delta_1^2\delta_3^{\frac{4}{n-2}} + c\int_\Omega \delta_1\delta_3^{\frac{n+2}{n-2}} + c\int_\Omega \delta_1^{\frac{n+2}{n-2}}\delta_3 + c\int_\Omega \delta_1^{\frac{4}{n-2}}\delta_3^2\\ &+ c\int_\Omega \delta_1^{\frac{n+2}{n-2}}|v_1| + c\int_\Omega \delta_3^{\frac{n+2}{n-2}}|v_1| + c\varepsilon\int_\Omega \delta_1^2 + c\varepsilon\int_\Omega \delta_3^2.\end{aligned}$$

Using Lemmas 2.1, 2.2, and Hölder's inequality, easy computations imply that

$$|I_2| \leq c\left(|A_1| + |A_3| + \frac{1}{\lambda_1^2} \times \underbrace{\ln(\lambda_1)}_{\text{if } n=4} + \varepsilon_{13}^{\inf(1,4/(n-2))} \times \underbrace{\left(\ln\varepsilon_{13}^{-1}\right)^{2/3}}_{\text{if } n=6}\right)(\delta_1(y)+\delta_3(y)). \tag{19}$$

Proposition 3.1, (16), and (19) prove Proposition 3.2. □

Note that Λ, defined in Proposition 3.2, is a very small constant. Then we deduce that $|v_1(y)| \leq \mu(\delta_1(y)+\delta_3(y))$ for all $y \in \Omega$; hence we obtain

Corollary 3.3 *The function v_1, defined in (11), satisfies*

$$\begin{cases} -\triangle v_1 = f_1(v_1) - \sum_{k=1}^{2n+4}\left(\int_\Omega f_1(v_1)\varphi_k\right)(-\triangle\varphi_k) & \text{in } \Omega, \quad (20)\\ v_1 = 0 & \text{on } \partial\Omega. \quad (21)\end{cases}$$

Let

$$v_2 = v - v_1. \tag{22}$$

Since $v \in E_1^\perp$, $v_1 \in E_1^\perp$, $\|v\| = o(1)$ and $\|v_1\| = o(1)$ imply that

$$v_2 \in E_1^\perp \quad \text{and} \quad \|v_2\| = o(1). \tag{23}$$

However, the function v_2 is not orthogonal to $P\delta_2, \partial P\delta_2/\partial\lambda_2$ and $\partial P\delta_2/\partial a_2^k$.

The following proposition is crucial in our proof. In fact, it is an estimate of $\|v_2\|$ in which we stipulate that λ_1 and λ_3 do not appear alone. This piece of information will be very useful in Proposition 4.1.

Proposition 3.4 *For $n \geq 4$ there holds*

$$\|v_2\| = O(|A_2| + \widetilde{R}), \quad \textit{where}$$

$$\widetilde{R} := \begin{cases} \dfrac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \varepsilon_{23} + \dfrac{\varepsilon}{\lambda_2^{\frac{n-2}{2}}} & \textit{if } n = 4,5, \\ \dfrac{\big(\ln(\lambda_2 d_2)\big)^{2/3}}{(\lambda_2 d_2)^4} + \varepsilon_{12}\big(\ln \varepsilon_{12}^{-1}\big)^{2/3} + \varepsilon_{23}\big(\ln \varepsilon_{23}^{-1}\big)^{2/3} + \dfrac{\varepsilon}{\lambda_2^2}\big(\ln \lambda_2\big)^{2/3} & \textit{if } n = 6, \\ \dfrac{1}{(\lambda_2 d_2)^{\frac{n+2}{2}}} + \varepsilon_{12}^{\frac{n+2}{2(n-2)}}\big(\ln \varepsilon_{12}^{-1}\big)^{\frac{n-2}{2n}} + \varepsilon_{23}^{\frac{n+2}{2(n-2)}}\big(\ln \varepsilon_{23}^{-1}\big)^{\frac{n-2}{2n}} + \dfrac{\varepsilon}{\lambda_2^2} & \textit{if } n > 6. \end{cases}$$

Proof The idea of the proof is the following: At the beginning, we introduce the equation that verifies v_2 (see (24)). Then, by multiplying the obtained equation by v_2 and by integrating it on Ω, we get a piece of information on $\|v_2\|$ (see (27)). As v_2 is not necessarily in $F^{\perp}$, we need to decompose it into two functions: $v_2 = \overline{v}_2 + \underline{v}_2$, so that $\overline{v}_2 \in F^{\perp}$ and what follows, the obtained quadratic form in (27) becomes definite positive. Concerning $\underline{v}_2$, we prove that its norm is very small compared to certain well chosen terms.

From (8), Corollary 3.3, and (22) some computations imply that

$$\begin{aligned} -\triangle v_2 = {} & \alpha_2 \delta_2^{\frac{n+2}{n-2}} - (\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}(\alpha_1 P\delta_1)^{\frac{4}{n-2}} v_2 + \frac{n+2}{n-2}(\alpha_3 P\delta_3)^{\frac{4}{n-2}} v_2 \\ & + O\left((\delta_1\delta_3)^{\frac{2}{n-2}}|v_2|\right) + O\left(|v_1|^{\frac{4}{n-2}}|v_2|\right) + O_{(\text{if } n=4,5)}\left(\delta_1^{\frac{6-n}{n-2}}|v_1 v_2| + \delta_3^{\frac{6-n}{n-2}}|v_1 v_2|\right) \\ & + \frac{n+2}{n-2}(\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} v_2 + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}(\delta_i^{\frac{4}{n-2}}\delta_2 + \delta_i \delta_2^{\frac{4}{n-2}})\right) \\ & + O_{(\text{if } n\geq 6)}\left(\sum_{i=1,3}(\delta_i\delta_2)^{\frac{n+2}{2(n-2)}}\right) + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}(\delta_i^{\frac{6-n}{n-2}}\delta_2 + \delta_i\delta_2^{\frac{6-n}{n-2}})|v_2|\right) \\ & + O_{(\text{if } n\geq 6)}\left(\sum_{i=1,3}(\delta_i\delta_2)^{\frac{2}{n-2}}|v_2|\right) + O_{(\text{if } n=4,5)}\left(\sum_{i=1}^{3}\delta_i^{\frac{6-n}{n-2}} v_2^2\right) + O\left(|v_2|^{\frac{n+2}{n-2}}\right) \\ & + \varepsilon\alpha_2 P\delta_2 - \varepsilon v_2 + \sum_{k=1}^{2n+4}\left(\int_\Omega f_1(v_1)\varphi_k\right)(-\triangle\varphi_k). \end{aligned} \tag{24}$$

We multiply (24) by v_2, and we integrate it on Ω; by Proposition 3.2 and the fact that $v_2 \in E_1^{\perp}$, we get

$$\|v_2\|^2 - \frac{n+2}{n-2}\sum_{i=1}^{3}\int_\Omega P\delta_i^{\frac{4}{n-2}} v_2^2 \leq$$

$$\alpha_2|A_2|\langle P\delta_2, v_2\rangle + c\sum_{i=1}^{3}|A_i|\int_\Omega \delta_i^{\frac{4}{n-2}}v_2^2 + O\left(\int_\Omega \theta_2\delta_2^{\frac{4}{n-2}}|v_2|\right) + O\left(\int_\Omega (\delta_1\delta_3)^{\frac{2}{n-2}}v_2^2\right)$$

$$+ c\Lambda\sum_{i=1,3}\int_\Omega \delta_i^{\frac{4}{n-2}}v_2^2 (\text{if } n=4,5) + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}\int_\Omega (\delta_i^{\frac{4}{n-2}}\delta_2 + \delta_i\delta_2^{\frac{4}{n-2}})|v_2|\right)$$

$$O_{(\text{if } n\geq 6)}\left(\sum_{i=1,3}\int_\Omega (\delta_i\delta_2)^{\frac{n+2}{2(n-2)}}|v_2|\right) + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}\int_\Omega (\delta_i^{\frac{6-n}{n-2}}\delta_2 v_2^2 + \delta_i\delta_2^{\frac{6-n}{n-2}}v_2^2)\right)$$

$$+ O_{(\text{if } n\geq 6)}\left(\sum_{i=1,3}\int_\Omega (\delta_i\delta_2)^{\frac{2}{n-2}}v_2^2\right) + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}\int_\Omega \delta_i^{\frac{6-n}{n-2}}v_2^3\right) + O\left(\int_\Omega |v_2|^{\frac{2n}{n-2}}\right)$$

$$+ c\|v_1\|^{\frac{4}{n-2}}\|v_2\|^2 + c\varepsilon\int_\Omega P\delta_2 v_2 + c\varepsilon\|v_2\|^2. \tag{25}$$

Observe that

$$|\langle P\delta_2, v_2\rangle| = |\langle P\delta_2, v\rangle - \langle P\delta_2, v_1\rangle| = \left|\int_\Omega \triangle(P\delta_2)v_1\right| \leq \Lambda(\varepsilon_{12}+\varepsilon_{23}). \tag{26}$$

(25), (26), Lemma 2.3, and Hölder's inequality imply that

$$\|v_2\|^2 - \frac{n+2}{n-2}\sum_{i=1}^{3}\int_\Omega P\delta_i^{\frac{4}{n-2}}v_2^2 \leq c|A_2|\Lambda(\varepsilon_{12}+\varepsilon_{23}) + c\widetilde{R}\|v_2\| + o(\|v_2\|^2), \tag{27}$$

where $\widetilde{R}$ is defined in the statement of the proposition. Let

$$v_2 := \underline{v}_2 + \overline{v}_2, \quad \text{where} \quad \underline{v}_2 \in F \text{ and } \overline{v}_2 \in F^\perp. \tag{28}$$

Before continuing to prove the proposition, we introduce the following lemma (whose proof will be presented at the end of this section).

Lemma 3.5 *The function $\underline{v}_2$ defined in (28) satisfies*

$$\|\underline{v}_2\| = O\left(\Lambda(\varepsilon_{12}+\varepsilon_{23})\right),$$

where Λ is defined in Proposition 3.2.

By Hölder's inequality we have

$$\int_\Omega P\delta_i^{\frac{4}{n-2}}v_2^2 = \int_\Omega P\delta_i^{\frac{4}{n-2}}(\overline{v}_2^2 + \underline{v}_2^2 + 2\underline{v}_2\overline{v}_2) = \int_\Omega P\delta_i^{\frac{4}{n-2}}\overline{v}_2^2 + O\left(\|\underline{v}_2\|^2 + \|\underline{v}_2\|\|\overline{v}_2\|\right). \tag{29}$$

(26) and (29) imply that

$$\begin{aligned}\|\overline{v}_2\|^2 - \frac{n+2}{n-2}\sum_{i=1}^{3}\int_\Omega P\delta_i^{\frac{4}{n-2}}\overline{v}_2^2 \\ \leq c\|\underline{v}_2\|^2 + c\|\underline{v}_2\|\|\overline{v}_2\| + c|A_2|\Lambda(\varepsilon_{12}+\varepsilon_{23}) + \widetilde{R}\left(\|\overline{v}_2\| + \|\underline{v}_2\|\right) + o\left(\|\overline{v}_2\|^2\right),\end{aligned} \tag{30}$$

where $\widetilde{R}$ is defined in Proposition 3.4.

Note that $\overline{v}_2 \in F^{\perp}$; then Lemma 2.5 ($N=3$) and (30) imply that there exists $\beta_0 > 0$ such that

$$(\beta_0 + o(1))\,\|\overline{v}_2\|^2 \le \gamma_1 \|\overline{v}_2\| + \gamma_2, \qquad \text{where} \tag{31}$$

$$\gamma_1 := c\|\underline{v}_2\| + \widetilde{R} \qquad \text{and} \qquad \gamma_2 := c\|\underline{v}_2\|^2 + c|A_2|\Lambda(\varepsilon_{12} + \varepsilon_{23}) + \widetilde{R}\|\underline{v}_2\|.$$

Thus, we obtain

$$\|\overline{v}_2\| \le c\gamma_1 + c\sqrt{\gamma_2}. \tag{32}$$

Lemma 3.5 and the fact that $\|v_2\| \le \|\overline{v}_2\| + \|\underline{v}_2\|$ complete the proof of Proposition 3.4. □

We note that A_2 appears in the estimation of $\|v_2\|$. However, the estimate of A_2, proved in [12] Proposition 1.2.2, is a defect. In what follows, we suggest improving this estimate by using the information already found.

Proposition 3.6 *The variable $A_2 := 1 - \alpha_2^{4/(n-2)}$ satisfies*

$$|A_2| = \begin{cases} O\left(\dfrac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \varepsilon_{23} + \dfrac{\varepsilon}{\lambda_2^2}(1 + \underbrace{\ln \lambda_2}_{if\ n=4})\right) & \textit{if } n \le 6, \\ O\left(\dfrac{1}{(\lambda_2 d_2)^{\inf(n-2, \frac{(n+2)^2}{2(n-2)})}} + \varepsilon_{12}^{\inf(1, \frac{(n+2)^2}{2(n-2)^2})} + \varepsilon_{23}^{\inf(1, \frac{(n+2)^2}{2(n-2)^2})} + \dfrac{\varepsilon}{\lambda_2^2}\right) & \textit{if } n > 6. \end{cases}$$

Proof We multiply (1) by $P\delta_2$; then we integrate it on Ω and we obtain

$$\begin{aligned} &\int_\Omega -\triangle(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v)P\delta_2 \\ &= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v)P\delta_2 \\ &- \varepsilon\alpha_1 \int_\Omega P\delta_1 P\delta_2 + \varepsilon\alpha_2 \int_\Omega P\delta_2^2 - \varepsilon\alpha_3 \int_\Omega P\delta_3 P\delta_2 - \varepsilon \int_\Omega P\delta_2 v. \end{aligned} \tag{33}$$

Lemmas 2.1 and 2.2 imply that the integral of the left-hand side of (33) is equal to

$$\begin{aligned} &\alpha_1 \int_\Omega \delta_1^{\frac{n+2}{n-2}} P\delta_2 - \alpha_2 \int_\Omega \delta_2^{\frac{2n}{n-2}} + \alpha_2 \int_\Omega \delta_2^{\frac{n+2}{n-2}}\theta_2 + \alpha_3 \int_\Omega \delta_3^{\frac{n+2}{n-2}} P\delta_2 \\ &= -\alpha_2 S + O\left(\varepsilon_{12} + \varepsilon_{23} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right). \end{aligned} \tag{34}$$

Lemmas 2.1, 2.2, and the fact that $|v_1(x)| \le c(\delta_1(x) + \delta_3(x)), \forall x \in \Omega,$ imply that the first integral of the right-hand side of (33) is equal to

$$
\begin{aligned}
I_1 = & \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3) P\delta_2 \\
& + \frac{n+2}{n-2}\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{\frac{4}{n-2}} vP\delta_2 \\
& + O_{(\text{if } n=4,5)}\left(\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{\frac{6-n}{n-2}} v^2 P\delta_2\right) + O\left(\int_\Omega P\delta_2 |v|^{\frac{n+2}{n-2}}\right) \\
= & -\alpha_2^{\frac{n+2}{n-2}}\int_\Omega P\delta_2^{\frac{2n}{n-2}} + O\left(\sum_{i=1,3}\int_\Omega (\delta_i^{\frac{n+2}{n-2}}\delta_2 + \delta_i\delta_2^{\frac{n+2}{n-2}})\right) + \frac{n+2}{n-2}\alpha_2^{\frac{4}{n-2}}\int_\Omega P\delta_2^{\frac{n+2}{n-2}} v \\
& + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}\int_\Omega (\delta_i\delta_2^{\frac{4}{n-2}} + \delta_i^{\frac{4}{n-2}}\delta_2)|v|\right) + O_{(\text{if } n\geq 6)}\left(\sum_{i=1,3}\int_\Omega (\delta_i\delta_2)^{\frac{n+2}{2(n-2)}}|v|\right) \\
& + O_{(\text{if } n=4,5)}\left(\sum_{i=1,3}\int_\Omega \delta_i^{\frac{6-n}{n-2}}\delta_2 v^2\right) + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}} v^2 + \int_\Omega \delta_2 |v|^{\frac{n+2}{n-2}}\right)
\end{aligned}
$$

Using the fact that $v = v_1 + v_2$, Proposition 3.2 and Lemmas 2.1 (claim (c)), 2.2, 2.3 (Claims (b) and (c)), we get

$$
I_1 = -\alpha_2^{\frac{n+2}{n-2}} S + O\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \varepsilon_{23}\right) + \frac{n+2}{n-2}\alpha_2^{\frac{4}{n-2}}\langle P\delta_2, v_2\rangle + c\|v_2\|^2 + c\|v_2\|^{\frac{n+2}{n-2}} \tag{35}
$$

$$
+c\|v_2\| \times \begin{cases} \dfrac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \varepsilon_{23} & \text{if } n = 4,5, \\ \dfrac{\big(\ln(\lambda_2 d_2)\big)^{2/3}}{(\lambda_2 d_2)^4} + \varepsilon_{12}\big(\ln \varepsilon_{12}^{-1}\big)^{2/3} + \varepsilon_{23}\big(\ln \varepsilon_{23}^{-1}\big)^{2/3} & \text{if } n = 6, \\ \dfrac{1}{(\lambda_2 d_2)^{\frac{n+2}{2}}} + \varepsilon_{12}^{\frac{n+2}{2(n-2)}}\big(\ln \varepsilon_{12}^{-1}\big)^{\frac{n-2}{2n}} + \varepsilon_{23}^{\frac{n+2}{2(n-2)}}\big(\ln \varepsilon_{23}^{-1}\big)^{\frac{n-2}{2n}} & \text{if } n > 6. \end{cases}
$$

Finally, Hölder's inequality (by decomposing v into $v_1 + v_2$ and by using the fact that $|v_1| \leq c(\delta_1 + \delta_3)$), Lemma 2.3, and (26), (34), (35) and easy computations imply the result. □

Proof of Lemma 3.5: Let $(\varphi_1, ..., \varphi_{2n+4})$ and $(\psi_1, ..., \psi_{n+2})$ be two orthonormal bases of E_1 and E_2 respectively, where E_1 is defined in (10) and $E_2 := \left\langle P\delta_2, \lambda_2\frac{\partial P\delta_2}{\partial \lambda_2}, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^1}, .,.,., \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^n}\right\rangle$. Thus the function $\underline{v}_2$ can be written as

$$
\underline{v}_2 = \sum_{k=1}^{2n+4}\nu_k\varphi_k + \sum_{k=1}^{n+2}\mu_k\psi_k.
$$

Note that, for all $k \in \{1, ..., n+2\}$, we claim that

$$
\mu_k = O\left(\Lambda(\varepsilon_{12} + \varepsilon_{23})\right) \quad \text{and} \quad \nu_k = O\left(\Lambda(\varepsilon_{12}^2 + \varepsilon_{23}^2)\right). \tag{36}
$$

In fact, for all $\psi \in E_2$, we have

$$
\langle \psi, v_2\rangle = \langle \psi, v - v_1\rangle = -\langle \psi, v_1\rangle = \int_\Omega \triangle\psi v_1. \tag{37}
$$

Moreover, an easy computation shows that $|\triangle\psi| \leq \|\psi\|\delta_2^{(n+2)/(n-2)}$ for every $\psi \in E_2$. Hence by Proposition 3.2 and Lemma 2.2 (Claim (a)) we deduce that for all $k \in \{1, ..., n+2\}$, we have $\langle\psi_k, v_2\rangle = O(\Lambda(\varepsilon_{12} + \varepsilon_{23}))$. Note that $\overline{v}_2 \in F^\perp$; then

$$\langle\psi_k, v_2\rangle = \langle\psi_k, \underline{v}_2\rangle = \mu_k + \sum_{i=1}^{2n+4} \nu_i \langle\varphi_i, \psi_k\rangle,$$

which implies that

$$\mu_k = -\sum_{i=1}^{2n+4} \nu_i \langle\varphi_i, \psi_k\rangle + O(\Lambda(\varepsilon_{12} + \varepsilon_{23})) = \sum_{i=1}^{2n+4} o(\nu_i) + O(\Lambda(\varepsilon_{12} + \varepsilon_{23})). \tag{38}$$

Since $v_2 \in E_1^\perp$ and $\overline{v}_2 \in F^\perp$, then for all $k \in \{1, ..., 2n+4\}$ we have

$$\langle\varphi_k, v_2\rangle = \sum_{i=1}^{2n+4} \nu_i \langle\varphi_i, \varphi_k\rangle + \sum_{i=1}^{n+2} \mu_i \langle\psi_i, \varphi_k\rangle = 0. \tag{39}$$

From (38) and (39), we deduce that for all $k \in \{1, ..., 2n+4\}$, and we have

$$\nu_k = \sum_{i=1}^{n+2} O(\mu_i(\varepsilon_{12} + \varepsilon_{23})) = \sum_{i=1}^{n+2} o(\mu_i), \tag{40}$$

and from (38) and (40), we deduce that for all $k \in \{1, ..., n+2\}$ we have

$$\mu_k = \sum_{j=1}^{2n+4} o(\mu_j) + O(\Lambda(\varepsilon_{12} + \varepsilon_{23})). \tag{41}$$

Thus, we obtain the following linear system: $AX = B$ on the variables $\mu_1, ..., \mu_{n+2}$,
where $A = (m_{ij})_{1\leq i,j\leq 2n+4}$ $m_{ii} = 1 + o(1)$ and $m_{ij} = o(1)$ for $i \neq j$, $X := (\mu_1, ..., \mu_{2n+4})^t$ and $B := (O(\Lambda(\varepsilon_{12} + \varepsilon_{23})), ..., O(\Lambda(\varepsilon_{12} + \varepsilon_{23})))^t$. Thus for all $k \in \{1, ..., 2n+4\}$, we have $\mu_k = O(\Lambda(\varepsilon_{12} + \varepsilon_{23}))$, by (40), we deduce that $\nu_k = O\left(\Lambda(\varepsilon_{12}^2 + \varepsilon_{23}^2)\right)$ and our claim follows.

Finally (36) implies that

$$\|\underline{v}_2\|^2 = c\sum_{k=1}^{2n+4} \nu_k^2 + c\sum_{k=1}^{n+2} \mu_k^2 + c\sum_{j,k} \nu_j \mu_k \langle\varphi_j, \psi_k\rangle = O\left(\Lambda^2(\varepsilon_{12} + \varepsilon_{23})^2\right). \tag{42}$$

The proof of the lemma follows. □

4. Improvement of Proposition 3.3

In the case $(\lambda_1 \simeq \lambda_3 \ll \lambda_2)$, the remaining of proposition 2.11 is inadequate for our situation, and we suggest using, in this section, the results already acquired in order to improve them. The new proposition will serve us, in the fifth section, to prove Theorem 1.1.

Proposition 4.1 *Let u_ε be a solution of $(P_{-\varepsilon})$ under the form (7) such that (5) is satisfied and $\lambda_1 \simeq \lambda_3 \ll \lambda_2$; then for all $n \geq 4$, we have*

$$\frac{n-2}{2}\widetilde{c}\frac{H_{22}}{\lambda_2^{n-2}} - \widetilde{c}\left(\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - \widetilde{c}\left(\lambda_2\frac{\partial \varepsilon_{23}}{\partial\lambda_2} + \frac{n-2}{2}\frac{H_{23}}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\right)$$
$$+ \frac{C_2\varepsilon}{\lambda_2^2}(\textit{if } n \geq 5) + C_3\frac{\varepsilon\ln(\lambda_2 d_2)}{\lambda_2^2}(\textit{if } n = 4)$$
$$= o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2} + d_2\varepsilon_{12} + d_2\varepsilon_{23} + \varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) + \varepsilon_{23}^{\frac{n}{n-2}}\ln(\varepsilon_{23}^{-1})\right),$$

where $\widetilde{c}$ is defined in Lemma 2.8, C_2 and C_3 are defined in Proposition 2.11.

Proof We multiply the equation (1) by $\varphi_2 := \lambda_2\partial P\delta_2/\partial\lambda_2$, and we get

$$\alpha_1\langle P\delta_1,\varphi_2\rangle - \alpha_2\langle P\delta_2,\varphi_2\rangle + \alpha_3\langle P\delta_3,\varphi_2\rangle$$
$$= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v)\varphi_2$$
$$- \varepsilon\alpha_1\int_\Omega P\delta_1\varphi_2 + \varepsilon\alpha_2\int_\Omega P\delta_2\varphi_2 - \varepsilon\alpha_3\int_\Omega P\delta_3\varphi_2 - \varepsilon\int_\Omega v\varphi_2. \tag{43}$$

Let $\Omega_1 := \{x \in \Omega,\ \alpha_1 P\delta_1(x) + \alpha_3 P\delta_3(x) \leq \alpha_2 P\delta_2(x)\}$, and the first integral of the right-hand side of (43) is equal to

$$I = \int_\Omega(\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{n+2}{n-2}}\varphi_2 - \int_\Omega(\alpha_2 P\delta_2)^{\frac{n+2}{n-2}}\varphi_2 + \frac{n+2}{n-2}\int_\Omega(\alpha_1 P\delta_1 + \alpha_3 P\delta_3)(\alpha_2 P\delta_2)^{\frac{4}{n-2}}\varphi_2$$
$$+ O\left(\int_{\Omega_1}(\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^2(\alpha_2 P\delta_2)^{\frac{4}{n-2}}\right) + O\left(\int_{\Omega_1^c}(\alpha_1 P\delta_1 + \alpha_3 P\delta_3)^{\frac{4}{n-2}}(\alpha_2 P\delta_2)^2\right)$$
$$+ O\left(\int_{\Omega_1^c}(\delta_1^{\frac{4}{n-2}} + \delta_3^{\frac{4}{n-2}})|v||\varphi_2|\right) + \frac{n+2}{n-2}\int_\Omega(\alpha_2 P\delta_2)^{\frac{4}{n-2}}v\varphi_2$$
$$+ O\left(\int_{\Omega_1}(\alpha_1 P\delta_1 + \alpha_3 P\delta_3)P\delta_2^{\frac{4}{n-2}}|v|\right) + O\left(\int_{|\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3| \leq 2|v|}|v|^{\frac{n+2}{n-2}}P\delta_2\right)$$
$$+ O\left(\int_{2|v| \leq |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|}|\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{\frac{6-n}{n-2}}P\delta_2 v^2\right)$$
$$:= I_1 - I_2 + I_3 + O(I_4 + I_5 + I_6) + I_7 + O(I_8 + I_9 + I_{10}). \tag{44}$$

For I_1 we have

$$I_1 = \sum_{i=1,3}\int_\Omega(\alpha_i P\delta_i)^{\frac{n+2}{n-2}}\varphi_2 + O\left(\int_\Omega P\delta_1^{\frac{4}{n-2}}P\delta_3|\varphi_2| + \int_\Omega P\delta_1 P\delta_3^{\frac{4}{n-2}}|\varphi_2|\right)$$
$$= \sum_{i=1,3}\alpha_i^{\frac{n+2}{n-2}}\langle P\delta_i,\varphi_2\rangle + O\left(\sum_{i=1,3}\int_\Omega \delta_i^{\frac{4}{n-2}}\theta_i P\delta_2\right) + O\left(\int_\Omega(P\delta_1^{\frac{4}{n-2}}P\delta_3 + P\delta_1 P\delta_3^{\frac{4}{n-2}})|\varphi_2|\right).$$

Note that, in view of (5), we have $d_i \geq c > 0$ for $i = 1, 3$; then, if $d_2 \nrightarrow 0$ we have

$$\int_\Omega \delta_i^{\frac{4}{n-2}} \theta_i P\delta_2 \leq \|\theta_i\|_\infty^{\min(1,4/(n-2))} \int_\Omega \delta_i^{\max(1,4/(n-2))} P\delta_2 \leq \frac{c}{\lambda_i^{\min(2,(n-2)/2)}} \varepsilon_{i2} = o(d_2 \varepsilon_{i2}).$$

and if $d_2 \to 0$, then $|a_i - a_2| \geq c > 0$ (for $i = 1, 3$); hence Lemma 2.14 implies that

$$\int_\Omega \delta_i^{\frac{4}{n-2}} \theta_i P\delta_2 \leq \|\theta_i\|_\infty^{\min(1,4/(n-2))} \int_\Omega \delta_i^{\max(1,4/(n-2))} P\delta_2$$
$$\leq \frac{c\varepsilon_{i2}}{\lambda_i^{\min(2,(n-2)/2)}} (d_2 + \frac{1}{(\lambda_2 d_2)^2}) = o\left(d_2 \varepsilon_{i2} + \varepsilon_{i2}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right).$$

On the other hand, we have

$$\int_\Omega P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| \leq \int_{\Omega \setminus B_{(a_1, |a_1 - a_3|/4)}} P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| + \int_{\Omega \setminus B_{(a_3, |a_1 - a_3|/4)}} P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| \tag{45}$$

Concerning the first integral of the right-hand side of (45), we have

$$\int_{\Omega \setminus B_{(a_1, |a_1 - a_3|/4)}} P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| = o\left(d_2 \varepsilon_{12} + d_2 \varepsilon_{23} + \varepsilon_{12}^{\frac{n}{n-2}} + \varepsilon_{23}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right),$$

(by using Lemma 2.2 if $d_2 \nrightarrow 0$ and Lemma 2.14 if $d_2 \to 0$). Concerning the second integral of the right-hand side of (45), two cases may occur:

The first case: if $n \leq 6$, then $4/(n-2) \geq 1$; hence

$$\int_{\Omega \setminus B_{(a_3, |a_1 - a_3|/4)}} P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| \leq \frac{1}{\lambda_3^{\frac{n-2}{2}}} \int_\Omega \delta_1^{\frac{4}{n-2}} P\delta_2 = o\left(d_2 \varepsilon_{12} + \varepsilon_{12}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^n}\right), \tag{46}$$

(by using Lemma 2.2 if $d_2 \nrightarrow 0$ and Lemma 2.14 if $d_2 \to 0$).

The second case: if $n \geq 7$, then $4/(n-2) < 1$. Note that, in this case, the last integral (46) will be $O(\varepsilon_{12}^{4/(n-2)})$. The power of ε_{12} (which is $4/(n-2) < 1$) is not adequate for our situation. Hence, to overcome this obstacle, we will need to preserve $P\delta_3^{(n-6)/(n-2)}$ in the integral and we will use $P\delta_1^{4/(n-2)} P\delta_3^{(n-6)/(n-2)} \leq \delta_1 + \delta_3$. The new power of δ_i (which is 1) will allow us to apply Lemma 2.14. In fact,

$$\int_{\Omega \setminus B_{(a_3, |a_1 - a_3|/4)}} P\delta_1^{\frac{4}{n-2}} P\delta_3 |\varphi_2| \leq \frac{1}{\lambda_3^2} \int_\Omega (\delta_1 + \delta_3) P\delta_2$$
$$= o\left(d_2 \varepsilon_{12} + d_2 \varepsilon_{23} + \varepsilon_{12}^{\frac{n}{n-2}} + \varepsilon_{23}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^n}\right),$$

(by using Lemma 2.2 if $d_2 \nrightarrow 0$ and Lemma 2.14 if $d_2 \to 0$). Thus

$$I_1 = \sum_{i=1,3} \alpha_i^{\frac{n+2}{n-2}} \langle P\delta_i, \varphi_2 \rangle + o\left(d_2 \varepsilon_{12} + d_2 \varepsilon_{23} + \varepsilon_{12}^{\frac{n}{n-2}} + \varepsilon_{23}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right). \tag{47}$$

For I_2, Lemma 2.8 implies that

$$I_2 = 2\alpha_2^{\frac{n+2}{n-2}} \langle P\delta_2, \varphi_2 \rangle + o\left(\frac{1}{(\lambda_2 d_2)^{n-2}}\right), \tag{48}$$

and Lemma 2.9 implies that

$$I_3 = \alpha_1 \alpha_2^{\frac{4}{n-2}} \langle P\delta_1, \varphi_2 \rangle + \alpha_3 \alpha_2^{\frac{4}{n-2}} \langle P\delta_3, \varphi_2 \rangle + o\left(\varepsilon_{12}^{\frac{n}{n-2}} \ln(\varepsilon_{12}^{-1}) + \varepsilon_{23}^{\frac{n}{n-2}} \ln(\varepsilon_{23}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}}\right). \tag{49}$$

Concerning I_4 and I_5, Lemma 2.2 implies that

$$I_k = o\left(\varepsilon_{12}^{\frac{n}{n-2}} \ln(\varepsilon_{12}^{-1}) + \varepsilon_{23}^{\frac{n}{n-2}} \ln(\varepsilon_{23}^{-1})\right), \quad k = 4, 5. \tag{50}$$

Note that $v := v_1 + v_2$. Proposition 3.2 implies that

$$I_6 \leq c \sum_{i=1,3} c\Lambda \langle P\delta_i, P\delta_2 \rangle + c \int_{\Omega_1^c} (\delta_1 + \delta_3)^{\frac{4}{n-2}} P\delta_2 |v_2|. \tag{51}$$

Note that $\forall i \in \{1, 3\}$ we have

$$\langle P\delta_i, P\delta_2 \rangle = O\left(d_2 \varepsilon_{i2} + \varepsilon_{i2}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^n}\right). \tag{52}$$

(by using Lemma 2.2 if $d_2 \nrightarrow 0$ and Lemma 2.14 if $d_2 \to 0$).

Concerning the second integral of the right-hand side of (51), three cases may occur:

The first case: if $n \leq 5$, then

$$\int_\Omega \sum_{i=1,3} \delta_i^{\frac{4}{n-2}} \delta_2 |v_2| \leq c \|v_2\| \sum_{i=1,3} \varepsilon_{i2} = O\left(\|v_2\|^2\right) + o\left(\sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right).$$

The second case: if $n = 6$, then

$$\int_\Omega \sum_{i=1,3} \delta_i^{\frac{4}{n-2}} \delta_2 |v_2| \leq c \|v_2\| \sum_{i=1,3} \varepsilon_{i2} \left(\ln \varepsilon_{i2}^{-1}\right)^{2/3} = o\left(\|v_2\|^2 + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right).$$

The third case: if $n \geq 7$, then $4/(n-2) < 1$; hence

$$\int_{\Omega_1^c} \sum_{i=1,3} \delta_i^{\frac{4}{n-2}} \delta_2 |v_2| \leq \int_\Omega \sum_{i=1,3} \delta_i^{\frac{n+2}{2(n-2)}} \delta_2^{\frac{n+2}{2(n-2)}} |v_2| = O\left(\|v_2\|^2\right) + o\left(\sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right).$$

Thus, $\forall n \geq 4$ we have

$$I_6 = o\left(\sum_{i=1,3} d_2 \varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O\left(\|v_2\|^2\right). \tag{53}$$

By using the fact that $v \in F^{\perp}$, we deduce that

$$I_7 \le c\Lambda \int_\Omega \sum_{i=1,3} \delta_i \delta_2^{\frac{4}{n-2}} \left(\theta_2 + \left|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right|\right) + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}} |v_2| \left(\theta_2 + \left|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right|\right)\right)$$
$$+ O\left(\int_{\Omega_1^c} \Lambda \sum_{i=1,3} (\delta_i\delta_2)^{\frac{n}{n-2}}\right) + O\left(\int_{\Omega_1^c} \sum_{i=1,3} (\delta_i\delta_2)^{\frac{n+2}{2(n-2)}} |v_2|\right)$$
$$:= J_1 + O(J_2 + J_3 + J_4).$$

The estimate of J_3 is done in Lemma 2.2 (Claim (b)). Concerning J_1, two cases may occur:

The first case: if $n \le 6$, then $4/(n-2) \ge 1$. Using Hölder's inequality and Lemma 2.2, we get

$$J_1 \le o(1) \sum_{i=1,3} \varepsilon_{i2} \times \underbrace{\left(\ln \varepsilon_{i2}^{-1}\right)^{2/3}}_{\text{if } n=6} \left(\|\theta_2\|_{L^{\frac{2n}{n-2}}} + \left\|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right\|_{L^{\frac{2n}{n-2}}}\right) = o\left(\sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right).$$

The second case: if $n \ge 7$, then $4/(n-2) < 1$. Using the fact that $\left(\theta_2 + \left|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right|\right)^{(n-6)/(n-2)} \le c\delta_2$ and arguing as in the previous case, we get

$$J_1 \le o(1) \sum_{i=1,3} \varepsilon_{i2} \left(\|\theta_2\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}} + \left\|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}}\right) = o\left(\varepsilon_{12}^{\frac{n}{n-2}} + \varepsilon_{23}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^n}\right).$$

Concerning the other integrals, using Hölder's inequality, we have

$$J_2 \le c\|v_2\| \left(\|\theta_2\| + \left\|\lambda_2 \frac{\partial \theta_2}{\partial \lambda_2}\right\|\right) \le \frac{c\|v_2\|}{(\lambda_2 d_2)^{\frac{n-2}{2}}},$$

$$J_4 \le c \sum_{i=1,3} \|v_2\| \varepsilon_{i2}^{\frac{n+2}{2(n-2)}} \left(\ln \varepsilon_{i2}^{-1}\right)^{\frac{n+2}{2n}} = o\left(\|v_2\|^{\frac{2(n-1)}{n}} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right).$$

Thus,

$$I_7 = o\left(\varepsilon_{12}^{\frac{n}{n-2}} \ln(\varepsilon_{12}^{-1}) + \varepsilon_{23}^{\frac{n}{n-2}} \ln(\varepsilon_{23}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O\left(\|v_2\|^{\frac{2(n-1)}{n}} + \frac{\|v_2\|}{(\lambda_2 d_2)^{\frac{n-2}{2}}}\right). \tag{54}$$

For I_8, by using Proposition 3.2 we get that

$$I_8 \le c\Lambda \int_{\Omega_1} \sum_{i=1,3} (\delta_i \delta_2)^{\frac{n}{n-2}} + c\int_{\Omega_1} (\alpha_1 P\delta_1 + \alpha_3 P\delta_3) P\delta_2^{\frac{4}{n-2}} |v_2|. \tag{55}$$

The estimate of the first integral of (55) follows from Lemma 2.2 (Claim (b)) and using $\Lambda = o(1)$. Concerning the second integral of (55), two cases may occur:

The first case: if $n \le 6$, then $4/(n-2) \ge 1$; hence

$$\int_{\Omega_1} (\alpha_1 P\delta_1 + \alpha_3 P\delta_3) P\delta_2^{\frac{4}{n-2}} |v_2| = o\left(\|v_2\|^2 + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right).$$

The second case: if $n \geq 7$, then $4/(n-2) < 1$, hence

$$\int_{\Omega_1} (\alpha_1 P\delta_1 + \alpha_3 P\delta_3) P\delta_2^{\frac{4}{n-2}} |v_2| \leq \sum_{i=1,3} \int_{\Omega} (\delta_i \delta_2)^{\frac{n+2}{2(n-2)}} |v_2| = o\left(\|v_2\|^{\frac{2(n-1)}{n}} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) \right).$$

Thus

$$I_8 = o\left(\sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) \right) + O_{(\text{if } n\leq 6)}\left(\|v_2\|^2\right) + O_{(\text{if } n\geq 7)}\left(\|v_2\|^{\frac{2(n-1)}{n}}\right). \tag{56}$$

For I_9, to simplify the presentation, we denote by

$$\Gamma_1 := \{x \in \Omega \text{ s.t. } \alpha_2 P\delta_2(x) \leq 4(\alpha_1 P\delta_1(x) + \alpha_3 P\delta_3(x))\}, \qquad \Gamma_2 := \Omega \setminus \Gamma_1,$$
$$\omega_1 := \{x \in \Omega \text{ s.t. } 2|v(x)| \leq |\alpha_1 P\delta_1(x) - \alpha_2 P\delta_2(x) + \alpha_3 P\delta_3(x)|\}, \qquad \omega_2 := \Omega \setminus \omega_1,$$
$$D_1 := \{x \in \Omega \text{ s.t. } |v(x)| \leq 4|v_1(x)|\}, \qquad D_2 := \Omega \setminus D_1.$$

Thus we have

$$I_9 \leq \int_{\omega_2 \cap D_1} |v|^{\frac{n+2}{n-2}} P\delta_2 + \int_{\omega_2 \cap D_2} |v|^{\frac{n+2}{n-2}} P\delta_2 := J_5 + J_6.$$

Concerning J_5, (52) and Proposition 3.2 imply that

$$J_5 \leq c \int_{\Omega} |v_1|^{\frac{n+2}{n-2}} P\delta_2 \leq c\Lambda^{\frac{n+2}{n-2}} \sum_{i=1,3} \langle P\delta_i, P\delta_2 \rangle = o\left(\sum_{i=1,3} d_2 \varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}} \right).$$

Concerning J_6 we have

$$J_6 = \int_{\omega_2 \cap D_2 \cap \Gamma_1} |v|^{\frac{n+2}{n-2}} P\delta_2 + \int_{\omega_2 \cap D_2 \cap \Gamma_2} |v|^{\frac{n+2}{n-2}} P\delta_2 \tag{57}$$

Note that $\forall x \in D_2$ we have $|v| \leq c|v_2|$ and $\forall x \in \Gamma_1$ we have $P\delta_2 \leq c \sum_{i=1,3} (\delta_i \delta_2)^{1/2}$; then

$$\int_{\omega_2 \cap D_2 \cap \Gamma_1} |v|^{\frac{n+2}{n-2}} P\delta_2 \leq c \sum_{i=1,3} \int_{\Omega} |v_2|^{\frac{n+2}{n-2}} (\delta_i \delta_2)^{1/2} = o\left(\sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) \right) + O(\|v_2\|^2).$$

Concerning the second integral of (57), note that $\forall x \in \omega_2 \cap D_2 \cap \Gamma_2$ we have
$\alpha_2 P\delta_2 \leq (4/3)|\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3| \leq (8/3)|v| \leq c|v_2|$, then

$$\int_{\omega_2 \cap D_2 \cap \Gamma_2} |v|^{\frac{n+2}{n-2}} P\delta_2 \leq c \int_{\Omega} |v_2|^{\frac{2n}{n-2}} = o(\|v_2\|^2).$$

Thus

$$I_9 = o\left(\sum_{i=1,3} d_2 \varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}} \right) + O(\|v_2\|^2). \tag{58}$$

For I_{10}, three cases may occur:

The first case: if $n=4,5,$ then $(6-n)/(n-2)\geq 0$; hence by using (52), we get

$$
\begin{aligned}
I_{10} &\leq c\int_\Omega(\alpha_1P\delta_1+\alpha_3P\delta_3)^{\frac{6-n}{n-2}}P\delta_2v^2+c\int_\Omega P\delta_2^{\frac{4}{n-2}}v^2\\
&\leq c\Lambda^2\int_\Omega\sum_{i=1,3}\delta_i^{\frac{n+2}{n-2}}P\delta_2+c\int_\Omega\sum_{i=1,3}\delta_i^{\frac{6-n}{n-2}}P\delta_2|v_2|^2+c\Lambda^2\int_\Omega\sum_{i=1,3}\delta_i^2P\delta_2^{\frac{4}{n-2}}+c\int_\Omega P\delta_2^{\frac{4}{n-2}}|v_2|^2\\
&=o\left(\sum_{i=1,3}d_2\varepsilon_{i2}+\sum_{i=1,3}\varepsilon_{i2}^{\frac{n}{n-2}}\ln(\varepsilon_{i2}^{-1})+\frac{\varepsilon}{\lambda_2^2}+\frac{1}{(\lambda_2d_2)^{n-2}}\right)+O(\|v_2\|^2).
\end{aligned}
$$

The second case: if $n=6$, (52) implies that

$$
\begin{aligned}
I_{10} \leq \int_\Omega v^2P\delta_2 &\leq c\Lambda^2\int_\Omega\sum_{i=1,3}\delta_i^2P\delta_2+c\int_\Omega\delta_2|v_2|^2\\
&\leq c\Lambda^2\sum_{i=1,3}\langle P\delta_i,P\delta_2\rangle+c\|v_2\|^2\\
&=o\left(\sum_{i=1,3}d_2\varepsilon_{i2}+\sum_{i=1,3}\varepsilon_{i2}^{\frac{n}{n-2}}\ln(\varepsilon_{i2}^{-1})+\frac{\varepsilon}{\lambda_2^2}+\frac{1}{(\lambda_2d_2)^{n-2}}\right)+O(\|v_2\|^2).
\end{aligned}
$$

The third case: if $n\geq 7$, then $(6-n)/(n-2)<0$.

$$
\begin{aligned}
I_{10}=&\int_{\omega_1\cap\Gamma_1}|\alpha_1P\delta_1(x)-\alpha_2P\delta_2(x)+\alpha_3P\delta_3(x)|^{\frac{6-n}{n-2}}P\delta_2v^2\\
&+\int_{\omega_1\cap\Gamma_2}|\alpha_1P\delta_1(x)-\alpha_2P\delta_2(x)+\alpha_3P\delta_3(x)|^{\frac{6-n}{n-2}}P\delta_2v^2:=J_7+J_8.
\end{aligned}
$$

Note that, since $(6-n)/(n-2)<0$, $\forall x\in\omega_1$ we have $|\alpha_1P\delta_1-\alpha_2P\delta_2+\alpha_3P\delta_3|^{(6-n)/(n-2)}\leq c|v|^{(6-n)/(n-2)}$; then by using (52), we get

$$
\begin{aligned}
J_7 &\leq c\int_{\omega_1\cap\Gamma_1}|v|^{\frac{n+2}{n-2}}P\delta_2\\
&\leq c\Lambda^{\frac{n+2}{n-2}}\sum_{i=1,3}\langle P\delta_i,P\delta_2\rangle+c\sum_{i=1,3}\int_\Omega|v_2|^{\frac{n+2}{n-2}}(\delta_i\delta_2)^{1/2}\\
&=o\left(\sum_{i=1,3}d_2\varepsilon_{i2}+\sum_{i=1,3}\varepsilon_{i2}^{\frac{n}{n-2}}\ln(\varepsilon_{i2}^{-1})+\frac{1}{(\lambda_2d_2)^{n-2}}\right)+O(\|v_2\|^2).
\end{aligned}
$$

Concerning J_8, we have

$$
J_8\leq c\int_{\omega_1\cap\Gamma_2}|\alpha_1P\delta_1-\alpha_2P\delta_2+\alpha_3P\delta_3|^{\frac{6-n}{n-2}}v_1^2P\delta_2+c\int_{\omega_1\cap\Gamma_2}|\alpha_1P\delta_1-\alpha_2P\delta_2+\alpha_3P\delta_3|^{\frac{6-n}{n-2}}v_2^2P\delta_2. \tag{59}
$$

Note that $\forall x\in\Gamma_2$ we have $|\alpha_1P\delta_1-\alpha_2P\delta_2+\alpha_3P\delta_3|^{(6-n)/(n-2)}\leq c(\delta_1+\delta_3)^{(6-n)/(n-2)}$ and $\forall x\in\Gamma_2$ we have

$|\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{(6-n)/(n-2)} \leq cP\delta_2^{(6-n)/(n-2)}$, then by using (52), we get

$$J_8 \leq c\Lambda^2 \sum_{i=1,3} \langle P\delta_i, P\delta_2\rangle + c\int_\Omega P\delta_2^{\frac{4}{n-2}} v_2^2$$
$$= o\left(\sum_{i=1,3} d_2\varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O(\|v_2\|^2).$$

Thus, $\forall n \geq 4$ we have

$$I_{10} = o\left(\sum_{i=1,3} d_2\varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1}) + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O(\|v_2\|^2). \tag{60}$$

At the end (44), (47)–(50), (53), (54), (56), (58), and (60) imply that

$$I = \sum_{i=1,3} \alpha_i^{\frac{n+2}{n-2}} \langle P\delta_i, \varphi_2\rangle - 2\alpha_2^{\frac{n+2}{n-2}} \langle P\delta_2, \varphi_2\rangle + \sum_{i=1,3} \alpha_i \alpha_2^{\frac{4}{n-2}} \langle P\delta_i, \varphi_2\rangle$$
$$+ o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2} + \sum_{i=1,3} d_2\varepsilon_{i2} + \sum_{i=1,3} \varepsilon_{i2}^{\frac{n}{n-2}} \ln(\varepsilon_{i2}^{-1})\right)$$
$$+ O_{(\text{if } n\leq 6)}(\|v_2\|^2) + O_{(\text{if } n\geq 7)}(\|v_2\|^{\frac{2(n-1)}{n}}) + O\left(\frac{\|v_2\|}{(\lambda_2 d_2)^{\frac{n-2}{2}}}\right). \tag{61}$$

Concerning the other integrals of the right-hand side of (43), note that

$$\forall i \in \{1,3\}, \quad \varepsilon\alpha_i \int_\Omega P\delta_i \varphi_2 = o\left(d_2\varepsilon_{i2} + \varepsilon_{i2}^{\frac{n}{n-2}} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right), \tag{62}$$

(by using Lemma 2.2 if $d_2 \nrightarrow 0$ and Lemma 2.14 if $d_2 \to 0$).

Moreover, we have

$$|\varepsilon \int_\Omega v\varphi_2| \leq c\Lambda \int_\Omega \sum_{i=1,3} \delta_i P\delta_2 + \varepsilon \int_\Omega |v_2|\delta_2 = o\left(\frac{\varepsilon}{\lambda_2^2} + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O\left(\varepsilon\|v_2\|^2\right). \tag{63}$$

Therefore (61)–(63), Lemmas 2.8, 2.10, and Proposition 3.4 complete the proof. □

5. Proof of the main theorem

The major objective of this section is to show the main theorem. By arguing by contradiction, we assume that there exists a solution u_ε of $(P_{-\varepsilon})$ under the form (7). Recall that $f = o(g)$ means that f/g goes to 0 as $\varepsilon \to 0$. Our goal is to find a relation of the type $x = o(x)$ or $A + B = o(A + B)$ (which is equivalent to $A(1 + o(1)) + B(1 + o(1)) = 0$ with $A > 0$ and $B > 0$), which leads to a contradiction. Note that, from [12], (5) is satisfied. Thus, two cases may occur:

The first case: If $\lambda_2 \simeq \lambda_1 \simeq \lambda_3$. Since $\varepsilon_{i2} \to 0$ (for $i = 1, 3$); then $\lambda_i|a_i - a_2| \to +\infty$; hence

$$\varepsilon_{12} = \frac{1 + o(1)}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{\frac{n-2}{2}}} \quad \text{and} \quad \varepsilon_{23} = \frac{1 + o(1)}{(\lambda_2\lambda_3|a_2 - a_3|^2)^{\frac{n-2}{2}}}. \tag{64}$$

Recall that $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{2-n}{2}}$ and $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-2}{2}\varepsilon_{ij}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^{\frac{2}{n-2}}\right)$; then

$$\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} = -\frac{n-2}{2}\varepsilon_{12}(1+o(1)) \qquad \text{and} \qquad \lambda_2 \frac{\partial \varepsilon_{23}}{\partial \lambda_2} = -\frac{n-2}{2}\varepsilon_{23}(1+o(1)). \tag{65}$$

Proposition 2.11, for $i = 2$, implies that

$$\begin{aligned} &\frac{n-2}{2}\widetilde{c}\frac{H_{22}}{\lambda_2^{n-2}} + \frac{n-2}{2}\widetilde{c}\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{n-2}{2}\widetilde{c}\frac{G_{23}}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}} + \frac{C_2\varepsilon}{\lambda_2^2}(\text{if } n \geq 5) + C_3\frac{\varepsilon\ln(\lambda_2 d_2)}{\lambda_2^2}(\text{if } n = 4) \\ &= o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{13}\right). \end{aligned} \tag{66}$$

★ If $|a_1 - a_2| \geq c > 0$ and $|a_2 - a_3| \geq c > 0$; then we have

$$0 < G_{12} \leq c, \quad 0 < G_{23} \leq c, \quad \text{and} \quad \varepsilon_{ij} \leq \frac{c}{(\lambda_2 d_2)^{n-2}}.$$

Thus, (66) leads to a contradiction.

★ If a_2 approaches one of the points a_1 or a_3, for example $|a_1 - a_2| \to 0$, then we have: $d_2 \nrightarrow 0$ and $|a_2 - a_3| \geq c > 0$, which imply that

$$\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = \varepsilon_{12}(1+o(1)), \quad \varepsilon_{13} \leq c\varepsilon_{12} \quad \text{and} \quad \varepsilon_{23} \leq c\varepsilon_{12}.$$

Thus, (66) leads to a contradiction.

The second case: If $\lambda_1 \simeq \lambda_3 \ll \lambda_2$, two subcases occur:

★ If $|a_1 - a_2| \geq c > 0$ and $|a_2 - a_3| \geq c > 0$, then

$$\text{for } n \geq 5, \text{ we have } \varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) = o\left(\frac{\varepsilon}{\lambda_2^2}\right), \tag{67}$$

$$\text{for } n = 4, \text{ we have } \varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + d_2\varepsilon_{12}\right). \tag{68}$$

In fact, since $|a_i - a_2| \geq c > 0$, $\forall i \in \{1, 3\}$, then, $\varepsilon_{i2} \simeq (\lambda_i\lambda_2)^{\frac{2-n}{2}}$; thus, for $n \geq 5$, we have $\varepsilon_{12}^{n/(n-2)}\ln(\varepsilon_{12}^{-1}) = o\left(\varepsilon_{12}^{(n-1)/(n-2)}\right) = o\left((\lambda_1\lambda_2)^{(1-n)/2}\right)$.

Note that, according to [12], for all $n \geq 5$, we have $\varepsilon \simeq \lambda_1^{4-n}$; then

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{-1}\lambda_2^2}{\lambda_1^{\frac{n-1}{2}}\lambda_2^{\frac{n-1}{2}}} = \lim_{\varepsilon \to 0^+} \frac{1}{\lambda_1}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n-5}{2}} = 0.$$

Thus, (67) is proved. Concerning (68), for $n = 4$, we have

$$\begin{aligned} \varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) = \varepsilon_{12}^2\ln(\varepsilon_{12}^{-1}) &= o\left(\frac{1}{(\lambda_1\lambda_2)^{3/2}}\right) = o\left(\frac{d_2}{(\lambda_1\lambda_2)^{1/2}(\lambda_2 d_2)}\right) \\ &= o\left(\frac{d_2^2}{\lambda_1\lambda_2} + \frac{1}{(\lambda_2 d_2)^2}\right) = o\left(\frac{1}{(\lambda_2 d_2)^2} + d_2\varepsilon_{12}\right). \end{aligned}$$

Note that the equations (64) and (65) remain true. Thus Proposition 4.1 can be written as

$$\frac{n-2}{2}\widetilde{c}\frac{H_{22}}{\lambda_2^{n-2}}+\frac{n-2}{2}\widetilde{c}\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}+\frac{n-2}{2}\widetilde{c}\frac{G_{23}}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}+\frac{C_2\varepsilon}{\lambda_2^2}(\text{if } n\geq 5)$$
$$+C_3\frac{\varepsilon\ln(\lambda_2 d_2)}{\lambda_2^2}(\text{if } n=4)=o\left(\frac{1}{(\lambda_2 d_2)^{n-2}}+\frac{\varepsilon}{\lambda_2^2}+d_2\varepsilon_{12}+d_2\varepsilon_{23}\right). \tag{69}$$

Note that according to Lemma 2.13 we have $G_{i2}\geq cd_2$, for $i=1,3$; thus, we obtain

$$d_2\varepsilon_{i2}\leq\frac{cd_2}{(\lambda_i\lambda_2)^{\frac{n-2}{2}}}\leq\frac{cG_{i2}}{(\lambda_i\lambda_2)^{\frac{n-2}{2}}}. \tag{70}$$

By combining (69) and (70), we get a contradiction.

★ If a_2 approaches one of the points a_1 or a_3, for example $|a_1-a_2|\to 0$, then $d_2\nrightarrow 0$; hence Proposition 4.1 implies that

$$\frac{n-2}{2}\widetilde{c}\frac{H_{22}}{\lambda_2^{n-2}}+\frac{n-2}{2}\widetilde{c}\varepsilon_{12}-\frac{n-2}{2}\widetilde{c}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}+\frac{n-2}{2}\widetilde{c}\frac{G_{23}}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}$$
$$+\frac{C_2\varepsilon}{\lambda_2^2}(\text{if } n\geq 5)+C_3\frac{\varepsilon\ln(\lambda_2 d_2)}{\lambda_2^2}(\text{if } n=4)=o\left(\frac{1}{\lambda_2^{n-2}}+\frac{\varepsilon}{\lambda_2^2}+\varepsilon_{12}+\varepsilon_{23}\right). \tag{71}$$

On the other hand, we have

- ⋆ $\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\ll\varepsilon_{12}$, because $|a_1-a_2|\to 0$,
- ⋆ $c^{-1}\leq G_{23}\leq c$, because $|a_2-a_3|\geq c>0$, $d_2\nrightarrow 0$ and $d_3\nrightarrow 0$; hence $\frac{G_{23}}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\ll\varepsilon_{12}$,
- ⋆ $\varepsilon_{23}\ll\varepsilon_{12}$, because $|a_1-a_2|\to 0$ and $|a_2-a_3|\geq c>0$.

Hence, (71) leads to a contradiction. Thus Theorem 1.1 is proved. □

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