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The localization theorem for finite-dimensional compact group actions

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Abstract: The localization theorem is known for compact G -spaces, where G is a compact Lie group. In this study, we show that the localization theorem remains true for finite-dimensional compact group actions, and Borel's fixed point theorem holds not only for torus actions but for arbitrary finite-dimensional compact connected abelian group actions.

Key words: Equivariant cohomology, fixed point, compact groups

1. Introduction

The abelian localization theorem of Borel, Atiyah-Segal, and Quillen was introduced in [1, 2, 13]. Furthermore, Hsiang [7] stated the general form of the localization theorem as follows:

Theorem 1.1 *Let G be a compact Lie group and X be a compact or a paracompact space of finite cohomological dimension and finitely many orbit types. Let $S \subseteq H^*(B_G, \mathbb{Q})$ be a multiplicative system (i.e. multiplicative semigroup with $\{1\} \subseteq S$) and*

$$X^S = \{x \in X : \text{no element of } S \text{ maps to zero in } H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})\}.$$

Then the localized restriction homomorphism

$$S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(X^S, \mathbb{Q})$$

is an isomorphism.

Remark 1.2 *The localization theorem above was extended by Deo et al. [4] for compact Lie group actions on finitistic spaces with finitely many orbit types.*

The localization theorem is a powerful tool for cohomology theory of compact, especially compact abelian, transformation groups. One of the most profound results of the localization theorem is to determine the cohomology structure of a fixed point set by the equivariant cohomology of the space for the torus or p -torus actions.

The following well-known Borel's fixed point theorem is one of the important consequences of the localization theorem.

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Theorem 1.3 *Let G be a finite-dimensional torus and X be a compact G -space. Then the fixed point set $X^G \neq \emptyset$ if and only if $H^*(B_G, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$ is injective [2].*

In this paper, the localization theorem has been proved for finite-dimensional compact group actions on compact spaces and as a corollary we extend Borel’s fixed point theorem to finite-dimensional compact connected abelian transformation groups.

2. Preliminaries

The Borel construction is a powerful tool for cohomology theory of topological transformation groups. It is defined as follows:

For any topological group G , we have a universal principal G -bundle $E_G \rightarrow B_G$ (see Milnor [9, 10]). B_G is unique up to homotopy equivalence and is called the classifying space of G .

Let X be a G -space. There is the diagonal action on $E_G \times X$ and the Borel construction is defined to be the orbit space $(E_G \times X)/G$ and denoted by X_G . This leads to the following commutative diagram:

$$\begin{array}{ccccc}
 X & \longleftarrow & E_G \times X & \longrightarrow & E_G \\
 \downarrow & \searrow^{i_G} & \downarrow & & \downarrow \\
 X/G & \xleftarrow{\pi_2} & X_G & \xrightarrow{\pi_1} & B_G
 \end{array}$$

where π_1 is a fiber bundle mapping with fiber X and structure group G/K where K is the ineffective kernel of the G action on X , π_2 is a mapping such that $\pi_2^{-1}(x^*) = B_{G_x}$, where $x^* \in X/G$, and $x \in x^*$. Moreover, $H^*(X_G, k)$ is an algebra over $H^*(B_G, k)$ by $\pi_1^* : H^*(B_G, k) \rightarrow H^*(X_G, k)$, which is called the equivariant graded cohomology algebra of X with coefficient k and denoted by $H_G^*(X; k)$.

Remark 2.1 *Some results in Alexander–Spanier cohomology are in need of some compactness assumptions (for example, Proposition 3.1.). Even if G is a compact group and X is a compact G -space, B_G and X_G are not necessarily compact. It is often convenient to consider N -classifying space B_G^N (compact) instead of B_G . Since the morphisms $H^i(B_G^{N+1}, \mathbb{Q}) \rightarrow H^i(B_G^N, \mathbb{Q})$ are isomorphisms for all $i < N$ [6, III; Proposition 1.8], then $H^i(B_G, \mathbb{Q})$ is canonically isomorphic to $H^i(B_G^N, \mathbb{Q})$ for all $i < N$. To prove any result for equivariant cohomology, $H^*(B_G^N, \mathbb{Q})$ can be taken into $H^*(B_G, \mathbb{Q})$ for sufficiently large N without loss of generality. Similarly, let E_G^N be the inverse image of B_G^N under the map $E_G \rightarrow B_G$. Let us define $X_G^N = X \times_G E_G^N$. Note that E_G^N is compact and $(N - 1)$ -connected (i.e. $H^i(E_G^N, \mathbb{Q}) = H^i(pt, \mathbb{Q})$ for all $i < N$), and so X_G^N is also compact. By Vietoris–Begle mapping theorem for sheaf cohomology [13], the inclusion $X_G^N \rightarrow X_G$ induces an isomorphism*

$$H^i(X_G, \mathbb{Q}) \rightarrow H^i(X_G^N, \mathbb{Q})$$

for $i < N$. One can assume $H^*(X_G^N, \mathbb{Q})$ instead of $H^*(X_G, \mathbb{Q})$ for sufficiently large N without loss of generality.

The Mayer–Vietoris exact sequence for equivariant cohomology follows from its ordinary nonequivariant version. If A and B are closed invariant subspaces of the G -space X , there is a long exact sequence

$$\cdots \rightarrow H_G^i(A \cup B, k) \rightarrow H_G^i(A, k) \oplus H_G^i(B, k) \rightarrow H_G^i(A \cap B, k) \rightarrow H_G^{i+1}(A \cup B, k) \rightarrow \cdots$$

of $H^*(B_G)$ -modules.

This paper deals with finite-dimensional compact groups. Therefore, we need to recall the dimension of a compact Hausdorff topological space.

Definition 2.2 *Let X be a compact Hausdorff space. We say that X has dimension $n \geq 0$ if the following two conditions are satisfied:*

1) *For every finite covering of X by open sets there exists a finite covering Σ by closed sets that refines the given covering and, for each $x \in X$, the number of the sets of Σ that contain x is at most $n + 1$.*

2) *There exists a finite covering of X by open sets such that if Σ is a finite closed refinement of this covering then the number of the sets of Σ that contain x exceeds n . If the dimension of X is n then we denote $\dim X = n$.*

If G is a compact group (we always include the Hausdorff separation axiom) then the dimension of G is the dimension of the space G . Of course, we say that G is finite-dimensional compact group if $\dim G = n$ for some $n \geq 0$.

Remark 2.3 1. *There are various dimension functions. Topological dimension, small inductive dimension, local large inductive dimension, cohomological dimension, and sheaf theoretical dimension are frequently used. For an arbitrary compact group G , all these dimensions agree, and when finite $\dim G = \text{rank} \widehat{G}$ where \widehat{G} is a character group of G .*

2. *An important fact about finite-dimensional compact groups is that they have a totally disconnected closed normal subgroup such that the factor group is a compact Lie group with the same dimension (see for details [12, Theorem 69]).*

We need a few facts about finite-dimensional compact groups.

Theorem 2.4 (Nagami, [11]) *Any closed subgroup of a locally compact, finite-dimensional group has a local cross section.*

According to the above, if G is a finite-dimensional compact group and N is a closed normal subgroup of G , then the quotient map $G \rightarrow G/N$ is a principal N -bundle.

The proof of the next theorem can also be found in the 1974 work of Bruner et al. (<https://www.math.uchicago.edu/~may/CHAR/charclasses.pdf>).

Theorem 2.5 *If G is a finite-dimensional compact group and N is a closed normal subgroup of G , then up to homotopy, the sequence $B_N \rightarrow B_G \rightarrow B_{G/N}$, induced by the inclusion $N \hookrightarrow G$ and the quotient map $G \rightarrow G/N$, is a fiber sequence.*

Proof Let $E_G \rightarrow B_G$ and $E_{G/N} \rightarrow B_{G/N}$ be universal bundles for G and G/N and take $E_G \rightarrow E_{G/N} = B_N$ to be the universal bundle for N . We may assume given a map $E_j : E_G \rightarrow E_{G/N}$ such that $(E_j)(yg) =$

$(Ej)(y)j(g)$. Since $j(n) = e$ for $n \in N$, Ej factors through B_N and we obtain a bundle map

$$\begin{array}{ccc} B_N & \longrightarrow & E_{G/N} \\ Bi \downarrow & & \downarrow \\ B_G & \xrightarrow{Bj} & B_{G/N} \end{array}$$

In particular, this square is a pullback. We can replace the map Bj by a fibration. The fiber of this fibration is the homotopy fiber $F(Bj)$ of Bj , where the homotopy fiber of Bj is defined by the pullback diagram

$$\begin{array}{ccc} F(Bj) & \longrightarrow & PB_{G/N} \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{Bj} & B_{G/N} \end{array}$$

We shall show that $F(Bj)$ is homotopy equivalence to B_N . Notice that $PB_{G/N}$ is homotopy equivalence to $E_{G/N}$ since $E_{G/N}$ is contractible.

We see by the universal property of pullbacks that there is a map $\theta : B_N \rightarrow F(Bj)$ such that the following diagram commutes.

$$\begin{array}{ccccc} G/N & \longrightarrow & B_N & \longrightarrow & B_G \\ \zeta \downarrow & & \downarrow \theta & & \parallel \\ \Omega B_{G/N} & \longrightarrow & F(Bj) & \longrightarrow & B_G \end{array}$$

By Zeeman’s comparison theorem [14], since ζ is homotopy equivalence, θ induces an isomorphism on homotopy groups and is thus a homotopy equivalence. The result follows. \square

Throughout the present article Alexander–Spanier cohomology with rational coefficients is used. Indeed, Alexander–Spanier cohomology, Čech cohomology, and Sheaf cohomology are naturally equivalent for paracompact spaces [3].

3. Main results

Since each finite-dimensional compact group has a totally disconnected (i.e. zero-dimensional) closed normal subgroup such that the factor group is a compact Lie group (see Remark 2.3.), the compact totally disconnected group actions play an important role for compact transformation group theory. We give a few well-known propositions for totally disconnected group actions.

Proposition 3.1 *If G is a totally disconnected compact group and X is a compact G -space, then the orbit map $\pi : X \rightarrow X/G$ induces an isomorphism*

$$H^*(X/G, \mathbb{Q}) \simeq (H^*(X, \mathbb{Q}))^G$$

where $(H^*(X, \mathbb{Q}))^G$ is the invariant subspace of $H^*(X, \mathbb{Q})$ under the induced action of G on $H^*(X, \mathbb{Q})$.

Proposition 3.2 *Let G be a compact totally disconnected group. Then $H^*(B_G, \mathbb{Q}) = \mathbb{Q}$.*

The proofs of Propositions 3.1. and 3.2. are trivial from continuity of cohomology and can be found in [8].

From now on, unless otherwise stated, G is a finite-dimensional compact group and N is a totally disconnected closed normal subgroup of G such that the factor group G/N is a compact Lie group with the same dimension as G . Let X be a compact G -space and $S \subseteq H^*(B_G, \mathbb{Q})$ be a multiplicative system (i.e. multiplicative semigroup with $\{1\} \subseteq S$).

Moreover, we set, in the sense of Hsiang’s book [7],

$$X^S = \{x \in X : \text{no element of } S \text{ maps to zero in } H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})\}.$$

Note that if X is a paracompact space and one is using Alexander–Spanier cohomology, then X^S is a closed invariant subspace of X .

Lemma 3.3 *The homomorphism $B\pi^* : H^*(B_{G/N}, \mathbb{Q}) \rightarrow H^*(B_G, \mathbb{Q})$ is induced by the quotient map $\pi : G \rightarrow G/N$ is an isomorphism.*

Proof We recall by Theorem 2.5. that there is a Serre fibration $B_N \xrightarrow{Bi} B_G \xrightarrow{B\pi} B_{G/N}$, induced by the inclusion $i : N \hookrightarrow G$ and the quotient map $\pi : G \rightarrow G/N$. Moreover, there is naturally a spectral sequence with $E_2^{p,q} \cong H^p(B_{G/N}, \mathcal{H}^q)$ where \mathcal{H}^q is the coefficient system $x \mapsto H^q((B\pi)^{-1}(x), \mathbb{Q})$. Since B_N is acyclic, i.e. $H^*(B_N, \mathbb{Q}) = H^*(pt, \mathbb{Q})$, then the fundamental group of $B_{G/N}$ acts trivially on the cohomology of the fiber B_N . Therefore, from the results of Leray–Serre spectral sequences $E_2^{p,q} \cong H^p(B_{G/N}, \mathbb{Q}) \otimes H^q(B_N, \mathbb{Q})$.

Since the spectral sequence converges to $H^*(B_G, \mathbb{Q})$ and $H^p(B_N, \mathbb{Q}) = 0$ for $p \neq 0$, then

$$H^n(B_G, \mathbb{Q}) \cong \bigoplus_{r+s=n} E_\infty^{r,s} \cong \bigoplus_{r+s=n} E_2^{r,s} \cong E_2^{n,0}.$$

On the other hand, since $B\pi^n : H^n(B_{G/N}, \mathbb{Q}) \rightarrow H^n(B_G, \mathbb{Q})$ is composite $H^n(B_{G/N}, \mathbb{Q}) = E_2^{n,0} \rightarrow E_3^{n,0} \rightarrow \dots \rightarrow E_{n+1}^{n,0} = E_\infty^{n,0} \cong H^n(B_G, \mathbb{Q})$ for each n and then the result follows. \square

Now consider the multiplicative system $R = (B\pi^*)^{-1}(S)$ in $H^*(B_{G/N}, \mathbb{Q})$, which is a copy of S and induced finite-dimensional compact Lie group G/N action on the orbit space X/N .

Lemma 3.4 *The orbit map $p : X \rightarrow X/N$ induces the map $X^S \rightarrow (X/N)^R$.*

Proof Let $x \in X^S$ and assume that $x^* = N(x) \notin (X/N)^R$. In this case, $Bj^*(r) = 0$ for some $r \in R$, where

$$Bj^* : H^*(B_{G/N}, \mathbb{Q}) \rightarrow H^*(B_{(G/N)_{x^*}}, \mathbb{Q}).$$

Put $s = (B\pi^*)(r) \in S$. From the following commutative diagram,

$$\begin{CD} H^*(B_{G/N}, \mathbb{Q}) @>Bj^*>> H^*(B_{(G/N)_{x^*}}, \mathbb{Q}) \\ @V B\pi^* VV @VVV \\ H^*(B_G, \mathbb{Q}) @>Bi^*>> H^*(B_{G_x}, \mathbb{Q}) \end{CD}$$

$s \in S$ maps to zero under $Bi^* : H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})$. This contradicts $x \in X^S$. \square

Remark 3.5 *The isotropy subgroups of the induced action of G/N on X/N are explicitly discussed in [5, Prop.10.31]. We have*

$$(G/N)_{x^*} = G_x N/N \simeq G_x/(G_x \cap N).$$

Furthermore, since $G_x \cap N$ is a totally disconnected closed normal subgroup of G_x , $H^*(B_{G_x \cap N}, \mathbb{Q}) = \mathbb{Q}$. Thus, as in Lemma 3.3., the right vertical map of the above diagram is induced by the canonical map $G_x \rightarrow G_x N/N$ is also an isomorphism by considering the Leray–Serre spectral sequence of the homotopy-fiber sequence $B_{G_x \cap N} \rightarrow B_{G_x} \rightarrow B_{G_x/G_x \cap N}$.

Corollary 3.6 $X^S/N = (X/N)^R$.

Proof Consider the previous diagram.

If $X^S = \emptyset$, then there exists an $s \in S$ such that s maps to zero in $H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})$. Set $r = (B\pi^*)^{-1}(s)$. Since $H^*(B_{(G/N)_{x^*}}, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})$ is an isomorphism from Remark 3.5., then $Bj^*(r) = 0$, which implies that $(X/N)^R = \emptyset$.

Suppose $X^S \neq \emptyset$. It is easy to see that Lemma 3.4. implies $X^S/N \subseteq (X/N)^R$. We take now $N(x) \in (X/N)^R$. Assume that $x \notin X^S$. Then $Bi^*(s) = 0$ for some $s \in S$. Set $r = (B\pi^*)^{-1}(s)$. Since $H^*(B_{(G/N)_{x^*}}, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})$ is an isomorphism, then $Bj^*(r) = 0$. This contradicts $N(x) \in (X/N)^R$. Hence, $x \in X^S$, which implies $N(x) \in X^S/N$. □

First we prove the localization theorem for finite-dimensional compact connected group action on a compact space.

Theorem 3.7 *Let G be a finite-dimensional compact connected group and X be a compact G -space. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(X^S, \mathbb{Q})$$

is an isomorphism.

Proof Since G is connected, its action (and hence that of N) on $H^*(X, \mathbb{Q})$ is trivial (see [3, II.10.6 cf. II.11.11]). Therefore, Proposition 3.1. implies that the orbit map $X \rightarrow X/N$ induces the isomorphism

$$H^*(X, \mathbb{Q}) \simeq H^*(X/N, \mathbb{Q}).$$

We now prove the theorem by reducing the G action on X to the G/N action on the compact space X/N .

By the localization theorem for compact Lie transformation groups,

$$R^{-1}H_{G/N}^*(X/N, \mathbb{Q}) \rightarrow R^{-1}H_{G/N}^*((X/N)^R, \mathbb{Q})$$

is an isomorphism.

On the other hand, from Zeeman’s comparison theorem [14], we have that

$$H_{G/N}^*(X/N, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$$

is an isomorphism by comparing the spectral sequence of the bundles $X \rightarrow X_G \rightarrow B_G$ and $X/N \rightarrow (X/N)_{G/N} \rightarrow B_{G/N}$. Thus,

$$R^{-1}H_{G/N}^*(X/N, \mathbb{Q}) \simeq S^{-1}H_G^*(X, \mathbb{Q}).$$

Similarly, we have

$$R^{-1}H_{G/N}^*(X^S/N, \mathbb{Q}) \simeq S^{-1}H_G^*(X^S, \mathbb{Q})$$

by considering restricted G -action on X^S . Therefore, we have

$$R^{-1}H_{G/N}^*((X/N)^R, \mathbb{Q}) \simeq S^{-1}H_G^*(X^S, \mathbb{Q})$$

from Corollary 3.6.

Because of the following commutative diagram,

$$\begin{CD} R^{-1}H_{G/N}^*(X/N, \mathbb{Q}) @>\simeq>> R^{-1}H_{G/N}^*((X/N)^R, \mathbb{Q}) \\ @V\simeq VV @VV\simeq V \\ S^{-1}H_G^*(X, \mathbb{Q}) @>>> S^{-1}H_G^*(X^S, \mathbb{Q}) \end{CD}$$

we obtain the isomorphism $S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(X^S, \mathbb{Q})$. □

The next theorem implies that the localization theorem is provided for general finite-dimensional compact group actions.

Theorem 3.8 *Let G be a finite-dimensional compact group, X be a compact G -space, and $S \subseteq H^*(B_G, \mathbb{Q})$ be a multiplicative system. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(X^S, \mathbb{Q})$$

is an isomorphism.

Proof Let $G_0 \subseteq G$ denote the connected component of the identity element, which is a closed normal subgroup of G and G/G_0 is a totally disconnected compact group (see [12, Section 22] for details).

Since G_0 acts freely on E_G , we may take for $E_{G_0} = E_G$ and B_{G_0} the quotient space E_G/G_0 . There are then obvious induced G/G_0 -actions on X_{G_0} and B_{G_0} . Since G/G_0 is totally disconnected, then, by Proposition 3.1., the orbit maps $X_{G_0} \rightarrow X_G$ and $B_{G_0} \rightarrow B_G$ induce the monomorphisms $H^*(X_G, \mathbb{Q}) \rightarrow H^*(X_{G_0}, \mathbb{Q})$ and $H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_0}, \mathbb{Q})$.

Case 1: First we assume that $X^S = \emptyset$. Let T be the direct image of S in $H^*(B_{G_0}, \mathbb{Q})$. Then, by considering restricted finite-dimensional compact connected group G_0 , action on X , and the following commutative diagram,

$$\begin{CD} H^*(B_G, \mathbb{Q}) @>>> H^*(B_{G_x}, \mathbb{Q}) \\ @VVV @VVV \\ H^*(B_{G_0}, \mathbb{Q}) @>>> H^*(B_{(G_0)_x}, \mathbb{Q}) \end{CD}$$

it is easy to see that $X^T = \emptyset$. Hence, Theorem 3.7. implies that there is an element t of T such that the image of t under $H^*(B_{G_0}, \mathbb{Q}) \rightarrow H^*(X_{G_0}, \mathbb{Q})$ is zero. Set s is the preimage of t under the monomorphism $H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_0}, \mathbb{Q})$. Now consider the following commutative diagram,

$$\begin{array}{ccc} H^*(B_G, \mathbb{Q}) & \longrightarrow & H^*(X_G, \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*(B_{G_0}, \mathbb{Q}) & \longrightarrow & H^*(X_{G_0}, \mathbb{Q}) \end{array}$$

which is induced by the following commutative diagram of the bundles:

$$\begin{array}{ccccc} X & \longrightarrow & X_{G_0} & \longrightarrow & B_{G_0} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X_G & \longrightarrow & B_G \end{array}$$

Since the orbit map $X_{G_0} \rightarrow X_G$ induces the monomorphism $H^*(X_G, \mathbb{Q}) \rightarrow H^*(X_{G_0}, \mathbb{Q})$, it is easy to see that s maps to zero under $H^*(B_G, \mathbb{Q}) \rightarrow H^*(X_G, \mathbb{Q})$, which means $S^{-1}H_G^*(X, \mathbb{Q}) = \{0\}$.

Case 2: Now assume that $X^S \neq \emptyset$. Let B be a closed invariant neighborhood of X^S in X and $B' = X - \text{int}(B)$. Thus, consider the long exact sequence

$$\dots \rightarrow H_G^*(X, \mathbb{Q}) \rightarrow H_G^*(B, \mathbb{Q}) \oplus H_G^*(B', \mathbb{Q}) \rightarrow H_G^*(B \cap B', \mathbb{Q}) \rightarrow \dots$$

Hence, by exactness of localization, we have

$$\dots \rightarrow S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(B, \mathbb{Q}) \oplus S^{-1}H_G^*(B', \mathbb{Q}) \rightarrow S^{-1}H_G^*(B \cap B', \mathbb{Q}) \rightarrow \dots$$

Since $(B')^S = \emptyset$ and $(B \cap B')^S = \emptyset$, we obtain that $S^{-1}H_G^*(B', \mathbb{Q}) = \{0\}$ and $S^{-1}H_G^*(B \cap B', \mathbb{Q}) = \{0\}$ from Case 1. Now $S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(B, \mathbb{Q})$ is an isomorphism. Taking the inverse limit as B run over all invariant closed neighborhoods of X^S , because of the continuity property of Alexander–Spanier cohomology, one has $H_G^*(X^S, \mathbb{Q}) = \varprojlim_{B \supset X^S} H_G^*(B, \mathbb{Q})$. Using the continuity property and the fact the localization commutes with the inverse limit, we obtain

$$\begin{aligned} S^{-1}H_G^*(X^S, \mathbb{Q}) &= S^{-1}(\varprojlim_{B \supset X^S} H_G^*(B, \mathbb{Q})) \\ &= \varprojlim_{B \supset X^S} S^{-1}H_G^*(B, \mathbb{Q}) \\ &\simeq \varprojlim_{B \supset X^S} S^{-1}H_G^*(X, \mathbb{Q}) = S^{-1}H_G^*(X, \mathbb{Q}). \end{aligned}$$

□

We will then shift our attention to the abelian case. In the next theorem, Hofmann and Mostert [6] give the cohomology algebra structure of the classifying space of compact (connected) abelian groups, which is irrespective of the choice of the classifying space.

Theorem 3.9 ([6, pp.206, 212]). *For any compact abelian group G , the homogeneous component $H^2(B_G, \mathbb{Z})$ of degree 2 is naturally isomorphic to the character group \widehat{G} . If G is connected, the entire cohomology algebra*

$H^*(B_G, \mathbb{Z})$ is the symmetric \mathbb{Z} -algebra $P(\widehat{G})$ generated by this component of homogeneous degree 2. Also, in this case, the homogeneous component $H^1(G, \mathbb{Z})$ of degree 1 of the space cohomology is also naturally isomorphic to \widehat{G} , and the entire cohomology algebra $H^*(G, \mathbb{Z})$ is the exterior algebra $\wedge(H^1(G, \mathbb{Z}))$.

In particular, for any commutative unital ring R and compact connected abelian G ,

$$H^*(B_G, R) \cong R \otimes P(\widehat{G}) \cong P(R \otimes \widehat{G}).$$

Remark 3.10 If G is a compact abelian group and N is any closed normal subgroup of G , then, by duality, the inclusion morphism $i : N \rightarrow G$ induces a quotient morphism $\widehat{i} : \widehat{G} \rightarrow \widehat{N}$ with kernel

$$\ker \widehat{i} = N^\perp = \{\chi \in \widehat{G} : \chi(n) = 0 \text{ for all } n \in N\}$$

where N^\perp is called the annihilator of N in \widehat{G} . Therefore, $H^2(Bi, \mathbb{Z}) : H^2(B_G, \mathbb{Z}) \rightarrow H^2(B_N, \mathbb{Z})$ is injective if and only if $N = G$, and if G is connected, this conclusion persists upon tensoring with \mathbb{Q} ; that is, $H^2(Bi, \mathbb{Q}) : H^2(B_G, \mathbb{Q}) \rightarrow H^2(B_N, \mathbb{Q})$ is injective if and only if $N = G$.

This observation applies to a compact connected abelian group G acting on X with $N = G_x$. Then one does obtain the statement that x is a fixed point of the action if and only if $Bi^* : H^*(B_G, \mathbb{Q}) \rightarrow H^*(B_{G_x}, \mathbb{Q})$ is injective for the inclusion map $i : G_x \rightarrow G$.

Thus, if a compact connected abelian group G acts on X , then $X^S = X^G$ for the multiplicative system $S = H^*(B_G, \mathbb{Q}) - \{0\} \subset H^*(B_G, \mathbb{Q})$.

Thus, we obtain an extension of Borel's fixed point theorem to finite-dimensional compact connected abelian group actions.

Corollary 3.11 Let G be a finite-dimensional compact connected abelian group and X be a compact G -space. Then the localized restriction homomorphism

$$S^{-1}H_G^*(X, \mathbb{Q}) \rightarrow S^{-1}H_G^*(X^G, \mathbb{Q}) = H^*(X^G, \mathbb{Q}) \otimes_{\mathbb{Q}} (S^{-1}H^*(B_G, \mathbb{Q}))$$

is an isomorphism where $S = H^*(B_G, \mathbb{Q}) - \{0\}$.

Corollary 3.12 Let G be a finite-dimensional compact connected abelian group and X be a compact G -space. Then $X^G \neq \emptyset$ if and only if $H^*(B_G, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$ is injective.

Proof \Rightarrow : Suppose that $X^G \neq \emptyset$ and $x \in X^G$. Thus, $B_G \rightarrow \{x\}_G \subseteq X_G$ is a cross-section of the fiber bundle $X_G \rightarrow B_G$ and consequently $H^*(pt, \mathbb{Q}) = H^*(B_G, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$ must be an injective map.

\Leftarrow : If $H^*(B_G, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$ is injective, then $1 \in H_G^*(X, \mathbb{Q})$ is torsion-free and hence $S^{-1}H_G^*(X, \mathbb{Q}) \neq \{0\}$. Therefore, it follows from Corollary 3.11. that $S^{-1}H_G^*(X^G, \mathbb{Q}) \simeq S^{-1}H_G^*(X, \mathbb{Q}) \neq \{0\}$. This clearly implies $X^G \neq \emptyset$. □

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References

- [1] Atiyah MF, Segal GB. Equivariant Cohomology and Localization. Lecture notes. Coventry, UK: University of Warwick, 1965.
- [2] Borel A. Seminar on Transformation Groups. Princeton, NJ, USA: Princeton University Press, 1960.
- [3] Bredon GE. Sheaf Theory. 2nd ed. Graduate Texts in Mathematics 170. New York, NY, USA: Springer, 1997.
- [4] Deo S, Singh TB, Shukla RA. On an extension of localization theorem and generalized Conner conjecture. T Am Math Soc 1982; 269: 395-402.
- [5] Hofmann KH, Morris SA. The Structure of Compact Groups. 3rd ed. Berlin, Germany: de Gruyter, 2013.
- [6] Hofmann KH, Mostert PS. Cohomology Theories for Compact Abelian Groups. Berlin, Germany: Springer, 1973.
- [7] Hsiang WY. Cohomology Theory of Transformation Groups. Berlin, Germany: Springer, 1975.
- [8] Ku MC. Some topics in compact transformation groups. PhD, Tulane University, New Orleans, LA, USA, 1967.
- [9] Milnor J. Construction of universal bundles I. Ann Math 1956; 63: 272-284.
- [10] Milnor J. Construction of universal bundles II. Ann Math 1956; 63: 430-436.
- [11] Nagami K. Cross sections in locally compact groups. J Math Soc Japan 1963; 15: 301-303.
- [12] Pontryagin LS. Topological Groups. 3rd ed. New York, NY, USA: Gordon and Breach Science Publishers, 1986.
- [13] Quillen D. The spectrum of an equivariant cohomology ring I. Ann Math 1971; 94: 549-572.
- [14] Zeeman EC. A proof of the comparison theorem for spectral sequence. Proc Cambridge Philos Soc 1957; 53: 57-62.