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On the coefficient problem for close-to-convex functions

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Abstract: This paper is concerned with the problem of estimating $|a_4 - a_2a_3|$, where a_k are the coefficients of a given close-to-convex function. The bounds of this expression for various classes of analytic functions have been applied to estimate the third Hankel determinant $H_3(1)$. The results for two subclasses of the class \mathcal{C} of all close-to-convex functions are sharp. This bound is equal to 2. It is conjectured that this number is also the exact bound of $|a_4 - a_2a_3|$ for the whole class \mathcal{C} .

Key words: Close-to-convex functions, coefficient problem

1. Introduction

Let \mathcal{A} be the family of all functions analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ given by the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let \mathcal{S}^* denote the class of starlike functions in \mathcal{A} and let \mathcal{P} denote the class of all analytic functions p with a positive real part in Δ satisfying the normalization condition $p(0) = 1$.

Given $\beta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is called close-to-convex with argument β with respect to g if

$$\Re \left\{ \frac{e^{i\beta} z f'(z)}{g(z)} \right\} > 0, \quad z \in \Delta. \quad (1.2)$$

Let $\mathcal{C}_\beta(g)$ be the class of all such functions. Moreover, let

$$\mathcal{C}(g) = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta(g) \quad \text{and} \quad \mathcal{C}_\beta = \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\beta(g).$$

Let \mathcal{C} denote the family of all close-to-convex functions (see [3, 5]). It is obvious that

$$\mathcal{C} = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta = \bigcup_{g \in \mathcal{S}^*} \mathcal{C}(g).$$

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The number $e^{i\beta}$ is necessary in (1.2) for the definition of close-to-convex function. In addition, this factor significantly complicates the task of estimating some coefficient functionals. Therefore, to simplify the calculation, many authors take $\beta = 0$ or use a specific starlike function, for example the Koebe function

$$k(z) = \frac{z}{(1-z)^2}, \quad z \in \Delta. \tag{1.3}$$

Then inequality (1.2) becomes:

$$\Re \left\{ \frac{zf'(z)}{g(z)} \right\} > 0, \quad z \in \Delta \tag{1.4}$$

or

$$\Re \{ e^{i\beta}(1-z)^2 f'(z) \} > 0, \quad z \in \Delta, \tag{1.5}$$

respectively, and defines the related subclass of close-to-convex functions \mathcal{C}_0 and $\mathcal{C}(k)$, respectively. Let us cite the most important results concerning the estimates of some coefficient functionals within the class \mathcal{C} . Keogh and Merkes in [6] solved the Fekete-Szegő problem in the class \mathcal{C}_0 . Koepf in [7] extended this result for the class \mathcal{C} . Kowalczyk and Lecko, in [9], studied the Fekete-Szegő problem in the class $\mathcal{C}(k)$ of all close-to-convex functions with respect to the Koebe function (and in [8], in the subclass of close-to-convex with respect to other starlike functions). Recently, several authors have extensively investigated the Hankel determinant for close-to-convex functions (see for example [13, 15, 16, 18]) and the logarithmic coefficients of close-to-convex functions (see for example [20]).

The main aim of this paper was to determine the estimates of the expression $|a_4 - a_2a_3|$ for the classes \mathcal{C}_0 , $\mathcal{C}(k)$, and \mathcal{C} . The functional $|a_4 - a_2a_3|$ has been estimated for many classes. Babalola, in [1], derived the exact bounds of $|a_4 - a_2a_3|$ for the class of starlike functions, for the class of convex functions and for the class of functions whose derivative has a positive real part; these values are equal to 2, $4/9\sqrt{3}$, and $5\sqrt{5}/18\sqrt{3}$, respectively. In [14], Mishra et al. proved that this bound in the class of starlike functions with respect to symmetric points is 1/2 and in the class of convex functions with respect to symmetric points is 4/27. Krishna et al. published the same results in [21]. In [17], Raza and Malik, found that $|a_4 - a_2a_3| \leq 1/6$ for the class of lemniscate starlike functions (for the definition of the class see [19]). All these authors used this functional $|a_4 - a_2a_3|$ to estimate the third Hankel determinant $H_3(1)$.

Taking into account (1.2), we can write

$$\frac{e^{i\beta}zf'(z)}{g(z)} = h(z)\cos\beta + i\sin\beta, \tag{1.6}$$

where $h \in \mathcal{P}$. If $g \in \mathcal{S}^*$ and $h \in \mathcal{P}$ are given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.7}$$

and

$$h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.8}$$

then (1.6) leads to

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n \right) \left(1 + e^{-i\beta} \cos \beta \sum_{n=1}^{\infty} p_n z^n \right). \tag{1.9}$$

Therefore,

$$na_n = b_n + e^{-i\beta} \cos \beta \left(p_{n-1} + \sum_{k=2}^{n-1} b_k p_{n-k} \right). \tag{1.10}$$

2. Preliminary results

We shall need the following results. The first one is known as Caratheodory’s lemma (for example see [2]). The second one is by Libera and Złotkiewicz [10, 11].

Lemma 2.1 ([2]) *If $h \in \mathcal{P}$ is given by (1.8), then the sharp estimate $|p_n| \leq 2$ holds for $n \geq 1$.*

Lemma 2.2 ([10, 11]) *Let h be given by (1.8) and $p_1 \in [0, 2]$. Then $h \in \mathcal{P}$ if and only if*

$$2p_2 = p_1^2 + (4 - p_1^2)x$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some complex numbers x, z such that $|x| \leq 1, |z| \leq 1$.

Remark. Given $g \in \mathcal{S}^*$ defined by (1.7), the functional $|b_4 - \mu b_2 b_3|$, $\mu \in \mathbb{R}$ is invariant under rotation. This means that for $g_\phi(z) = e^{-i\phi}g(ze^{i\phi})$, $\phi \in \mathbb{R}$ of the form $g_\phi(z) = z + c_2z^2 + c_3z^3 + \dots$ we have $|b_4 - \mu b_2 b_3| = |c_4 - \mu c_2 c_3|$. Similarly, it can be proved that $|p_3 - \mu p_1 p_2|$ for $h \in \mathcal{P}$ defined by (1.8) is invariant under rotation.

To obtain our results, we also need a few sharp estimates.

Lemma 2.3 *Let $h \in \mathcal{P}$ be given by (1.8) and $\mu \in [1/2, 1]$, then*

$$|p_3 - \mu p_1 p_2| \leq \begin{cases} \frac{1}{4}\mu^2 p^3 - \frac{1}{2}\mu(2 - \mu)p^2 + 2, & p \in \left[0, \frac{2}{2-\mu}\right], \\ (3 - 2\mu)p - (1 - \mu)p^3, & p \in \left[\frac{2}{2-\mu}, 2\right], \end{cases}$$

where $p = |p_1|$.

Proof From Lemma 2.2, we have

$$|p_3 - \mu p_1 p_2| = \frac{1}{4} \left| (1 - 2\mu)p_1^3 + 2(1 - \mu)(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z \right|. \tag{2.1}$$

Applying the triangle inequality in (2.1) with $|x| = \varrho$, $\varrho \in [0, 1]$ and $|p_1| = p$, $p \in [0, 2]$, we obtain

$$\begin{aligned} |p_3 - \mu p_1 p_2| &\leq \frac{1}{4} \left[(2\mu - 1)p^3 + 2(1 - \mu)(4 - p^2)p\varrho + (4 - p^2)p\varrho^2 + 2(4 - p^2)(1 - \varrho^2) \right] \\ &= \frac{1}{4} \left[(2\mu - 1)p^3 + 2(4 - p^2) + 2(1 - \mu)(4 - p^2)p\varrho - (4 - p^2)(2 - p)\varrho^2 \right] \equiv w(\varrho) \end{aligned}$$

(with the equality when $x = -\varrho$ and $z = -1$). If $p < \frac{2}{2-\mu}$, then

$$w(\varrho) \leq w\left(\frac{2}{2-\mu}\right) = \frac{1}{4}\mu^2 p^3 - \frac{1}{2}\mu(2-\mu)p^2 + 2.$$

If $p \geq \frac{2}{2-\mu}$, then

$$w(\varrho) \leq w(1) = (3-2\mu)p - (1-\mu)p^3.$$

Therefore, for $\mu \in [1/2, 1]$, we get the desired result. □

In the second case, the equality holds when $\varrho = 1$, i.e., $x = -1$. Then $p_2 = p_1^2 - 2$. This means that the extremal function is

$$h_t(z) = \frac{1-z^2}{1-2tz+z^2}, \quad t \in [-1, 1].$$

It is easy to check that

$$\begin{aligned} \max\{|p_3 - \mu p_1 p_2| : h \in \mathcal{P}\} &= \max\{|p_3 - \mu p_1 p_2| : |p_1| \in [0, 2]\} \\ &= \max\{2, 4\mu - 2\} = 2, \end{aligned} \tag{2.2}$$

which is the result obtained by Hayami and Owa [4].

From Lemma 2.3, we can easily get the following corollary:

Corollary 2.4 *Let $h \in \mathcal{P}$ be given by (1.8), then $|p_3 - \frac{2}{3}p_1 p_2| \leq G(p)$, where*

$$G(p) = \begin{cases} \frac{1}{9}p^3 - \frac{4}{9}p^2 + 2, & p \in [0, 3/2], \\ \frac{5}{3}p - \frac{1}{3}p^3, & p \in [3/2, 2] \end{cases} \tag{2.3}$$

and $p = |p_1|$.

Lemma 2.5 *Let $g \in \mathcal{S}^*$ be given by (1.7), then $|b_4 - \frac{2}{3}b_2 b_3| \leq H(q)$, where*

$$H(q) = \begin{cases} \frac{1}{48}(2+q)(9q^2 - 8q + 16), & q \in [0, 4/5], \\ \frac{1}{3}q(4 - q^2), & q \in [4/5, 2] \end{cases} \tag{2.4}$$

and $q = |b_2|$, $q \in [0, 2]$.

Proof Every function $g \in \mathcal{S}^*$ satisfies in Δ the equality

$$zg'(z) = g(z)Q(z), \tag{2.5}$$

where $Q \in \mathcal{P}$. Let $Q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$. Equating the coefficients in (2.5) gives

$$b_2 = q_1, \quad b_3 = \frac{1}{2}(q_2 + q_1^2), \quad b_4 = \frac{1}{3}q_3 + \frac{1}{2}q_1 q_2 + \frac{1}{6}q_1^3. \tag{2.6}$$

Applying (2.6), we get

$$\left|b_4 - \frac{2}{3}b_2 b_3\right| = \frac{1}{3} \left|q_3 + \frac{1}{2}q_1 q_2 - \frac{1}{2}q_1^3\right|. \tag{2.7}$$

Now, we use Lemma 2.2 to get

$$\left| b_4 - \frac{2}{3}b_2b_3 \right| = \frac{1}{12} \left| (4 - q_1^2) [3q_1x - q_1x^2 + 2(1 - |x|^2)z] \right|.$$

Since the functional $\left| b_4 - \frac{2}{3}b_2b_3 \right|$ is invariant under rotation, we may write $q_1 = q$, $q \in [0, 2]$. Hence, applying the triangle inequality with $|x| = \varrho$, $\varrho \in [0, 1]$, we get

$$\begin{aligned} \left| b_4 - \frac{2}{3}b_2b_3 \right| &= \frac{1}{12}(4 - q^2) |3q\varrho - q\varrho^2 + 2(1 - \varrho^2)z| \\ &\leq \frac{1}{12}(4 - q^2) [3q\varrho + q\varrho^2 + 2(1 - \varrho^2)] = \frac{1}{12}(4 - q^2) [(q - 2)\varrho^2 + 3q\varrho + 2] \end{aligned}$$

(with the equality when $x = -\varrho$ and $z = -1$).

Let $w(\varrho) = (q - 2)\varrho^2 + 3q\varrho + 2$. If $q \in [0, 4/5]$, then

$$w(\varrho) \leq w\left(\frac{3q}{4 - 2q}\right) = \frac{9q^2 - 8q + 16}{4(2 - q)}.$$

If $q \in [4/5, 2]$, then

$$w(\varrho) \leq w(1) = 4q.$$

Therefore,

$$\left| b_4 - \frac{2}{3}b_2b_3 \right| \leq \begin{cases} \frac{1}{48}(2 + q)(9q^2 - 8q + 16), & q \in [0, 4/5], \\ \frac{1}{3}q(4 - q^2), & q \in [4/5, 2]. \end{cases}$$

This completes the proof of Lemma 2.5. □

In the second case, the equality holds when $\varrho = 1$, i.e., $x = -1$. Then $b_3 = b_2^2 - 1$, which means that the extremal function is

$$g_t(z) = \frac{z}{1 - 2tz + z^2}, \quad t \in [-1, 1].$$

3. Main results

Theorem 3.1 *If $f \in \mathcal{C}_0$ is given by (1.1), then*

$$|a_4 - a_2a_3| \leq 2.$$

This result is sharp.

Proof From (1.4), we can write

$$zf'(z) = g(z)h(z), \tag{3.1}$$

where $h \in \mathcal{P}$. Let g and h be given by (1.7) and (1.8), respectively. Equating the coefficients in (3.1) gives

$$2a_2 = b_2 + p_1, \quad 3a_3 = b_3 + b_2p_1 + p_2, \quad 4a_4 = b_4 + b_3p_1 + b_2p_2 + p_3. \tag{3.2}$$

Therefore, using (3.2), we have

$$\begin{aligned} |a_4 - a_2a_3| &= \left| \frac{1}{4} \left(b_4 - \frac{2}{3}b_2b_3 \right) + \frac{1}{4} \left(p_3 - \frac{2}{3}p_1p_2 \right) + \frac{1}{12}p_1 \left(b_3 - b_2^2 \right) \right. \\ &\quad \left. + \frac{1}{12}b_2 \left(p_2 - p_1^2 \right) - \frac{1}{12}p_1b_2 \left(p_1 + b_2 \right) \right|. \end{aligned}$$

Applying the triangle inequality, we obtain

$$|a_4 - a_2a_3| \leq \frac{1}{4} |b_4 - \frac{2}{3}b_2b_3| + \frac{1}{4} |p_3 - \frac{2}{3}p_1p_2| + \frac{1}{12} |p_1| |b_3 - b_2^2| + \frac{1}{12} |b_2| |p_2 - p_1^2| + \frac{1}{12} |p_1| |b_2| |p_1 + b_2|. \tag{3.3}$$

It is well known (see [12]) that

$$|p_2 - p_1^2| \leq 2 \tag{3.4}$$

for $h \in \mathcal{P}$. Moreover, the Fekete-Szegő inequality for $g \in \mathcal{S}^*$ (see for example [7]) gives

$$|b_3 - b_2^2| \leq 1. \tag{3.5}$$

From (3.3), using (3.4), (3.5), Lemma 2.5, and Corollary 2.4 and writing $|p_1| = p$ and $|b_2| = q$, $p, q \in [0, 2]$, we get

$$|a_4 - a_2a_3| \leq F(p, q),$$

where

$$F(p, q) = \frac{1}{4}H(q) + \frac{1}{4}G(p) + \frac{1}{12}p + \frac{1}{6}q + \frac{1}{12}pq(p + q), \quad p, q \in [0, 2] \tag{3.6}$$

and $H(q), G(p)$ are given by (2.4) and (2.3).

We will show that $F(p, q) \leq 2$ for $p, q \in [0, 2]$. It is easy to check that H is increasing for $q \in [0, 2/\sqrt{3}]$. Suppose that $q \in [2/\sqrt{3}, 2]$. Then

$$F(p, q) = \frac{1}{12}q(4 - q^2) + \frac{1}{4}G(p) + \frac{1}{12}p + \frac{1}{6}q + \frac{1}{12}p^2q + \frac{1}{12}pq^2,$$

so

$$\frac{\partial F}{\partial q} = \frac{1}{12} (6 - 3q^2 + p^2 + 2pq).$$

Thus, for $q \in [2/\sqrt{3}, \sqrt{2}]$, $p \in [0, 2]$, we have $\frac{\partial F}{\partial q} \geq 0$, so $F(p, q)$, with a fixed p , is increasing as a function of q .

For this reason,

$$\max \{F(p, q) : (p, q) \in [0, 2] \times [0, 2]\} = \max \left\{ F(p, q) : (p, q) \in [0, 2] \times [\sqrt{2}, 2] \right\}. \tag{3.7}$$

Assume now that $q \in [\sqrt{2}, 2]$. If $p \in [0, 3/2]$, then

$$F(p, q) = \frac{1}{36}(p^3 - 4p^2 + 3p + 18 - 3q^3 + 18q + 3p^2q + 3pq^2).$$

Therefore,

$$\begin{aligned} \frac{\partial F}{\partial p} &= \frac{1}{36}(3p^2 - 8p + 3 + 6pq + 3q^2) \geq \frac{1}{36} (3p^2 - 8p + 3 + 6\sqrt{2}p + 6) \\ &= \frac{1}{36} (3p^2 + (6\sqrt{2} - 8)p + 9) > 0. \end{aligned}$$

So (3.7) remains true even for $(p, q) \in [3/2, 2] \times [\sqrt{2}, 2]$.

Now, let $p \in [3/2, 2]$. In this case,

$$F(p, q) = \frac{1}{12}(p + q) [6 - (p - q)^2] \leq 2$$

with the equality if $p = q = 2$.

The equalities $|p_1| = 2$ and $|b_2| = 2$ hold only for the functions

$$h(z) = \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} \quad \text{and} \quad g(z) = \frac{z}{(1 - ze^{i\phi})^2}, \quad \theta, \phi \in \mathbb{R}$$

respectively. This means that the equality in Theorem 3.1 holds for f given by

$$f'(z) = \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} \frac{1}{(1 - ze^{i\phi})^2}.$$

For this f , we have

$$a_4 - a_2a_3 = -e^{i(2\theta+\phi)} - \frac{5}{6}e^{i(\theta+2\phi)} - \frac{1}{6}e^{3i\theta}.$$

Thus,

$$|a_4 - a_2a_3| = \left| 1 + \cos(\phi - \theta) + \frac{2}{3}i \sin(\phi - \theta) \right|.$$

This expression is less than or equal to 2; the equality holds only when $\theta = \phi$. So we obtain that equality in Theorem 3.1 holds only for the function $f(z) = \frac{z}{(1 - ze^{i\phi})^2}$ with arbitrary $\phi \in \mathbb{R}$. □

Theorem 3.2 *If $f \in \mathcal{C}$ is given by (1.1), then*

$$|a_4 - a_2a_3| < 2.5.$$

Proof From (1.10) we have

$$2a_2 = b_2 + p_1 e^{-i\beta} \cos \beta, \quad 3a_3 = b_3 + (b_2 p_1 + p_2) e^{-i\beta} \cos \beta, \tag{3.8}$$

$$4a_4 = b_4 + (b_3 p_1 + b_2 p_2 + p_3) e^{-i\beta} \cos \beta.$$

Hence, using (3.8), we get

$$\begin{aligned} |a_4 - a_2a_3| &= \left| \frac{1}{4} (b_4 - \frac{2}{3}b_2b_3) + \frac{1}{4} (p_3 - \frac{2}{3}p_1p_2 e^{-i\beta} \cos \beta) e^{-i\beta} \cos \beta \right. \\ &\quad \left. + \frac{1}{12}p_1 (b_3 - b_2^2) e^{-i\beta} \cos \beta + \frac{1}{12}b_2 (p_2 - p_1^2 e^{-i\beta} \cos \beta) e^{-i\beta} \cos \beta \right. \\ &\quad \left. - \frac{1}{12}p_1b_2 (b_2 + p_1 e^{-i\beta} \cos \beta) e^{-i\beta} \cos \beta \right|. \end{aligned} \tag{3.9}$$

Applying the triangle inequality, we obtain

$$\begin{aligned} |a_4 - a_2a_3| &\leq \frac{1}{4} |b_4 - \frac{2}{3}b_2b_3| + \frac{1}{4} |p_3 - \frac{2}{3}p_1p_2 e^{-i\beta} \cos \beta| + \frac{1}{12} |p_1| |b_3 - b_2^2| \\ &\quad + \frac{1}{12} |b_2| |p_2 - p_1^2 e^{-i\beta} \cos \beta| + \frac{1}{12} |p_1| |b_2| |b_2 + p_1 e^{-i\beta} \cos \beta|. \end{aligned} \tag{3.10}$$

Using Lemma 2.2, we have

$$\begin{aligned} |p_2 - p_1^2 e^{-i\beta} \cos \beta| &= |(p_2 - p_1^2) \cos \beta + ip_2 \sin \beta| \\ &= |(\frac{1}{2}(4 - p_1^2)x - \frac{1}{2}p_1^2) \cos \beta + i(\frac{1}{2}(4 - p_1^2)x + \frac{1}{2}p_1^2) \sin \beta| \\ &= |\frac{1}{2}(4 - p_1^2)x e^{i\beta} - \frac{1}{2}p_1^2 e^{-i\beta}|. \end{aligned}$$

Since the functional $|p_2 - p_1^2 e^{-i\beta} \cos \beta|$ is invariant under rotation, we can assume (for a moment) that p_1 is a positive real number. In this case,

$$|\frac{1}{2}(4 - p_1^2)x e^{i\beta} - \frac{1}{2}p_1^2 e^{-i\beta}| \leq \frac{1}{2}(4 - p_1^2) + \frac{1}{2}p_1^2 = 2.$$

Hence, in general (for an arbitrary p_1)

$$|p_2 - p_1^2 e^{-i\beta} \cos \beta| \leq 2. \tag{3.11}$$

From Corollary 2.4 and Lemma 2.1, we get

$$\begin{aligned} |p_3 - \frac{2}{3}p_1 p_2 e^{-i\beta} \cos \beta| &= |(p_3 - \frac{2}{3}p_1 p_2) \cos \beta + ip_3 \sin \beta| \\ &\leq G(p) \cos \beta + 2|\sin \beta| \\ &\leq G(p) + 2, \end{aligned} \tag{3.12}$$

where $G(p)$ is given by (2.3) and $p = |p_1|$. Taking into account (3.5), (3.11), (3.12), and Lemma 2.5 and writing $|p_1| = p$, $|b_2| = q$, from (3.10), we obtain

$$|a_4 - a_2 a_3| \leq \frac{1}{4}H(q) + \frac{1}{4}G(p) + \frac{1}{2} + \frac{1}{12}p + \frac{1}{6}q + \frac{1}{12}pq(p + q) = F(p, q) + \frac{1}{2},$$

where $F(p, q)$ is given by (3.6). Since $F(p, q) \leq 2$ (see proof of Theorem 3.1), we obtain the declared bound. \square

However, the result in Theorem 3.2 is not sharp, it is the best known estimate for the whole class \mathcal{C} . Moreover, we conjecture that the exact bound is 2. This presumption is supported by the following theorem.

Theorem 3.3 *If $f \in \mathcal{C}(k)$ is given by (1.1), then*

$$|a_4 - a_2 a_3| \leq 2.$$

This result is sharp.

Proof From (3.9) for the Koebe function given by (1.3), we have

$$\begin{aligned} |a_4 - a_2 a_3| &= \cos \beta \left| \frac{1}{4}(p_3 - \frac{2}{3}p_1 p_2 e^{-i\beta} \cos \beta) - \frac{1}{12}p_1 + \frac{1}{6}(p_2 - p_1^2 e^{-i\beta} \cos \beta) \right. \\ &\quad \left. - \frac{1}{6}p_1 (2 + p_1 e^{-i\beta} \cos \beta) \right| \\ &= \cos \beta \left| \left[\frac{1}{4}(p_3 - \frac{2}{3}p_1 p_2) + \frac{1}{6}(p_2 - p_1^2) - \frac{1}{6}p_1^2 - \frac{5}{12}p_1 \right] \cos \beta \right. \\ &\quad \left. + i \left(\frac{1}{4}p_3 + \frac{1}{6}p_2 - \frac{5}{12}p_1 \right) \sin \beta \right|. \end{aligned} \tag{3.13}$$

Using Lemma 2.1 it is easy to check that

$$\left| \frac{1}{4}p_3 + \frac{1}{6}p_2 - \frac{5}{12}p_1 \right| \leq \frac{5}{3}. \tag{3.14}$$

Applying Corollary 2.4 and (3.4), we get

$$\left| \frac{1}{4} \left(p_3 - \frac{2}{3}p_1p_2 \right) + \frac{1}{6} \left(p_2 - p_1^2 \right) - \frac{1}{6}p_1^2 - \frac{5}{12}p_1 \right| \leq \frac{1}{4}G(p) + \frac{1}{3} + \frac{1}{6}p^2 + \frac{5}{12}p \equiv w(p),$$

where $G(p)$ is given by (2.3) and $|p_1| = p, p \in [0, 2]$. We have

$$w(p) = \begin{cases} \frac{1}{36}p^3 + \frac{1}{18}p^2 + \frac{5}{12}p + \frac{5}{6}, & p \in [0, 3/2], \\ -\frac{1}{12}p^3 + \frac{1}{6}p^2 + \frac{5}{6}p + \frac{1}{3}, & p \in [3/2, 2]. \end{cases}$$

The function w is increasing, so $w(p) \leq w(2) = 2$. Therefore,

$$\max \{w(p) : p \in [0, 2]\} = 2. \tag{3.15}$$

Applying (3.14) and (3.15) in (3.13), we obtain

$$|a_4 - a_2a_3|^2 \leq \cos^2 \beta \left(4 \cos^2 \beta + \frac{25}{9} \sin^2 \beta \right) \leq \cos^2 \beta \left(4 \cos^2 \beta + 4 \sin^2 \beta \right) = 4 \cos^2 \beta$$

Hence,

$$|a_4 - a_2a_3| \leq 2 \cos \beta \leq 2$$

and we get the desired result. □

Remark. In [15], Prajapat et al. proved that $|a_2a_3 - a_4| \leq 3$ in the class \mathcal{C}_0 . Our results $|a_2a_3 - a_4| \leq 2$ for the classes \mathcal{C}_0 and $\mathcal{C}(k)$ are sharp. Obtaining a sharp estimate for the class \mathcal{C} is still an open problem.

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