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Global attractors for the semilinear beam equation with localized viscosity

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Abstract: In this paper, we deal with the semilinear beam equation with localized viscosity. Under mild conditions on the viscous coefficient, we establish the well-posedness and boundedness of the weak solutions. Then we prove that the semigroup generated by this problem has a smooth global attractor in $H^3(0,1) \times H_0^1(0,1)$.

Key words: Beam equation, Boussinesq equation, Cahn–Hilliard equation, hyperbolic relaxation, global attractor

1. Introduction

This paper is devoted to the long-time behavior of the following semilinear beam equation with localized viscosity:

$$u_{tt} + u_{xxxx} - (a(x)\, u_t)_{xx} - (f(u))_{xx} = g(x), \quad (t,x) \in (0,\infty) \times (0,1). \tag{1.1}$$

Beam equations attract great interest from mathematicians due to their wide range of applications in diverse fields of science. For example, the Boussinesq equation,

$$u_{tt} + \alpha u_{xxxx} - u_{xx} - \beta \left(u^2\right)_{xx} = 0,\ \alpha > 0,\ \beta \in \mathbb{R}, \tag{1.2}$$

was introduced by Boussinesq (see [3]) to model shallow water wave propagation. Boussinesq equations find applications in numerous areas of physics, extending from wave propagation in shallow water to systems of nonlinear elastic beams. Among many papers related to (1.2), we confine ourselves to citing [7, 26–27]. The original Boussinesq equation can be generalized as follows:

$$u_{tt} + \alpha u_{xxxx} - u_{xx} - (f(u))_{xx} = 0. \tag{1.3}$$

Papers related to long-time dynamics of (1.3) can be exemplified by [1] and [19]. Due to the importance of viscosity in real processes, the following damped Boussinesq equation,

$$u_{tt} + \alpha u_{xxxx} - 2b u_{xxt} - u_{xx} - \beta \left(u^2\right)_{xx} = 0,\ b > 0,\ \alpha > 0,\ \beta \in \mathbb{R}, \tag{1.4}$$

is studied by some authors (see [6, 10, 23–25]). Equation (1.1) can be seen as a generalized version of the Boussinesq equation with localized internal damping.

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We also mention that by replacing the localized internal damping term $(a(x)u_t)_{xx}$ with frictional damping u_t in (1.1), we obtain the hyperbolic relaxation of the Cahn–Hilliard equation having the form

$$u_{tt} + u_{xxxx} + u_t - (f(u))_{xx} = g(x), \tag{1.5}$$

which was proposed by Galenko [8] to describe the rapid spinodal decomposition in materials like glasses. For contributions related to (1.5), we refer to [2, 8, 9, 11–15, 18, 21, 28].

The main novelty of this paper is that equation (1.1) includes localized viscosity, unlike the papers mentioned above. Namely, in the aforementioned papers viscosity is effective on the whole domain, but in this paper viscosity coefficient $a(x)$ can vanish on a set of nonzero measures (see (2.3)–(2.4)). This situation bears some difficulties. We overcome these obstacles by applying a multiplication technique and with the help of compact embedding theorems. This paper is organized as follows: In the second section, we give the statement of the problem and the main results. In the third section, the well-posedness and the boundedness of the weak solutions are proved. Finally, in the last section, the existence of the regular attractor is established.

2. Statement of the problem and main results

This paper is devoted to the following initial boundary value problem:

$$u_{tt} + u_{xxxx} - (a(x)u_t)_{xx} - (f(u))_{xx} = g(x), \quad (t,x) \in (0,\infty) \times (0,1),$$

$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0, \quad t \in (0,\infty),$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1).$$

Defining the linear positive operator $A : D(A) \to L^2(0,1)$ where $A = -\frac{d^2}{dx^2}$ and $D(A) = H^2(0,1) \cap H_0^1(0,1)$, we can restate the original problem as follows:

$$u_{tt} + A(Au + a(x)u_t + f(u)) = g(x), \quad (t,x) \in (0,\infty) \times (0,1), \tag{2.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1). \tag{2.2}$$

Here, $g \in L^2(\Omega)$ and the viscous coefficient $a(\cdot)$ satisfies the following conditions:

$$a \in W^{2,\infty}(0,1), \quad a(x) \geq 0 \text{ a.e. in } (0,1), \tag{2.3}$$

$$a(x) \geq \alpha_0 > 0 \text{ a.e. in } (r_0, r_1), \text{ for some } 0 \leq r_0 < r_1 \leq 1, \tag{2.4}$$

and there exists a constant $c > 0$ such that

$$a''(x) \leq c\sqrt{a(x)} \text{ a.e. in } (0,1). \tag{2.5}$$

Additionally, we assume that for the nonlinear function f the following holds:

$$f \in C^3(\mathbb{R}), \quad \liminf_{|s|\to\infty} \frac{f(s)}{s} > -\pi^2. \tag{2.6}$$

The following well-posedness theorem is the first main result of our paper.

Theorem 2.1 *Let assumptions (2.3)–(2.6) hold. Then, for every $T>0$ and $(u_0,u_1)\in(H^2\ (0,1)\cap H_0^1\ (0,1))\times L^2(0,1)$, the problem (2.1)–(2.2) has a unique weak solution u belonging to the class $C([0,T];H^2\ (0,1)\cap H_0^1\ (0,1))\cap C^1\left([0,T];L^2(0,1)\right)$ and satisfying the inequality*

$$\|u(t)\|_{H^2(0,1)}+\|u_t(t)\|_{L^2(0,1)}\leq c_1\left(\|(u_0,u_1)\|_{H^2(0,1)\times L^2(0,1)}\right),\ \ \forall t\geq 0,$$

where $c_1:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing function. Moreover, if $v,\ w\in C([0,T];H^2\ (0,1)\cap H_0^1\ (0,1))\cap C^1\left([0,T];L^2(0,1)\right)$ are the weak solutions of the problem (2.1)–(2.2) with initial data $(v_0,v_1)\in(H^2\ (0,1)\cap H_0^1\ (0,1))\times L^2(0,1)$ and $(w_0,w_1)\in(H^2\ (0,1)\cap H_0^1\ (0,1))\times L^2(0,1)$, then

$$\|v(t)-w(t)\|_{H^1(0,1)}+\|v_t(t)-w_t(t)\|_{H^{-1}(0,1)}$$

$$\leq c_2(T,r)\left[\|v_0-w_0\|_{H^1(0,1)}+\|v_1-w_1\|_{H^{-1}(0,1)}\right],\ \forall t\in[0,T],$$

where $c_2:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing function with respect to the each variable and

$$r=\max\left\{\|(v_0,v_1)\|_{H^2(0,1)\times L^2(0,1)},\|(w_0,w_1)\|_{H^2(0,1)\times L^2(0,1)}\right\}.$$

Hence, we observe that the problem (2.1)–(2.2) generates a weakly continuous semigroup $\{S(t)\}_{t\geq 0}$ in $\left(H^2(0,1)\cap H_0^1(0,1)\right)\times L^2(0,1)$, given by the formula $S(t)(u_0,u_1)=(u(t),u_t(t))$, where $u(t,x)$ is a weak solution determined by Theorem 2.1.

The second main result of our paper is as follows:

Theorem 2.2 *Assume that assumptions (2.3)–(2.6) hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by the problem (2.1)–(2.2) possesses a global attractor $\mathcal{A}$ in $\left(H^2(0,1)\cap H_0^1(0,1)\right)\times L^2(0,1)$. Moreover, $\mathcal{A}$ is bounded in $H^3(0,1)\times H_0^1(0,1)$.*

3. Well-posedness

We start with the following theorem.

Theorem 3.1 *Let assumptions (2.3)–(2.6) hold. Then for every $T>0$ and $(u_0,u_1)\in\left(H^2(0,1)\cap H_0^1(0,1)\right)\times L^2(0,1)$, the problem (2.1)–(2.2) has a weak solution u belonging to the class $C\left([0,T];\left(H^2(0,1)\cap H_0^1(0,1)\right)\right)\cap C^1\left([0,T];L^2(0,1)\right)$ and satisfying the inequality*

$$\|u(t)\|_{H^2(0,1)}+\|u_t(t)\|_{L^2(0,1)}\leq c_1\left(\|(u_0,u_1)\|_{H^2(0,1)\times L^2(0,1)}\right),\ \ \forall t\geq 0,$$

where $c_1:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing function.

Proof First, we will deal with the strong solutions, and then by using the density argument we will obtain the desired result for weak solutions. Namely, assume that $(u_0,u_1)\in\mathcal{H}_1:=\{(u,v)\in(H^4(0,1)\cap H_0^1(0,1))\times$

$\left(H^{2}(0,1)\cap H_{0}^{1}(0,1)\right):\ u_{xx}(t,0)=u_{xx}(t,1)=0,\quad t\in(0,\infty)\}$. Then the problem (2.1)–(2.2) can be reduced to the following initial value problem in $\mathcal{H}:=\left(H^{2}(0,1)\cap H_{0}^{1}(0,1)\right)\times L^{2}(0,1)$:

$$\begin{cases} \frac{d}{dt}\left(u(t),u_{t}(t)\right)=\mathcal{B}\left(u(t),u_{t}(t)\right)+\Phi\left(u(t),u_{t}(t)\right), & \forall t>0,\\ \left(u(0),u_{t}(0)\right)=\left(u_{0},u_{1}\right), \end{cases} \tag{3.1}$$

where $D(\mathcal{B})=\mathcal{H}_{1}$, $\mathcal{B}(w_{1},w_{2})=\left(w_{2},-A^{2}w_{1}-A\left(a(x)w_{2}\right)\right)$ and $\Phi(w_{1},w_{2})=\left(0,-A\left(f(w_{1})\right)+g\right)$. It is easy to show that $\mathcal{B}$ is a maximal dissipative operator in $\mathcal{H}$. Thus, due to the Lumer–Phillips theorem (see [20, Theorem 4.3]), it generates a linear continuous semigroup $\left\{e^{t\mathcal{B}}\right\}_{t\geq0}$ in $\mathcal{H}$ and $\mathcal{H}_{1}$. Also, one can easily prove that the operator $\Phi:\mathcal{H}\to\mathcal{H}$ is Lipschitz continuous. Hence, applying semigroup theory (see [4, Theorem 4.3.4 and Proposition 4.3.9]), for every $(u_{0},u_{1})\in\mathcal{H}_{1}$ the problem (3.1), and consequently (2.1)–(2.2), have a unique strong solution (u,u_{t}) belonging to the class $C\left([0,T_{\max});\mathcal{H}_{1}\right)\cap C^{1}\left([0,T_{\max});\mathcal{H}\right)$. Let $u(t,x)$ be a local strong solution of (2.1)–(2.2) in $(0,T_{\max})\times(0,1)$; then, multiplying (2.1) by $A^{-1}u_{t}$ and integrating over $(0,t)\times(0,1)$, we obtain

$$\frac{1}{2}\left\|A^{-\frac{1}{2}}u_{t}(t)\right\|_{L^{2}(0,1)}^{2}+\frac{1}{2}\left\|A^{\frac{1}{2}}u(t)\right\|_{L^{2}(0,1)}^{2}+\int_{0}^{1}\left(F\left(u(t,x)\right)-f(0)\,u(t,x)\right)dx$$

$$-\int_{0}^{1}g(x)\,A^{-1}u(t,x)\,dx+\int_{0}^{t}\int_{0}^{1}a(x)\left|u_{t}(\tau,x)\right|^{2}dxd\tau$$

$$=\frac{1}{2}\left\|A^{-\frac{1}{2}}u_{1}\right\|_{L^{2}(0,1)}^{2}+\frac{1}{2}\left\|A^{\frac{1}{2}}u_{0}\right\|_{L^{2}(0,1)}^{2}+\int_{0}^{1}\left(F\left(u_{0}(x)\right)-f(0)\,u_{0}(x)\right)dx$$

$$-\int_{0}^{1}g(x)\,A^{-1}u_{0}(x)\,dx,\quad 0\leq t<T_{\max}, \tag{3.2}$$

where $F(z)=\int_{0}^{z}f(t)\,dt$. By the conditions of the theorem, we infer from (3.2) that

$$\left\|u_{t}(t)\right\|_{H^{-1}(0,1)}^{2}+\left\|u(t)\right\|_{H^{1}(0,1)}^{2}$$

$$+\int_{0}^{t}\int_{0}^{1}a(x)\left|u_{t}(\tau,x)\right|^{2}dxd\tau\leq Q\left(\left\|(u_{0},u_{1})\right\|_{\mathcal{H}}\right),\qquad\forall t\in[0,T_{\max}), \tag{3.3}$$

where $Q:\mathbb{R}_{+}\to\mathbb{R}_{+}$ is a nondecreasing function. Next, multiplying (2.1) by $2u_{t}$ and integrating over $(0,1)$, with the help of (3.3) and the embedding $H^{1}(0,1)\hookrightarrow L^{\infty}(0,1)$, we have

$$\frac{d}{dt}\left(\left\|u_{t}(t)\right\|_{L^{2}(0,1)}^{2}+\left\|u_{xx}(t)\right\|_{L^{2}(0,1)}^{2}+\int_{0}^{1}f'\left(u(t,x)\right)\left|u_{x}(t,x)\right|^{2}dx\right)+2\int_{0}^{1}a(x)\left|u_{xt}(t,x)\right|^{2}dx$$

$$\leq\int_{0}^{1}f''\left(u(t,x)\right)u_{t}(t,x)\left|u_{x}(t)\right|^{2}dx+\int_{0}^{1}a''(x)\left|u_{t}(t,x)\right|^{2}dx+2\int_{0}^{1}h(x)\,u_{t}(t,x)\,dx$$

$$\leq c_1 \|u_t(t)\|_{L^2(0,1)} \|u_x(t)\|^2_{L^4(0,1)} + c_1 \left(1 + \|au_t(t)\|_{L^2(0,1)}\right) \|u_t(t)\|_{L^2(0,1)}$$

$$\leq c_2 \|u_t(t)\|_{L^2(0,1)} \|u_x(t)\|^2_{H^{1/4}(0,1)} + c_1 \left(1 + \|au_t(t)\|_{L^2(0,1)}\right) \|u_t(t)\|_{L^2(0,1)}$$

$$\leq c_2 \|u_t(t)\|_{L^2(0,1)} \|u_x(t)\|^{3/2}_{L^2(0,1)} \|u_x(t)\|^{1/2}_{H^1(0,1)} + c_1 \left(1 + \|au_t(t)\|_{L^2(0,1)}\right) \|u_t(t)\|_{L^2(0,1)}$$

Then, recalling (3.3) again, we deduce

$$\frac{d}{dt}\left(\|u_t(t)\|^2_{L^2(0,1)} + \|u_{xx}(t)\|^2_{L^2(0,1)} + \int_0^1 f'(u(t,x))|u_x(t,x)|^2 dx\right) + 2\int_0^1 a(x)|u_{xt}(t,x)|^2 dx$$

$$\leq c_3 \|u_t(t)\|_{L^2(0,1)} \|u(t)\|^{1/2}_{H^2(0,1)} + c_1 \left(1 + \|au_t(t)\|_{L^2(0,1)}\right) \|u_t(t)\|_{L^2(0,1)}. \tag{3.4}$$

Let $\zeta \in C^\infty([0,1])$, $0 \leq \zeta(x) \leq 1$ and $\zeta(x) = \begin{cases} 1, & 0 \leq x \leq r_0 \\ 0, & \frac{r_0+r_1}{2} \leq x \leq r_1 \end{cases}$. Now, multiplying (2.1) by $\varepsilon\zeta^2 x u_x$ and integrating over $(0,1)$, we get

$$\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\zeta^2(x)xu_x(t,x)dx - \frac{1}{2}\int_0^1 a'(x)\zeta^2(x)x|u_x(t,x)|^2 dx\right)$$

$$+\frac{\varepsilon}{2}\|\zeta u_t(t)\|^2_{L^2(0,1)} + \varepsilon\int_0^1 \zeta(x)\zeta'(x)x|u_t(t,x)|^2 dx$$

$$-\frac{\varepsilon}{2}\int_0^1 \left(\zeta^2(x)x\right)'''|u_x(t,x)|^2 dx + 3\varepsilon\int_0^1 \zeta(x)\zeta'(x)x|u_{xx}(t,x)|^2 dx$$

$$+\varepsilon\frac{3}{2}\|\zeta u_{xx}(t)\|^2_{L^2(0,1)} + \varepsilon\int_0^1 f'(u(t,x))|u_x(t,x)|^2\left(2\zeta(x)\zeta'(x)x + \zeta^2(x)\right)dx$$

$$+\varepsilon\int_0^1 f'(u(t,x))u_x(t,x)\zeta^2(x)xu_{xx}(t,x)dx + \varepsilon\int_0^1 a'(x)u_t(t,x)2\zeta(x)\zeta'(x)xu_x(t,x)dx$$

$$+\varepsilon\int_0^1 a(x)u_{tx}(t,x)2\zeta(x)\zeta'(x)xu_x(t,x)dx + \varepsilon\int_0^1 a'(x)u_t(t,x)\zeta^2(x)u_x(t,x)dx$$

$$+\varepsilon\int_0^1 a(x)u_{tx}(t,x)\zeta^2(x)u_x(t,x)dx - \varepsilon\int_0^1 \left(a'(x)\zeta^2(x)x\right)_x u_t(t,x)u_x(t,x)dx$$

$$+\varepsilon\int_0^1 a(x)u_{xt}(t,x)\zeta^2(x)xu_{xx}(t,x)dx = \varepsilon\int_0^1 h(x)\zeta^2(x)xu_x(t,x)dx.$$

Hence, by (2.3), (2.4), and (2.6), and using the embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$, applying the Young inequality, we deduce

$$\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\zeta^2(x)\,xu_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\zeta^2(x)\,x\,|u_x(t,x)|^2\,dx\right)$$

$$+\varepsilon\|\zeta u_{xx}(t)\|^2_{L^2(0,1)}+\varepsilon\|\zeta u_t(t)\|^2_{L^2(0,1)}$$

$$\le\varepsilon c_4\left(1+\|u_t(t)\|^2_{L^2(r_0,r_1)}+\left\|\sqrt{a}u_{xx}(t)\right\|^2_{L^2(0,1)}+\left\|\sqrt{a}u_{tx}(t)\right\|^2_{L^2(0,1)}\right). \tag{3.5}$$

Next, assume that $\eta\in C^\infty([0,1])$, $0\le\eta(x)\le1$ and $\eta(x)=\begin{cases}0, & 0\le x\le\frac{r_0+r_1}{2}\\ 1, & r_1\le x\le1\end{cases}$. Then, multiplying (2.1) by $\varepsilon\eta^2(x-1)\,u_x$ and applying similar arguments used for the multiplier ζ^2xu_x, we get

$$\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\eta^2(x-1)\,u_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\eta^2(x-1)\,|u_x(t,x)|^2\,dx\right)$$

$$+\varepsilon\|\eta u_{xx}(t)\|^2_{L^2(0,1)}+\varepsilon\|\eta u_t(t)\|^2_{L^2(0,1)}$$

$$\le\varepsilon c_5\left(1+\|u_t(t)\|^2_{L^2(r_0,r_1)}+\left\|\sqrt{a}u_{xx}(t)\right\|^2_{L^2(0,1)}+\left\|\sqrt{a}u_{tx}(t)\right\|^2_{L^2(0,1)}\right).$$

Then, adding the last inequality to (3.5), we find that

$$\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\zeta^2(x)\,xu_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\zeta^2(x)\,x\,|u_x(t,x)|^2\,dx\right)$$

$$\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\eta^2(x-1)\,u_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\eta^2(x-1)\,|u_x(t,x)|^2\,dx\right)$$

$$+\varepsilon\|u_{xx}(t)\|^2_{L^2((0,1)\backslash(r_0,r_1))}+\varepsilon\|u_t(t)\|^2_{L^2((0,1)\backslash(r_0,r_1))}$$

$$\le\varepsilon c_6\left(1+\|u_t(t)\|^2_{L^2(r_0,r_1)}+\left\|\sqrt{a}u_{xx}(t)\right\|^2_{L^2(0,1)}+\left\|\sqrt{a}u_{tx}(t)\right\|^2_{L^2(0,1)}\right). \tag{3.6}$$

Next, multiplying (2.1) by $a(x)\,u$ and integrating over $(0,1)$, we get

$$\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,a(x)\,u(t,x)\,dx+\|a'u(t)\|^2_{L^2(0,1)}\right)+\left\|\sqrt{a}u_{xx}(t)\right\|^2_{L^2(0,1)}$$

$$\le\left\|\sqrt{a}u_t(t)\right\|^2_{L^2(0,1)}+\left|\int_0^1 u_{xx}(t,x)\,a''(x)\,u(t,x)\,dx\right|$$

$$+\left\|\sqrt{|a''|}u_x(t)\right\|^2_{L^2(0,1)}+\left|\int_0^1 f'(u(t,x))\,u_x(t,x)\,a(x)\,u_x(t,x)\,dx\right|$$

$$+\left|\int_0^1 f'(u(t,x))\,u_x(t,x)\,a'(x)\,u(t,x)\,dx\right|+\left|\int_0^1 a'(x)\,a(x)\,u_t(t,x)\,u_x(t,x)\,dx\right|$$

$$+\left|\int_0^1 a(x)\,a'(x)\,u(t,x)\,u_{tx}(t,x)\,dx\right|+\left|\int_0^1 a^2(x)\,u_x(t,x)\,u_{tx}(t,x)\,dx\right|.$$

Taking into account (2.4) and (2.6), and again with the help of the embedding $H^1(0,1)\hookrightarrow L^\infty(0,1)$, from (3.3) and the last inequality, it follows that

$$\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,a(x)\,u(t,x)\,dx+\|a'u(t)\|^2_{L^2(0,1)}\right)+\left\|\sqrt{a}u_{xx}(t)\right\|^2_{L^2(0,1)}$$

$$\le c_7\left(1+\left\|\sqrt{a}u_t(t)\right\|^2_{L^2(0,1)}\right)+c_7\left\|\sqrt{a}u_{tx}(t)\right\|_{L^2(0,1)}. \tag{3.7}$$

Then, summing (3.3), (3.4), (3.6), and (3.7), choosing ε small enough and applying the Young inequality, we have

$$\frac{d}{dt}\left(\|u_t(t)\|^2_{L^2(0,1)}+\|u_{xx}(t)\|^2_{L^2(0,1)}+\int_0^1 f'(u(t,x))\,|u_x(t,x)|^2\,dx\right)$$

$$+\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\zeta^2(x)\,xu_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\zeta^2(x)\,x\,|u_x(t,x)|^2\,dx\right)$$

$$+\varepsilon\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,\eta^2(x-1)\,u_x(t,x)\,dx-\frac{1}{2}\int_0^1 a'(x)\,\eta^2(x-1)\,|u_x(t,x)|^2\,dx\right)$$

$$+\frac{d}{dt}\left(\int_0^1 u_t(t,x)\,a(x)\,u(t,x)\,dx+\|a'u(t)\|^2_{L^2(0,1)}\right)$$

$$+\|u_t(t)\|^2_{L^2(0,1)}+\|u_{xx}(t)\|^2_{L^2(0,1)}+\int_0^1 a(x)\,|u_{tx}(t,x)|^2\,dx$$

$$\le c_8\left(1+\|u_t(t)\|^2_{L^2(r_0,r_1)}+\left\|\sqrt{a}u_t(t)\right\|^2_{L^2(0,1)}\right). \tag{3.8}$$

From the last inequality and (2.4), it follows that

$$\frac{d}{dt}\Psi(u(t))+c_9E(u(t))\le c_{10}\left(1+\left\|\sqrt{a}u_t(t)\right\|^2_{L^2(0,1)}\right), \tag{3.9}$$

where $\Psi(u(t)) = \|u_t(t)\|^2_{L^2(0,1)} + \|u_{xx}(t)\|^2_{L^2(0,1)} + \|a'u(t)\|^2_{L^2(0,1)} + \int_0^1 f'(u(t,x))|u_x(t,x)|^2 dx$
$+\varepsilon\int_0^1 u_t(t,x)\zeta^2(x)xu_x(t,x)dx - \frac{\varepsilon}{2}\int_0^1 a'(x)\zeta^2(x)x|u_x(t,x)|^2 dx + \varepsilon\int_0^1 u_t(t,x)\eta^2(x-1)u_x(t,x)dx -$
$\frac{\varepsilon}{2}\int_0^1 a'(x)\eta^2(x-1)|u_x(t,x)|^2 dx + \int_0^1 u_t(t,x)a(x)u(t,x)dx + \|a'u(t)\|^2_{L^2(0,1)}$ and $E(u(t)) = \|u_{xx}(t)\|^2_{L^2(0,1)} +$
$\|u_t(t)\|^2_{L^2(0,1)}$. It is easy to show that

$$\mu_1 E(u(t)) - K \le \Psi(u(t)) \le \mu_2 E(u(t)) + K, \tag{3.10}$$

for some $0 < \mu_1 < \mu_2$ and $K > 0$. Then considering (3.10) in (3.9), we infer

$$\frac{d}{dt}\Psi(u(t)) + c_{11}\Psi(u(t)) \le c_{12}\left(1 + \left\|\sqrt{a}u_t(t)\right\|^2_{L^2(0,1)}\right)$$

which yields

$$\Psi(u(t)) \le e^{-c_{11}t}\Psi(u(0)) + c_{12}e^{-c_{11}t}\int_0^t e^{-c_{11}s}\left(1 + \left\|\sqrt{a}u_t(s)\right\|^2_{L^2(0,1)}\right) ds$$

and consequently, by (3.3), the following holds:

$$E(u(t)) \le c_{13}. \tag{3.11}$$

Now, by using the density argument, we will prove (3.11) for the weak solutions of (2.1)–(2.3). Let $(u_0, u_1) \in \mathcal{H}$. Since $\mathcal{H}_1$ is dense in $\mathcal{H}$, there exists a sequence $\{(u_{0n}, u_{1n})\}_{n=1}^{\infty} \subset \mathcal{H}_1$ such that $(u_{0n}, u_{1n}) \to (u_0, u_1)$ strongly in $\mathcal{H}$. Hence, following steps similar to those outlined above, for the problems

$$\begin{cases} u_{ntt} + A(Au_n + f(u_n) + a(x)u_{nt}) = h(x), \quad (t,x) \in (0,\infty) \times (0,1), \\ u_n(0,x) = u_{0n}(x), \quad u_{nt}(0,x) = u_{1n}(x), \quad x \in (0,1), \end{cases} \tag{3.12}$$

we readily obtain that

$$E(u_n(t)) \le M, \quad \forall t \ge 0, \tag{3.13}$$

where M only depends on $\|(u_0, u_1)\|_{\mathcal{H}_1}$ and is independent of n. Also, with the help of (3.13), from $(3.12)_1$ we infer that

$$\left\|A^{-1}u_{ntt}\right\|_{L^2(0,1)} \le \widehat{M}, \quad \forall t \ge 0.$$

Then from (3.13) and (3.14), it follows that the sequence $\{(u_n, u_{nt})\}_{n=1}^{\infty}$ has a weakly star convergent subsequence in $L^\infty(0,\infty;\mathcal{H}) \cap W^{1,\infty}\left(0,\infty; L^2(\Omega) \times D\left(A^{-1}\right)\right)$. Without loss of generality, denote this subsequence again by $\{(u_n, u_{nt})\}_{n=1}^{\infty}$. Then we have

$$\begin{cases} u_n \to u \quad \text{weakly star in } L^\infty\left(0,\infty; H^2(0,1) \cap H_0^1(0,1)\right), \\ u_{nt} \to u_t \quad \text{weakly star in } L^\infty\left(0,\infty; L^2(0,1)\right), \\ u_{ntt} \to u_{tt} \quad \text{weakly star in } L^\infty\left(0,\infty; D\left(A^{-1}\right)\right), \end{cases} \tag{3.14}$$

where $(u,u_t) \in L^{\infty}(0,\infty;\mathcal{H}) \cap W^{1,\infty}\left(0,\infty;L^2(\Omega)\times D\left(A^{-1}\right)\right)$. From $(3.14)_1$ and $(3.14)_2$, in virtue of [22, Corallary 4], it follows that

$$u_n \to u \quad \text{strongly in } C\left([0,T];H^{2-\varepsilon}(0,1)\right),\ \forall\varepsilon>0.$$

Moreover, from the last limit we can conclude that

$$u_n \to u \quad \text{strongly in } C\left([0,T]\times[0,1]\right), \tag{3.15}$$

for every $T \geq 0$. On the other hand, we get the following equation from $(3.12)_1$:

$$(u_n-u_m)_{tt} + A\left(A\left(u_n-u_m\right)+f\left(u_n\right)-f\left(u_m\right)+a\left(x\right)\left(u_{nt}-u_{mt}\right)\right)=0.$$

Testing the last equation with $2\left(u_{nt}-u_{mt}\right)$, and taking into account (3.13), we then obtain that

$$\left\|\left(u_{nxx}-u_{mxx}\right)(t)\right\|^2_{L^2(0,1)} + \left\|\left(u_{nt}-u_{mt}\right)(t)\right\|^2_{L^2(0,1)}$$

$$\leq \left\|\left(u_{0n}-u_{0m}\right)_{xx}\right\|^2_{L^2(0,1)} + \left\|\left(u_{1n}-u_{1m}\right)_{t}\right\|^2_{L^2(0,1)}$$

$$+c_{14}\int_0^t \left(\left\|\left(u_{nxx}-u_{mxx}\right)(s)\right\|^2_{L^2(0,1)} + \left\|\left(u_{nt}-u_{mt}\right)(s)\right\|^2_{L^2(0,1)}\right)ds$$

$$+c_{14}T\left(\left\|f_n'\left(u_n\right)-f_m'\left(u_m\right)\right\|_{C([0,T]\times[0,1])} + \left\|f_n''\left(u_n\right)-f_m''\left(u_m\right)\right\|_{C([0,T]\times[0,1])}\right),\ \forall t\in[0,T].$$

Hence, with the help of Gronwall's lemma and (3.15), it readily follows that

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \left\|E\left(u_n-u_m\right)\right\|_{C([0,T])}=0,\ \forall T\geq 0,$$

and consequently

$$\left(u_n,u_{nt}\right)\to\left(u,u_t\right) \quad \text{strongly in } C\left([0,T];\mathcal{H}\right). \tag{3.16}$$

Thus, considering $(3.14)_3$ and (3.16) and passing to the limit in (3.12) and (3.13), we complete the proof of the theorem. □

Theorem 3.2 *Let $v(t,x)$ and $w(t,x)$ be the weak solutions of (2.1)–(2.3) in $[0,T]\times(0,1)$, with the initial data (v_0,v_1) and (w_0,w_1). Then the following inequality holds:*

$$\left\|v(t)-w(t)\right\|_{H^1(0,1)} + \left\|v_t(t)-w_t(t)\right\|_{H^{-1}(0,1)}$$

$$\leq c(T,r)\left[\left\|v_0-w_0\right\|_{H^1(0,1)} + \left\|v_1-w_1\right\|_{H^{-1}(0,1)}\right],\ \forall t\in[0,T], \tag{3.17}$$

where $c:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing function with respect to the each variable and

$$r=\max\left\{\left\|(v,v_t)\right\|_{C([0,T];\mathcal{H})},\ \left\|(w,w_t)\right\|_{C([0,T];\mathcal{H})}\right\}.$$

Proof Let us set the function $u := v - w$. Then $u \in C\left([0,T]; H^2(0,1) \cap H_0^1(0,1)\right)$ $\cap C^1\left([0,T]; L^2(0,1)\right) \cap W^{2,\infty}\left(0,\infty; D\left(A^{-1}\right)\right)$ and u is the solution of the following problem:

$$\begin{cases} u_{tt} + A^2 u + A\left(f(v) - f(w)\right) + A\left(a(x)\,u_t\right) = 0, \\ u(0) = v_0 - w_0, \qquad u_t(0) = v_1 - w_1. \end{cases} \tag{3.18}$$

Testing $(3.18)_1$ with $A^{-1}u_t$ and considering (3.18), we obtain that

$$\left\|A^{-\frac{1}{2}}u_t(t)\right\|^2_{L^2(0,1)} + \left\|A^{\frac{1}{2}}u(t)\right\|^2_{L^2(0,1)} \le \left\|A^{-\frac{1}{2}}(v_1 - w_1)\right\|^2_{L^2(0,1)}$$

$$+\left\|A^{\frac{1}{2}}(v_0 - w_0)\right\|^2_{L^2(0,1)} + c_1\int_0^t \left\|A^{-\frac{1}{2}}u_t(\tau)\right\|^2_{L^2(0,1)} d\tau + c_1\int_0^t \left\|A^{\frac{1}{2}}u(\tau)\right\|^2_{L^2(0,1)} d\tau. \tag{3.19}$$

Therefore, by applying Gronwall's inequality, we obtain (3.17) and the proof of the theorem is complete. □

Consequently, Theorem 3.1, together with Theorem 3.2, proves Theorem 2.1.

4. Existence of the smooth global attractor

Firstly, we prove the following asymptotic compactness lemma by using the idea in [16].

Lemma 4.1 *Let conditions (2.4)–(2.6) hold and let B be a bounded subset of $\mathcal{H}$. Then every sequence of the form $\{S(t_k)\varphi_k\}_{k=1}^{\infty}$, where $\{\varphi_k\}_{k=1}^{\infty} \subset B$, $t_k \to \infty$, has a convergent subsequence in $\mathcal{H}$.*

Proof Due to Theorem 3.1, the sequence $\{S(\cdot)\varphi_k\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,\infty;\mathcal{H})$. Thus, for every $T > 0$ there exists a subsequence $\{k_m\}_{m=1}^{\infty}$ such that $t_{k_m} \ge T$ and

$$\begin{cases} u_m \to u \text{ weakly star in } L^\infty\left(0,\infty; H^2(0,1) \cap H_0^1(0,1)\right), \\ u_{mt} \to u_t \text{ weakly star in } L^\infty\left(0,\infty; L^2(0,1)\right), \\ u_m \to u \text{ strongly in } C\left([0,T]; H^{2-\varepsilon}(0,1)\right), \end{cases} \tag{4.1}$$

for some $u \in L^\infty\left(0,\infty; H^2(0,1) \cap H_0^1(0,1)\right) \cap W^{1,\infty}\left(0,\infty; L^2(0,1)\right)$, where $(u_m(t), u_{mt}(t)) = S(t + t_{k_m} - T)\varphi_{k_m}$. With the help of (3.3), we find that

$$\int_0^t \left\|\sqrt{a}u_{mt}(s)\right\|^2_{L^2(0,1)} ds \le c_1, \ \forall t \ge 0. \tag{4.2}$$

Now, by (2.1), we have

$$(u_{ntt} - u_{mtt}) + A^2(u_n - u_m) + A\left(f(u_n) - f(u_m)\right) + A\left(a(x)(u_{nt} - u_{mt})\right) = 0. \tag{4.3}$$

With multiplication of (4.3) by $(u_{nt} - u_{mt})$ and integrating over $(0,1)\times(0,t)$, we readily get

$$\int_0^t \left\|\sqrt{a}\left(u_{ntx}(s) - u_{mtx}(s)\right)\right\|^2_{L^2(0,1)} ds \le c_2\left(1 + \int_0^t \left\|u_{nt}(s) - u_{mt}(s)\right\|_{L^2(0,1)} ds\right).$$

Then, multiplying (4.3) by $\varepsilon\left(\zeta^2 x+\eta^2(x-1)\right)(u_{nx}-u_{mx})$, integrating over $(0,1)\times(0,t)$, and considering the last estimate, we have

$$\varepsilon\int_0^t\left(\|u_{nxx}(s)-u_{mxx}(s)\|^2_{L^2((0,1)\backslash(r_0,r_1))}+\|u_{nt}(s)-u_{mt}(s)\|^2_{L^2((0,1)\backslash(r_0,r_1))}\right)ds$$

$$\leq\varepsilon\int_0^t\left(\left\|\sqrt{a}\left(u_{nxx}(s)-u_{mxx}(s)\right)\right\|^2_{L^2(0,1)}+\|u_{nt}(s)-u_{mt}(s)\|^2_{L^2(r_0,r_1)}\right)ds$$

$$+\varepsilon c_3 F\left(u_n(t),u_m(t)\right)+\varepsilon c_3\left(1+\int_0^t\|u_{nt}(s)-u_{mt}(s)\|_{L^2(0,1)}ds\right), \tag{4.4}$$

where $F\left(u_n(t),u_m(t)\right)=\int_0^t(\|f''(u_n(s))-f''(u_m(s))\|_{L^\infty(0,1)}+\|f'(u_n(s))-f'(u_m(s))\|_{L^\infty(0,1)})ds+$ $\int_0^t\|u_n(s)-u_m(s)\|_{H^1(0,1)}ds+\|u_n-u_m\|_{C([0,T];H^1(0,1))}$ and from (4.1)

$$\limsup_{n\to\infty}\ \limsup_{m\to\infty}F\left(u_n(t),u_m(t)\right)=0. \tag{4.5}$$

After that, multiplying (4.3) by $a(x)(u_n-u_m)$ and integrating over $(0,1)\times(0,t)$, the following holds:

$$\int_0^t\left\|\sqrt{a}\left(u_{nxx}(s)-u_{mxx}(s)\right)\right\|^2_{L^2(0,1)}ds\leq c_4 F\left(u_n(t),u_m(t)\right)$$

$$+c_4\int_0^t\left\|\sqrt{a}\left(u_{nt}(s)-u_{mt}(s)\right)\right\|^2_{L^2(0,1)}ds. \tag{4.6}$$

Thus, summing (4.4) and (4.6), and exploiting (4.2) and (4.5), for sufficiently small ε, we deduce that

$$\limsup_{n\to\infty}\ \limsup_{m\to\infty}\int_0^t E\left(u_n(s)-u_m(s)\right)ds\leq c_5,\ \forall t\geq 0. \tag{4.7}$$

Now, multiplying (4.2) by $2t(u_{nt}-u_{mt})$ and integrating over $(0,T)\times(0,1)$, from (2.5), it follows that

$$TE\left(u_n(T)-u_m(T)\right)+2\int_0^T t\left\|\sqrt{a}\left(u_{ntx}(t)-u_{mtx}(t)\right)\right\|^2_{L^2(0,1)}dt$$

$$\leq\int_0^T E\left(u_n(t)-u_m(t)\right)dt+c_6(T+1)F\left(u_n(t),u_m(t)\right)$$

$$+c_6\int_0^T t\left\|\sqrt{a}\left(u_{nt}\left(t\right)-u_{mt}\left(t\right)\right)\right\|_{L^2(0,1)}\left\|\left(u_{nt}\left(s\right)-u_{mt}\left(s\right)\right)\right\|_{L^2(0,1)}dt. \tag{4.8}$$

Then, multiplying (4.2) by $\varepsilon\left(\zeta^2 x+\eta^2\left(x-1\right)\right)t\left(u_{nx}-u_{mx}\right)$ and integrating over $(0,T)\times(0,1)$, we infer that

$$\varepsilon\int_0^T t\left(\left\|u_{nxx}\left(t\right)-u_{mxx}\left(t\right)\right\|^2_{L^2((0,1)\setminus(r_0,r_1))}+\left\|u_{nt}\left(t\right)-u_{mt}\left(t\right)\right\|^2_{L^2((0,1)\setminus(r_0,r_1))}\right)$$

$$\leq\varepsilon c_7\int_0^T t\left(\left\|u_{nxx}\left(t\right)-u_{mxx}\left(t\right)\right\|^2_{L^2(r_0,r_1)}+\left\|u_{nt}\left(t\right)-u_{mt}\left(t\right)\right\|^2_{L^2(r_0,r_1)}\right)dt$$

$$+\varepsilon c_7 F\left(u_n\left(T\right),u_m\left(T\right)\right)+\varepsilon c_7\int_0^T t\left\|\sqrt{a}\left(u_{ntx}\left(t\right)-u_{mtx}\left(t\right)\right)\right\|^2_{L^2(0,1)}dt$$

$$+\varepsilon c_7\int_0^T t\left\|\sqrt{a}\left(u_{nxx}\left(t\right)-u_{mxx}\left(t\right)\right)\right\|^2_{L^2(0,1)}dt. \tag{4.9}$$

Next, multiplying (4.2) by $ta\left(x\right)u$ and integrating over $(0,T)\times(0,1)$, the following inequality holds:

$$\int_0^T t\left\|u_{nxx}\left(t\right)-u_{mxx}\left(t\right)\right\|^2_{L^2((0,1))}\leq c_8 F\left(u_n\left(T\right),u_m\left(T\right)\right)$$

$$+c_8\int_0^T t\left\|\sqrt{a}\left(u_{nt}\left(t\right)-u_{mt}\left(t\right)\right)\right\|^2_{L^2(0,1)}dt+c_8\int_0^T t\left\|\sqrt{a}\left(u_{ntx}\left(t\right)-u_{mtx}\left(t\right)\right)\right\|_{L^2(0,1)}dt. \tag{4.10}$$

Then, summing (4.8), (4.9), and (4.10), we obtain

$$TE\left(u_n\left(T\right)-u_m\left(T\right)\right)$$

$$\leq\int_0^T E\left(u_n\left(t\right)-u_m\left(t\right)\right)dt+c_9\left(T+1\right)F\left(u_n\left(t\right),u_m\left(t\right)\right)$$

$$+c_9\int_0^T t\left\|\sqrt{a}\left(u_{nt}\left(t\right)-u_{mt}\left(t\right)\right)\right\|^2_{L^2(0,1)}dt. \tag{4.11}$$

At this point, to estimate the last term on the right-hand side of (4.11), a multiplication of (4.3) by $2tA^{-1}\left(u_{nt}-u_{mt}\right)$ entails that

$$T\left\|A^{-\frac{1}{2}}\left(u_{nt}\left(T\right)-u_{mt}\left(T\right)\right)\right\|^2_{L^2(0,1)}+T\left\|A^{\frac{1}{2}}\left(u_n\left(T\right)-u_m\left(T\right)\right)\right\|^2_{L^2(0,1)}$$

$$+\int_0^T\int_0^1 ta(x)\left|(u_{nt}(t,x)-u_{mt}(t,x))\right|^2 dxdt$$

$$\leq \int_0^T \left\|A^{-\frac{1}{2}}(u_{nt}(t)-u_{mt}(t))\right\|^2_{L^2(0,1)}dt+\int_0^T\left\|A^{\frac{1}{2}}(u_n(t)-u_m(t))\right\|^2_{L^2(0,1)}dt$$

$$+c_{10}T\left\|u_n-u_m\right\|_{C([0,T];L^2(0,1))},$$

from (4.1) and (4.7), which yields

$$\limsup_{n\to\infty}\ \limsup_{m\to\infty}\int_0^T t\left\|\sqrt{a}\,(u_{nt}(t)-u_{mt}(t))\right\|^2_{L^2(0,1)}dt\leq c_{11},\quad \text{for all } T>0. \tag{4.12}$$

Then, taking into account (4.5), (4.7), and (4.12) in (4.11), we infer

$$\limsup_{n\to\infty}\ \limsup_{m\to\infty}E(u_n(T)-u_m(T))\leq\frac{c_{12}}{T},\ \text{for all } T>0,$$

which gives

$$\limsup_{n\to\infty}\ \limsup_{m\to\infty}\left\|S(t_k)\varphi_k-S(t_m)\varphi_m\right\|_{\mathcal{H}}\leq\frac{c_{13}}{\sqrt{T}}.$$

Consequently, we deduce

$$\liminf_{n\to\infty}\ \liminf_{m\to\infty}\left\|S(t_k)\varphi_k-S(t_m)\varphi_m\right\|_{\mathcal{H}}=0,$$

and in view of the proof of [17, Lemma 3.4], the last equality completes the proof of the lemma. □

Now we are in a position to prove the existence of the global attractor.

Theorem 4.1 *Under the conditions (2.4)–(2.6), the semigroup* $\{S(t)\}_{t\geq0}$ *possesses a global attractor* $\mathcal{A}$ *in* $\mathcal{H}$.

Proof Let B be a bounded subset of $\mathcal{H}$. Then, as a consequence of the previous asymptotic compactness lemma,

$$\omega(B)=\underset{\tau\geq0}{\cap}\overline{\underset{t\geq\tau}{\cup}S(t)B}$$

is a nonempty, compact set. Also, it is invariant with respect to $S(t)$ and attracts B. Let $\theta\in\omega(B)$ and $(u(t),u_t(t))=S(t)\theta$. Then, from (3.2), it follows that the Lyapunov function $L(u(t),u_t(t))$, defined by the formula

$$L(u(t),u_t(t)):=\frac{1}{2}\left\|A^{-\frac{1}{2}}u_t(t)\right\|^2_{L^2(0,1)}+\frac{1}{2}\left\|A^{\frac{1}{2}}u(t)\right\|^2_{L^2(0,1)}$$

$$+\int_0^1(F(u(t,x))-f(0)u(t,x))\,dx-\int_0^1 g(x)A^{-1}u(t,x)(x)\,dx,$$

is a nonincreasing function with respect to t . Therefore, since $L((u(t),u_t(t)))$ is also bounded, we infer

$$\lim_{t\to-\infty}L(u(t),u_t(t))=l. \tag{4.13}$$

On the other hand, let us establish an α-limit set as follows:

$$\alpha(\theta) := \{\varphi \in \omega(B) : \text{ there exists a sequence } \{(u(t_k), u_t(t_k))\}_{k=1}^{\infty}, \text{ such that } t_k \searrow -\infty$$
$$\text{and } (u(t_k), u_t(t_k)) \to \varphi \text{ strongly in } \mathcal{H}\}.$$

One can readily deduce that $\alpha(\theta)$ is a compact and invariant subset of $\omega(B)$. Then, in view of the definition of the set $\alpha(\theta)$ and (4.13), the following holds:

$$L(\varphi) = l, \quad \forall \varphi \in \alpha(\theta).$$

Hence, recalling the invariance of $\alpha(\theta)$, we also find

$$L(S(t)\varphi) = l, \quad \forall \varphi \in \alpha(\theta), \quad \forall t \geq 0.$$

Now, assume that $\varphi \in \alpha(\theta)$ and establish $(v(t), v_t(t)) = S(t)\varphi$. Then, considering the last equality in (3.2), we have

$$\int_0^t \int_0^1 a(x) |v_t(\tau, x)|^2 \, dx d\tau = 0, \quad \forall t \geq 0,$$

and recalling (2.3) and (2.4), from the last equality it follows that

$$v_t(\tau, x) = 0 \text{ a. e. in } [0, \infty) \times (r_0, r_1). \tag{4.14}$$

Our objective is to show that

$$v_t(\tau, x) = 0 \text{ a. e. in } [0, \infty) \times (0, 1). \tag{4.15}$$

Defining $w(t, x) := v_t(t, x)$, from (4.14), $w \in C\left(0, \infty; L^2(0,1)\right) \cap C^1\left(0, \infty; D\left(A^{-1}\right)\right)$ and it is the solution of the following problem:

$$\begin{cases} A^{-3/2} w_{tt} + A^{1/2} w + A^{-1/2}\left(f'(v) w\right) = 0, \ (t, x) \in [0, \infty) \times (0, 1), \\ w = 0, \ (t, x) \in [0, \infty) \times (r_0, r_1). \end{cases} \tag{4.16}$$

Then, testing $(4.16)_1$ by $x^n \zeta^2 w$ and taking into account (2.6), we have

$$\frac{n(n-1)(n-2)}{2} \int_0^t \int_0^1 x^{n-3} \zeta^2 \left| A^{-3/2} w_t(t, x) \right|^2 dx dt + \frac{n}{2} \int_0^t \int_0^1 x^{n-1} \zeta^2 |w(t, x)|^2 \, dx dt$$

$$\leq c_1 \int_0^t \int_0^1 x^n \zeta^2 |w(t, x)|^2 \, dx dt.$$

Since $x \in (0, 1)$, for sufficiently large n, from the last inequality, it follows that

$$\int_0^t \int_0^{r_0} x^n |w(t, x)|^2 \, dx dt = 0.$$

Therefore, from the last one, we deduce

$$w(\tau,x)=0 \text{ a. e. in } [0,\infty)\times(0,r_0). \tag{4.17}$$

Similarly, a testing of $(4.16)_1$ by $(1-x)^n\mu^2 w$ yields

$$\int_0^t\int_{r_1}^1 (1-x)^n |w(t,x)|^2\,dxdt=0,$$

and so

$$w(\tau,x)=0 \text{ a. e. in } [0,\infty)\times(r_0,r_1),$$

which proves (4.15), together with (4.14) and (4.17). Since (4.15) is satisfied, there holds

$$S(t)\varphi=\varphi,\ \forall t\in[0,\infty),$$

and so $\alpha(\theta)$ is the subset of the stationary points $\mathcal{N}$ (for definition, see [5, p. 35]). Then, from the definition of the set $\alpha(\theta)$, $\omega(B)$ is the subset of the unstable manifold $\mathcal{M}^u(\mathcal{N})$ emanating from $\mathcal{N}$. On the other hand, since $\{S(t)\}_{t\geq 0}$ is weakly continuous and asymptotically compact, and $\mathcal{N}$ is bounded in $\mathcal{H}$, we readily obtain that $\mathcal{M}^u(\mathcal{N})$ is invariant and compact. Hence, we conclude that $\mathcal{A}:=\mathcal{M}^u(\mathcal{N})$ is a global attractor. □

Finally, we establish the regularity of the global attractor in the following theorem.

Theorem 4.2 *The global attractor $\mathcal{A}$ is a bounded subset of $H^3(0,1)\times H^1(0,1)$.*

Proof Assume that $(u_0,u_1)\in\mathcal{A}$. Due to the invariance of $\mathcal{A}$, there exists a full trajectory $\{(u(t),u_t(t)),\ t\in\mathbb{R}\}\subset\mathcal{A}$, such that $(u(0),u_t(0)))=(u_0,u_1)$. Denoting $v(t):=\frac{u(t+h)-u(t)}{h}$, $h>0$, by (2.1), v solves the following equation:

$$v_{tt}+A^2v+\frac{1}{h}A\left(f(u(t+h))-f(u(t))\right)+A(a(x)v_t)=0,\quad (t,x)\in\mathbb{R}\times(0,1). \tag{4.18}$$

A multiplication of (4.18) by $2A^{-1}v_t$ and then integration on $(0,1)$ entail that

$$\frac{d}{dt}\left(\left\|A^{-\frac{1}{2}}v_t(t)\right\|^2_{L^2(0,1)}+\left\|A^{\frac{1}{2}}v(t)\right\|^2_{L^2(0,1)}+\int_0^1\int_0^1 f'(u(t)+\tau hv(t,x))\,d\tau\,|v(t,x)|^2\,dx\right)$$

$$+\left\|\sqrt{a}v_t(t)\right\|^2_{L^2(0,1)}\leq c_1\|v(t)\|^2_{L^2(\Omega)},\ \ \forall t\in\mathbb{R}. \tag{4.19}$$

Now, assume that $\kappa\in C^\infty([0,1])$, $0\leq\kappa(x)\leq 1$ and $\kappa(x)=\begin{cases}0, & x\in(0,1)\setminus(r_0,r_1)\\ 1, & \widetilde{r}_0<x<\widetilde{r}_1\end{cases}$ where $\widetilde{r}_0,\widetilde{r}_1\in(r_0,r_1)$.

Then we test equation (4.18) by $A^{-1}(\varepsilon\kappa(x)v)$, so as to get

$$\varepsilon\frac{d}{dt}\left(\int_0^1 A^{\frac{-1}{2}}v_t(t,x)\,\kappa(x)\,A^{\frac{-1}{2}}v(t,x)\,dx\right)+\varepsilon\left\|A^{\frac{1}{2}}v(t)\right\|^2_{L^2(\widetilde{r}_0,\widetilde{r}_1)}$$

$$\leq\varepsilon c_2\|v(t)\|^2_{L^2(0,1)}+\varepsilon c_2\|v_t(t)\|^2_{L^2(r_0,r_1)}. \tag{4.20}$$

Let $\widetilde{\zeta}\in C^{\infty}\left([0,1]\right),\ 0\leq\widetilde{\zeta}\left(x\right)\leq 1$ and $\widetilde{\zeta}\left(x\right)=\left\{\begin{array}{c} 1,\ 0\leq x\leq\widetilde{r}_0 \\ 0,\ \frac{\widetilde{r}_0+\widetilde{r}_1}{2}\leq x\leq 1\end{array}\right.$. A further testing of equation (4.18) by $A^{-1}\left(\varepsilon^2 x\widetilde{\zeta}^2 v_x\right)$ gives us

$$\varepsilon^2\frac{d}{dt}\left(\int\limits_0^1 A^{\frac{-1}{2}}v_t\left(t,x\right)x\widetilde{\zeta}^2 A^{\frac{-1}{2}}v_x\left(t,x\right)dx+\int\limits_0^1 A^{-1}v_t\left(t,x\right)A^{\frac{1}{2}}\left(x\widetilde{\zeta}^2\right)A^{\frac{-1}{2}}v_x\left(t,x\right)dx\right)$$

$$+\varepsilon^2\left\|A^{\frac{-1}{2}}v_t\left(t\right)\right\|^2_{L^2(0,\widetilde{r}_0)}+\varepsilon^2\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2(0,\widetilde{r}_0)}$$

$$\leq\varepsilon^2 c_3\left\|v\left(t\right)\right\|^2_{L^2(0,1)}+\varepsilon^2 c_3\left\|\sqrt{a}v_t\left(t\right)\right\|^2_{L^2(0,1)}+\varepsilon^2 c_3\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2((\widetilde{r}_0,\widetilde{r}_1))}. \tag{4.21}$$

Similarly defining $\widetilde{\eta}\in C^{\infty}\left([0,1]\right),\ 0\leq\widetilde{\eta}\left(x\right)\leq 1$ and $\widetilde{\eta}\left(x\right)=\left\{\begin{array}{cc} 0, & 0\leq x\leq\frac{\widetilde{r}_0+\widetilde{r}_1}{2} \\ 1, & \widetilde{r}_1\leq x\leq 1\end{array}\right.$ and testing equation (4.18) by $A^{-1}\left(\varepsilon^2\left(1-x\right)\widetilde{\eta}^2 v_x\right)$, the following holds:

$$\varepsilon^2\frac{d}{dt}\left(\int\limits_0^1 A^{\frac{-1}{2}}v_t\left(t,x\right)\left(1-x\right)\widetilde{\eta}^2 A^{\frac{-1}{2}}v_x\left(t,x\right)dx+\int\limits_0^1 A^{-1}v_t\left(t,x\right)A^{\frac{1}{2}}\left(\left(1-x\right)\widetilde{\eta}^2\right)A^{\frac{-1}{2}}v_x\left(t,x\right)dx\right)$$

$$+\varepsilon^2\left\|A^{\frac{-1}{2}}v_t\left(t\right)\right\|^2_{L^2(\widetilde{r}_1,1)}+\varepsilon^2\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2(\widetilde{r}_1,1)}$$

$$\leq\varepsilon^2 c_4\left\|v\left(t\right)\right\|^2_{L^2(0,1)}+\varepsilon^2 c_4\left\|\sqrt{a}v_t\left(t\right)\right\|^2_{L^2(0,1)}+\varepsilon^2 c_4\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2((\widetilde{r}_0,\widetilde{r}_1))}. \tag{4.22}$$

Finally, summing inequalities (4.19)–(4.22) and picking ε sufficiently small, with the help of (2.4), we obtain the estimate

$$\frac{d}{dt}\widetilde{\Psi}\left(v\left(t\right)\right)+c_4\widetilde{E}\left(v\left(t\right)\right)\leq c_5\left\|v\left(t\right)\right\|^2_{L^2(0,1)} \tag{4.23}$$

where $\widetilde{\Psi}\left(v\left(t\right)\right)=\left\|A^{-\frac{1}{2}}v_t\left(t\right)\right\|^2_{L^2(0,1)}+\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2(0,1)}+\int\limits_0^1 A^{\frac{-1}{2}}v_t\left(t,x\right)\kappa\left(x\right)A^{\frac{-1}{2}}v\left(t,x\right)dx$
$+\int\limits_0^1 A^{\frac{-1}{2}}v_t\left(t,x\right)x\widetilde{\zeta}^2 A^{\frac{-1}{2}}v_x\left(t,x\right)dx+\int\limits_0^1 A^{-1}v_t\left(t,x\right)A^{\frac{1}{2}}\left(x\widetilde{\zeta}^2\right)A^{\frac{-1}{2}}v_x\left(t,x\right)dx$
$+\int\limits_0^1 A^{\frac{-1}{2}}v_t\left(t,x\right)\left(1-x\right)\widetilde{\eta}^2 A^{\frac{-1}{2}}v_x\left(t,x\right)dx+\int\limits_0^1 A^{-1}v_t\left(t,x\right)A^{\frac{1}{2}}\left(\left(1-x\right)\widetilde{\eta}^2\right)A^{\frac{-1}{2}}v_x\left(t,x\right)dx$
$+\int\limits_0^1\int\limits_0^1 f'\left(u\left(t\right)+\tau hv\left(t,x\right)\right)d\tau\left|v\left(t,x\right)\right|^2 dx$ and $\widetilde{E}\left(v\left(t\right)\right)=\left\|A^{-\frac{1}{2}}v_t\left(t\right)\right\|^2_{L^2(0,1)}+\left\|A^{\frac{1}{2}}v\left(t\right)\right\|^2_{L^2(0,1)}$.
At this point, it is worth mentioning that estimate (4.23) is first justified for strong solutions, and then it can be extended to weak solutions by using the density argument.
On the other hand, by the conditions of the theorem, it is easy to show that

$$\lambda_1\widetilde{E}\left(u\left(t\right)\right)-M\leq\widetilde{\Psi}\left(u\left(t\right)\right)\leq\lambda_2\widetilde{E}\left(u\left(t\right)\right), \tag{4.24}$$

for some $0 < \lambda_1 < \lambda_2$ and $M > 0$. Considering (4.24) in (4.23), we find that

$$\frac{d}{dt}\widetilde{\Psi}\left(v\left(t\right)\right) + c_6\widetilde{\Psi}\left(v\left(t\right)\right) \leq c_7\left\|v\left(t\right)\right\|^2_{L^2(0,1)},$$

and in particular

$$\widetilde{\Psi}\left(v\left(t\right)\right) \leq e^{-c_6(t-s)}\widetilde{\Psi}\left(v\left(s\right)\right) + c_7\int\limits_s^t e^{-c_6(t-\tau)}\left\|v\left(\tau\right)\right\|^2_{L^2(\Omega)}d\tau,\ s \leq t. \tag{4.25}$$

At this point, recalling the definition of v, we have

$$\left\|v\left(t\right)\right\|_{L^2(\Omega)} \leq \int\limits_0^1 \left\|u_t\left(t+\tau h\right)\right\|_{L^2(\Omega)}d\tau \leq c_8,\ \forall t \in \mathbb{R},$$

which, together with (4.23), yields that

$$\widetilde{\Psi}\left(v\left(t\right)\right) \leq e^{-c_6(t-s)}\widetilde{\Psi}\left(v\left(s\right)\right) + c_9,\ s \leq t. \tag{4.26}$$

Then passing to the limit in (4.24) as $s \to -\infty$ and recalling (4.24), we find

$$\widetilde{E}\left(v\left(t\right)\right) \leq c_{10},\ \forall t \in \mathbb{R}.$$

Consequently, passing to the limit as $h \to 0^+$ in the last inequality, by using the definition of v, the following holds:

$$\left\|u_{tt}\left(t\right)\right\|_{H^{-1}(\Omega)} + \left\|u_t\left(t\right)\right\|_{H^1(\Omega)} \leq c_{11},\ \forall t \in \mathbb{R}.$$

Taking into account the previous estimate in (2.1)–(2.2), we infer that

$$\left\|u\left(t\right)\right\|_{H^3(\Omega)} + \left\|u_t\left(t\right)\right\|_{H^1(\Omega)} \leq c_{12},\ \forall t \in \mathbb{R}.$$

Hence, choosing $t = 0$ in the last inequality, we eventually obtain that

$$\left\|u_0\right\|_{H^3(\Omega)} + \left\|u_1\right\|_{H^1(\Omega)} \leq c_8,$$

which concludes the proof of the theorem. □

In conclusion, Theorem 2.1 follows from Theorem 4.1 and Theorem 4.2.

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