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An exponential method to solve linear Fredholm–Volterra integro-differential equations and residual improvement

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Abstract: In this paper, a collocation approach based on exponential polynomials is introduced to solve linear Fredholm–Volterra integro-differential equations under the initial boundary conditions. First, by constructing the matrix forms of the exponential polynomials and their derivatives, the desired exponential solution and its derivatives are written in matrix forms. Second, the differential and integral parts of the problem are converted into matrix forms based on exponential polynomials. Later, the main problem is reduced to a system of linear algebraic equations by aid of the collocation points, the matrix operations, and the matrix forms of the conditions. The solutions of this system give the coefficients of the desired exponential solution. An error estimation method is also presented by using the residual function and the exponential solutions are improved by the estimated error function. Numerical examples are solved to show the applicability and the effectiveness of the method. In addition, the results are compared with the results of other methods.

Key words: Collocation method, exponential polynomials, exponential solutions, Fredholm–Volterra integro-differential equations, initial boundary conditions, residual improvement

1. Introduction

Differential, integral, and integro-differential equations contribute to the modeling of many problems in science and engineering. In this study, we introduce an exponential method together with residual error estimation and residual correction method for solutions of the delay linear Fredholm integro-differential equations. In recent years, integro-differential equations have solved semianalytical methods such as the homotopy perturbation method [7, 26], the Taylor collocation method [11], the Haar functions method, [14, 15], He's variational iteration technique [8], the power series method [24], the Chebyshev technique [22], the Legendre-spectral method [9], the Tau method [20], the Legendre multiwavelets method [13], the finite-difference scheme [5], the variational iteration method [21], the CAS wavelet operational matrix method [3], the trigonometric wavelets method [12], the Legendre matrix method [25], the Taylor polynomial approach [18], the Adomian method [1], the differential transformation method [4], the Galerkin method [16], the Bessel matrix method [30], the Legendre collocation method [32], the improved homotopy perturbation method [27], the modified homotopy perturbation method [10], and the moving least square method [6, 17]. Yübaşı and Sezer [31] gave a matrix method based on exponential polynomials for solutions of systems of differential equations.

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In this study, we have three goals for the mth order linear Fredholm–Volterra integro-differential equation with variable coefficient in the form of

$$L[y(x)] = \sum_{k=0}^{R} F_k(x)y^{(k)}(x) + \lambda \int_a^b \sum_{n=0}^{S} K_n(x,t)y^{(n)}(t)dt + \gamma \int_a^x \sum_{r=0}^{J} P_r(x,t)y^{(r)}(t)dt = g(x), \tag{1.1}$$

$m = \max\{R, S, J\}$, $0 \le a \le x, t \le b < \infty$, under the initial boundary conditions

$$\sum_{k=0}^{m-1} \left(a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b)\right) = \mu_j, \quad j = 0, 1, ..., m-1. \tag{1.2}$$

Here a_{jk}, b_{jk}, λ, γ, and μ_j are suitable constants; $y^{(0)}(x) = y(x)$ is the unknown function; $F_k(x)$, $g(x)$, $K_n(x,t)$, and $P_r(x,t)$ are the defined functions for $(x,t) \in [a,b] \times [a,b]$; and the kernel functions $K_n(x,t)$ and $P_n(x,t)$, $(n = 0, 1, ..., S)$ can be expanded to Maclaurin series.

The first goal is to obtain the approximate solutions of the problem (1.1)–(1.2) in the form

$$y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}, \tag{1.3}$$

where the exponential basis set is defined by $\{1, e^{-x}, e^{-2x}, ..., e^{-Nx}\}$ and a_n, $(n = 0, 1, 2, ..., N)$ are unknown coefficients.

The second goal is to make an error estimation for the problem and the method by using the residual function.

The third goal of the study is to compute the improved exponential solutions in the form

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x). \tag{1.4}$$

Here, $y_N(x)$ is the exponential solution given by (1.3) and

$$e_{N,M}(x) = \sum_{n=0}^{M} a_n^* e^{-nx} \tag{1.5}$$

is the exponential solution of the error problem obtained by using the residual error function.

Here, a_n^*, $n = 0, 1, 2, ..., N$, are the unknown coefficients; M are chosen as any positive integers such that $M \ge N \ge m$.

This paper is organized as follows: the required matrix relations for the solution method are given in Section 2. In Section 3, the exponential approach is presented for linear Fredholm–Volterra integro-differential equations. In Section 4, an error estimation scheme and the residual correction technique are given for the approximate solutions by using the residual function. In Section 5, we solve some numerical examples to clarify the method, error estimation, and residual correction scheme. Section 6 concludes the paper with a brief summary.

2. Main matrix relations required for the solution method

First, let us show Eq. (1.1) in the form

$$D(x) + \lambda I(x) + \gamma V(x) = g(x), \tag{2.1}$$

where the differential part is

$$D(x) = \sum_{k=0}^{R} F_k(x) y^{(k)}(x), \tag{2.2}$$

the Fredholm integral part is

$$I(x) = \int_a^b \sum_{n=0}^{S} K_n(x,t) y^{(n)}(t) dt, \tag{2.3}$$

and the Volterra integral part is

$$V(x) = \int_a^x \sum_{r=0}^{J} P_r(x,t) y^{(r)}(t) dt. \tag{2.4}$$

In the following subsections, we will find the matrix forms of the solution $y(x)$ and its kth order derivative $y^{(k)}(x)$ and then we will obtain the matrix forms of the parts $D(x)$ and $I(x)$, and the mixed conditions (1.2).

2.1. Matrix relations for the part $D(x)$

First, we can express the approximate solution (1.3) by the matrix form

$$y_N(x) = \mathbf{E}(x)\mathbf{A}. \tag{2.5}$$

Here,

$$\mathbf{E}(x) = \begin{bmatrix} 1 & e^{-x} & e^{-2x} & \cdots & e^{-Nx} \end{bmatrix}, \quad \textit{and} \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T.$$

kth order derivative $\mathbf{E}^{(k)}(x)$ can be expressed by means of the matrix $\mathbf{E}(x)$ as follows:

$$\mathbf{E}^{(k)}(x) = \mathbf{E}(x)\mathbf{M}^k, k = 0, 1, 2, ... \tag{2.6}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & -2 & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -N \end{bmatrix} \quad \text{and} \quad \mathbf{M}^0 \quad \text{is the unit matrix in the dimension} (N+1) \times (N+1).$$

Clearly, $\mathbf{M}^k$ is written as

$$\mathbf{M}^k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^k & 0 & \cdots & 0 \\ \vdots & \vdots & (-2)^k & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & (-N)^k \end{bmatrix}.$$

By using (2.6), the kth order derivative of (2.5) becomes

$$y_N^{(k)}(x) = \mathbf{E}(x)\mathbf{M}^k\mathbf{A}. \tag{2.7}$$

We substitute expression (2.14) into (2.5) and so we have the matrix form

$$D(x)=\sum_{k=0}^{R}F_k(x)\mathbf{E}(x)\mathbf{M}^k\mathbf{A}. \tag{2.8}$$

2.2. Matrix relations for the integral part $I(x)$

By using the following procedure, the kernel functions $K_n(x,t)$, $(n=0,1,...,S)$ can be converted to the matrix forms.

In this subsection, we will find the matrix form of integral part $I(x)$. For this purpose, we will benefit from the truncated Maclaurin series of the kernel functions $K_n(x,t)$ for $n=0,1,...,S$. The kernel functions $K_n(x,t)$, $(n=0,1,...,S)$ can be expressed approximately by the truncated Maclaurin series

$$K_n(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}k_{r,s}^{T,n}x^r t^s, \tag{2.9}$$

where

$$k_{r,s}^{T,n}=\frac{1}{r!s!}\frac{\partial^{r+s}K_n(0,0)}{\partial x^r\partial t^s};\quad r,s=0,1,...,N,\quad (n=0,1,...,S).$$

On the order hand, the kernel functions $K_n(x,t)$, $(n=0,1,...,S)$ can be expressed by the truncated exponential series form

$$K_n(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}k_{r,s}^{E,n}e^{-rx}e^{-st},\quad (n=0,1,...,S). \tag{2.10}$$

Here, $k_{r,s}^{E,n}$ $(n=0,1,...,S)$ are the coefficients of the exponential series form and they are determined by the following procedure.

The truncated series (2.9) and (2.10) can be transformed to the matrix forms

$$K_n(x,t)=\mathbf{X}(x)\mathbf{K}_n^T\mathbf{X}^T(t),\quad \mathbf{K}_n^T=\left[k_{r,s}^{T,n}\right], \tag{2.11}$$

and

$$K_n(x,t)=\mathbf{E}(x)\mathbf{K}_n^E\mathbf{E}^T(t),\quad \mathbf{K}_n^E=\left[k_{r,s}^{E,n}\right], \tag{2.12}$$

where

$$\mathbf{X}(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix},\quad r,s=0,1,...,N,\quad n=0,1,...S.$$

The relation between the standard basis matrix $\mathbf{X}(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix}$ and the exponential basis matrix $\mathbf{E}(x)=\begin{bmatrix}1 & e^{-x} & e^{-2x} & \cdots & e^{-Nx}\end{bmatrix}$ is given by

$$\mathbf{E}(x)=\mathbf{X}(x)\mathbf{Ş}^T, \tag{2.13}$$

where

$$\mathbf{Ş} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \frac{(-1)}{1!} & \frac{(-1)^2}{2!} & \cdots & \frac{(-1)^N}{N!} \\ 1 & \frac{(-2)}{1!} & \frac{(-2)^2}{2!} & \cdots & \frac{(-2)^N}{N!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{(-N)}{1!} & \frac{(-N)^2}{2!} & \cdots & \frac{(-N)^N}{N!} \end{bmatrix}.$$

By equaling relation (2.11) to relation (2.12) and by using relation (2.13), we get the matrix relation

$$\mathbf{K}_n^T = \mathbf{Ş}^T \mathbf{K}_n^E \mathbf{Ş}$$

or

$$\mathbf{K}_n^E = \left(\mathbf{Ş}^T\right)^{-1} \mathbf{K}_n^T \mathbf{Ş}^{-1}. \tag{2.14}$$

By placing relation (2.14) into Eq. (2.12), we gain

$$K_n(x,t) = \mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1} \mathbf{K}_n^T \mathbf{Ş}^{-1} \mathbf{E}^T(t), \tag{2.15}$$

where $\mathbf{K}_n^T = \left[k_{r,s}^{T,n}\right]$, $r,s = 0,1,...,N$, $n = 0,1,...,S$ are the Taylor coefficients matrix of the function $K_n(x,t)$ at the point $(0,0)$. By putting Eqs. (2.7) and (2.15) into the part $I(x)$ of Eq. (2.1), we obtain the matrix form

$$I(x) = \int_a^b \sum_{n=0}^{S} \mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1} \mathbf{K}_n^T \mathbf{Ş}^{-1} \mathbf{E}^T(t) \mathbf{E}(t) \mathbf{M}^n \mathbf{A} dt.$$

This equation can be simplified as follows:

$$I(x) = \mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1} \left\{\sum_{n=0}^{S} \mathbf{K}_n^T \mathbf{Ş}^{-1} \mathbf{Q} \mathbf{M}^n \right\} \mathbf{A}, \tag{2.16}$$

where

$$\mathbf{Q} = \int_a^b \mathbf{E}^T(t) \mathbf{E}(t) dt.$$

Here, the matrix $\mathbf{Q}$ is computed by

$$\mathbf{Q} = \int_a^b \mathbf{E}^T(t) \mathbf{E}(t) dt = \left[q_{r,s}\right], \quad r,s = 0,1,...,N,$$

where

$$\begin{cases} q_{r,s} = b - a, & \text{for} \quad r = s = 0, \\ q_{r,s} = \frac{e^{-(r+s)b} - e^{-(r+s)a}}{-(r+s)}, & \text{for} \quad r \neq s \quad \text{and} \quad r,s = 0,1,...,N. \end{cases}$$

2.3. Matrix relations for the integral part $V(x)$

In this subsection, we first obtain matrix forms of the kernel functions $P_n(x,t)$, $(n=0,1,...,J)$. Similar to the finding of the matrix forms of the kernel functions in the previous section, the matrix form $P_n(x,t)$ becomes

$$P_n(x,t)=\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\mathbf{P}_n^T\mathbf{Ş}^{-1}\mathbf{E}^T(t), \tag{2.17}$$

where

$$\mathbf{P}_n^T=\left[p_{r,s}^{T,n}\right],\quad P_n(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}p_{r,s}^{T,n}x^rt^s;\quad p_{r,s}^{T,n}=\frac{1}{r!s!}\frac{\partial^{r+s}P_n(0,0)}{\partial x^r\partial t^s};\quad r,s=0,1,...,N,\quad (n=0,1,...,J).$$

We put Eqs. (2.7) and (2.17) into the part $V(x)$ of Eq. (2.1) and thus we get

$$V(x)=\int_a^x\sum_{r=0}^{J}\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\mathbf{P}_r^T\mathbf{Ş}^{-1}\mathbf{E}^T(t)\mathbf{E}(t)\mathbf{M}^r\mathbf{A}dt.$$

If this equation is simplified, it becomes

$$V(x)=\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\sum_{r=0}^{J}\mathbf{P}_r^T\mathbf{Ş}^{-1}\int_a^x\mathbf{E}^T(t)\mathbf{E}(t)dt\mathbf{M}^r\mathbf{A}=\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\sum_{r=0}^{J}\mathbf{P}_r^T\mathbf{Ş}^{-1}\mathbf{Q}_V(x)\mathbf{M}^r\mathbf{A}. \tag{2.18}$$

Here,

$$\mathbf{Q}_V(x)=\int_a^x\mathbf{E}^T(t)\mathbf{E}(t)dt=\left[q_{r,s}^v\right],\quad r,s=0,1,...,N;$$

$$\begin{cases} q_{r,s}^v=x-a, & \text{for}\quad r=s=0,\\ q_{r,s}^v=\frac{e^{-(r+s)x}-e^{-(r+s)a}}{-(r+s)}, & \text{for}\quad r\neq s\quad\text{and}\quad r,s=0,1,...,N. \end{cases}$$

3. The solution of the problem

In Section 2, we constructed the required matrix relations for the solution of the problem. Now we will obtain the approximate solution by computing the unknown coefficients.

First, we substitute the relations (2.8), (2.16) and (2.18), into Eq.(2.1) and thus we get the matrix equation

$$\sum_{k=0}^{R}F_k(x)\mathbf{E}(x)\mathbf{M}^k\mathbf{A}+\lambda\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\left\{\sum_{n=0}^{S}\mathbf{K}_n^T\mathbf{Ş}^{-1}\mathbf{Q}\mathbf{M}^n\right\}\mathbf{A}+\gamma\mathbf{E}(x)\left(\mathbf{Ş}^T\right)^{-1}\left\{\sum_{r=0}^{J}\mathbf{P}_r^T\mathbf{Ş}^{-1}\mathbf{Q}_V(x)\mathbf{M}^r\right\}\mathbf{A}=g(x). \tag{3.1}$$

The collocation points, defined by

$$x_i=a+\frac{(b-a)}{N}i,\quad i=0,1,...,N, \tag{3.2}$$

are placed in Eq. (3.1), and thus we obtain a system of the matrix equations

$$\sum_{k=0}^{R} F_k(x_i)\mathbf{E}(x_i)\mathbf{M}^k\mathbf{A}+\lambda\mathbf{E}(x_i)\left(\mathbf{Ş}^T\right)^{-1}\left\{\sum_{n=0}^{S}\mathbf{K}_n^T\mathbf{Ş}^{-1}\mathbf{Q}\mathbf{M}^n\right\}\mathbf{A}+\gamma\mathbf{E}(x_i)\left(\mathbf{Ş}^T\right)^{-1}\left\{\sum_{r=0}^{J}\mathbf{P}_r^T\mathbf{Ş}^{-1}\mathbf{Q}_V(x_i)\mathbf{M}^r\right\}\mathbf{A}=g(x_i),$$

$i=0,1,...,N$. This system can be represented by the matrix form

$$\left\{\sum_{k=0}^{R}\mathbf{F}_k\mathbf{E}\mathbf{M}^k+\lambda\mathbf{E}\left(\mathbf{Ş}^T\right)^{-1}\sum_{n=0}^{S}\mathbf{K}_n^T\mathbf{Ş}^{-1}\mathbf{Q}\mathbf{M}^n+\gamma\bar{\mathbf{E}}\bar{\mathbf{Ş}}\sum_{r=0}^{J}\mathbf{P}_r\tilde{\mathbf{Ş}}\mathbf{Q}_V\mathbf{M}_r\right\}\mathbf{A}=\mathbf{G}, \tag{3.3}$$

where

$$\mathbf{F}_k=\begin{bmatrix} F_k(x_0) & 0 & \cdots & 0\\ 0 & F_k(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & F_k(x_N)\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}\mathbf{E}(x_0)\\ \mathbf{E}(x_1)\\ \vdots\\ \mathbf{E}(x_N)\end{bmatrix}=\begin{bmatrix}1 & e^{-x_0} & e^{-2x_0} & \cdots & e^{-Nx_0}\\ 1 & e^{-x_1} & e^{-2x_1} & \cdots & e^{-Nx_1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & e^{-x_N} & e^{-2x_N} & \cdots & e^{-Nx_N}\end{bmatrix},$$

$$\bar{\mathbf{E}}=\begin{bmatrix}\mathbf{E}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{E}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{E}(x_N)\end{bmatrix},\quad \bar{\mathbf{Ş}}=\begin{bmatrix}\left(\mathbf{Ş}^T\right)^{-1} & 0 & \cdots & 0\\ 0 & \left(\mathbf{Ş}^T\right)^{-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \left(\mathbf{Ş}^T\right)^{-1}\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathbf{P}_n=\begin{bmatrix}\mathbf{P}_n^T & 0 & \cdots & 0\\ 0 & \mathbf{P}_n^T & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{P}_n^T\end{bmatrix}_{(N+1)\times(N+1)},\quad \tilde{\mathbf{Ş}}=\begin{bmatrix}\mathbf{Ş}^{-1} & 0 & \cdots & 0\\ 0 & \mathbf{Ş}^{-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Ş}^{-1}\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathbf{G}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix},\quad \mathbf{Q}_V=\begin{bmatrix}\mathbf{Q}_V(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{Q}_V(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Q}_V(x_N)\end{bmatrix},\quad \mathbf{M}_n=\begin{bmatrix}\mathbf{M}^n\\ \mathbf{M}^n\\ \vdots\\ \mathbf{M}^n\end{bmatrix}_{(N+1)\times 1}.$$

Briefly, we can write Eq. (3.3) in the form

$$\mathbf{W}\mathbf{A}=\mathbf{G}\quad or\quad [\mathbf{W};\mathbf{G}], \tag{3.4}$$

where

$$\mathbf{W}=[W_{pq}]=\sum_{k=0}^{R}\mathbf{F}_k\mathbf{E}\mathbf{M}^k+\lambda\mathbf{E}\left(\mathbf{Ş}^T\right)^{-1}\sum_{n=0}^{S}\mathbf{K}_n^T\mathbf{Ş}^{-1}\mathbf{Q}\mathbf{M}^n+\gamma\bar{\mathbf{E}}\bar{\mathbf{Ş}}\sum_{r=0}^{J}\mathbf{P}_r\tilde{\mathbf{Ş}}\mathbf{Q}_V\mathbf{M}_r;\quad p,q=0,1,...,N.$$

Let us compute the matrix form of the conditions (1.2). For this purpose, by putting a and b instead of x in Eq. (2.7) and by using them in conditions (1.2), we have the matrix form

$$\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{E}(a)+b_{jk}\mathbf{E}(b)\right)\mathbf{M}^k\mathbf{A}=\mu_j,\quad j=0,1,...,m-1. \tag{3.5}$$

Briefly, the matrix forms (3.5) can be expressed in the form

$$\mathbf{UA} = \mu \quad or \quad [\mathbf{U};\mu], \tag{3.6}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{m-1} \end{bmatrix}, \quad \mathbf{U}_j = \sum_{k=0}^{m-1} \big(a_{jk}\mathbf{E}(a) + b_{jk}\mathbf{E}(b)\big)\mathbf{M}^k\mathbf{A}, \quad j = 0,1,...,m-1, \quad \mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{m-1} \end{bmatrix}.$$

To solve Eq. (1.1) under conditions (1.2), the unknown coefficients should be determined by using Eq. (3.4) and Eq. (3.6). Therefore, we replace the rows of the matrix (3.6) by any m rows of the matrix (3.4) and so we get the new augmented matrix:

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} w_{0\,0} & w_{0\,1} & \cdots & w_{0\,N} & ; & g(x_0) \\ w_{1\,0} & w_{1\,1} & \cdots & w_{1\,N} & ; & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-m\,0} & w_{N-m\,1} & \cdots & w_{N-m\,N} & ; & g(x_{N-m}) \\ u_{0\,0} & u_{0\,1} & \cdots & u_{0\,N} & ; & \mu_0 \\ u_{1\,0} & u_{1\,1} & \cdots & u_{1\,N} & ; & \mu_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{m-1\,0} & u_{m-1\,1} & \cdots & u_{m-1\,N} & ; & \mu_{m-1} \end{bmatrix}. \tag{3.7}$$

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = N+1$, then we can write

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Hence, the coefficients a_n, $(n = 0,1,...,N)$ are uniquely determined by Eq. (3.7). By placing the determined coefficients into Eq. (1.3), we obtain the exponential series solution

$$y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}. \tag{3.8}$$

We note that, when $|\widetilde{\mathbf{W}}| = 0$, if rank $\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] < N+1$, then we may find a particular solution. If rank $\widetilde{\mathbf{W}} \neq rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] < N+1$, then a solution is not available.

4. Error estimation and residual improvement

In this section, we introduce an error estimation technique based on the residual function. Oliveira [19] presented the residual correction method for boundary value problems of linear differential equations, çelik [2] presented a collocation method together with residual correction with Chebyshev series, Shahmorad [23] presented the residual error estimation for solution of integro-differential equations by the Tau method, and Yübaşı[28, 29] presented the Laguerre approach and the Bessel collocation method together with residual error estimation.

First, let us define the residual function of Eq. (1.1) for the approximate solution (3.8) as

$$R_N(x) = L[y_N(x)] - g(x). \tag{4.1}$$

Also, the approximate solution provides conditions (1.2) as follows:

$$\sum_{k=0}^{m-1}\left(a_{jk}y^{(k)}(a)+b_{jk}y^{(k)}(b)\right)=\mu_j,\quad j=0,1,...,m-1.$$

Thus, $y_N(x)$ satisfies the problem

$$\begin{cases} L[y_N(x)]=\sum_{k=0}^{R}F_k(x)y_N^{(k)}(x)+\lambda\int_a^b\sum_{n=0}^{S}K_n(x,t)y_N^{(n)}(t)dt+\gamma\int_a^x\sum_{r=0}^{J}P_r(x,t)y_N^{(r)}(t)dt=g(x)+R_N(x),\\ \sum_{k=0}^{m-1}\left(a_{jk}y_N^{(k)}(a)+b_{jk}y_N^{(k)}(b)\right)=\mu_j,\quad j=0,1,...,m-1.\end{cases} \tag{4.2}$$

$y(x)$ and $y_N(x)$ represent the exact solution and the approximate solution (3.8) of the problem (1.1)–(1.2), respectively. Then the error function can be estimated as

$$e_N(x)=y(x)-y_N(x).$$

By subtracting the integral equation in Eq. (4.2) from Eq. (1.1), we get the error differential equation

$$L[e_N(x)]=\sum_{k=0}^{R}F_k(x)e_N^{(k)}(x)+\lambda\int_a^b\sum_{n=0}^{S}K_n(x,t)e_N^{(n)}(t)dt+\gamma\int_a^x\sum_{r=0}^{J}P_r(x,t)e_N^{(r)}(t)dt=-R_N(x), \tag{4.3}$$

and by subtracting the conditions in (4.2) from the conditions in (1.2), we have the homogeneous mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{jk}e_N^{(k)}(a)+b_{jk}e_N^{(k)}(b)\right)=0,\quad j=0,1,...,m-1 \tag{4.4}$$

for the error function. By solving the error differential equation (4.3) under the conditions in (4.4) by using the suggested method in Section 3, we gain the approximation

$$e_{N,M}(x)=\sum_{n=0}^{M}a_n^{*}e^{-nx} \tag{4.5}$$

to the error function $e_N(x)$. By summing the approximate solution $y_N(x)$ and the estimated error function, $e_{N,M}(x)$, we gain the corrected approximate solution:

$$y_{N,M}(x)=y_N(x)+e_{N,M}(x).$$

5. Illustrative examples

In this section, the applications of the method, the error estimation, and the residual correction are given by the some examples. In this part, $y(x)$, $y_N(x)$, $y_{N,M}(x)$, $\left|e_N(x)\right|=\left|y(x)-y_N(x)\right|$, $y_{N,M}(x)=y_N(x)+e_{N,M}(x)$, and $\left|E_{N,M}(x)\right|=\left|y(x)-y_{N,M}(x)\right|$ show the exact solution, the exponential solution, the corrected exponential solution, the absolute error function, the estimated absolute error function, and the corrected absolute error function, respectively, in the considered interval. We have done all numerical calculations by using a code written in MATLAB.

Example 5.1 First, let us solve the linear Fredholm integro-differential equation

$$y^{(2)}(x)-xy^{(1)}(x)+xy(x)=(x-1)sin(x)-(x+1)cos(x)+\int_0^1\big[sin(x+t)y(t)+cos(x+t)y^{'}(t)\big]dt, 0\leq x,t\leq 1 \quad (5.1)$$

with the initial conditions $y(0)=0$, $y^{(1)}(0)=1$, $y^{(2)}(0)=0$, $y^{(3)}(0)=-1$. Here, the exact solution of the problem is $y(x)=sin(x)$, $m=2$, $F_2=1$, $F_1=-x$, $F_0=x$, $F_i=0$ for values of i, $\lambda=1$, $g(x)=(x-1)sin(x)-(x+1)cos(x)$, $K_0(x,t)=sin(x+t)$, $K_1(x,t)=cos(x+t)$.

By applying the method for $N=3$, we will obtain the approximate solution in the form

$$y_3(x)=\sum_{n=0}^{3}a_n e^{-nx}.$$

Now we compute the unknown coefficients a_n, $(n=0,1,...,N)$ by following the procedure in Section 3.

The set of collocation points for $N=3$ is

$$\big\{x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_1=1\big\}.$$

The fundamental matrix equation of the problem from Eq. (3.3) becomes

$$\big\{\mathbf{F_0E}+\mathbf{F_1EM}+\mathbf{F_2EM^2}-\lambda\mathbf{E}(\mathbf{Ş}^T)^{-1}(\mathbf{K}_0^T\mathbf{Ş}^{-1}\mathbf{Q}+\mathbf{K}_1^T\mathbf{Ş}^{-1}\mathbf{QM})\big\}\mathbf{A}=\mathbf{G},$$

where

$$\mathbf{F}_0=\begin{bmatrix}0&0&0&0\\0&1/3&0&0\\0&0&2/3&0\\0&0&0&1\end{bmatrix},\quad \mathbf{F}_1=\begin{bmatrix}0&0&0&0\\0&-1/3&0&0\\0&0&-2/3&0\\0&0&0&-1\end{bmatrix},\quad \mathbf{F}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$\mathbf{E}=\begin{bmatrix}\mathbf{E}(0)\\\mathbf{E}(1/3)\\\mathbf{E}(2/3)\\\mathbf{E}(1)\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&1413/1972&287/559&536/1457\\1&287/559&916/3475&329/2431\\1&536/1457&329/2431&152/3053\end{bmatrix},\mathbf{K}_0^T=\begin{bmatrix}0&1&0&-1/6\\1&0&-1/2&0\\0&-1/2&0&1/12\\-1/6&0&1/12&0\end{bmatrix},$$

$$\mathbf{K}_1^T=\begin{bmatrix}1&0&-1/2&0\\0&-1&0&1/6\\-1/2&0&1/4&0\\0&1/6&0&-1/36\end{bmatrix},\mathbf{M}=\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&-2&0\\0&0&0&-3\end{bmatrix},\mathbf{Ş}=\begin{bmatrix}1&0&0&0\\1&-1&1/2&-1/6\\1&-2&2&-4/3\\1&-3&9/2&-9/2\end{bmatrix},$$

$$\mathbf{Q}=\begin{bmatrix}1&1-e^{-1}&\frac{1}{2}-\frac{1}{2}e^{-2}&\frac{1}{3}-\frac{1}{3}e^{-3}\\1-e^{-1}&\frac{1}{2}-\frac{1}{2}e^{-2}&\frac{1}{3}-\frac{1}{3}e^{-3}&\frac{1}{4}-\frac{1}{4}e^{-4}\\\frac{1}{2}-\frac{1}{2}e^{-2}&\frac{1}{3}-\frac{1}{3}e^{-3}&\frac{1}{4}-\frac{1}{4}e^{-4}&\frac{1}{5}-\frac{1}{5}e^{-5}\\\frac{1}{3}-\frac{1}{3}e^{-3}&\frac{1}{4}-\frac{1}{4}e^{-4}&\frac{1}{5}-\frac{1}{5}e^{-5}&\frac{1}{6}-\frac{1}{6}e^{-6}\end{bmatrix},\mathbf{G}=\begin{bmatrix}-1\\-3067/2075\\-761/502\\-429/397\end{bmatrix}.$$

Hence, the augmented matrix (3.4) is computed as

$$[\mathbf{W};\mathbf{G}]=\left[\begin{array}{ccccc} -4954/10633 & 1418/1073 & 1766/379 & 1835/187 & ; & -1 \\ -11113/28153 & 538/437 & 1256/423 & 742/169 & ; & -3067/2075 \\ -2155/8201 & 892/937 & 435/256 & 1419/746 & ; & -761/502 \\ -139/1785 & 473/782 & 494/621 & 357/503 & ; & -429/397 \end{array}\right].$$

The augmented matrix form of the conditions from (3.6) is calculated as

$$[\mathbf{U};\mu]=\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & ; & 0 \\ 0 & -1 & -2 & -3 & ; & 1 \end{array}\right]$$

and thus the new augmented matrix becomes

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\left[\begin{array}{cccccc} -4954/10633 & 1418/1073 & 1766/379 & 1835/187 & ; & -1 \\ -11113/28153 & 538/437 & 1256/423 & 742/169 & ; & -3067/2075 \\ 1 & 1 & 1 & 1 & ; & 0 \\ 0 & -1 & -2 & -3 & ; & 1 \end{array}\right].$$

By solving the system corresponding to $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$, the coefficients matrix is found as

$$\mathbf{A}=\left[\begin{array}{cccc} 335/192 & -778/289 & 599/521 & -181/894 \end{array}\right]^T.$$

Finally, the determined coefficients are substituted into Eq. (1.3) and thus we get the approximate solution

$$y_3(x)=-\frac{1275}{1346}e^{-x}+\frac{599}{521}e^{-2x}-\frac{181}{894}e^{-3x}.$$

For the problem in this example, the error problem from Equations (4.3)–(4.4) is written as

$$\left\{\begin{array}{l} e^{(2)}(x)-xe^{(1)}(x)+xe(x)-\int_0^1\left[sin(x+t)e(t)+cos(x+t)e^{'}(t)\right]dt=-R_N(x) \\ e(0)=0,\quad e^{(1)}(0)=0,\quad e^{(2)}(0)=0,\quad e^{(3)}(0)=0. \end{array}\right. ,0\leq x,t\leq 1 \tag{5.2}$$

This error problem for $M=4$ is solved by using the method defined in Section 3 and the coefficient matrix of the error problem is computed as

$$\mathbf{A}^*=\left[\begin{array}{ccccc} -333/1328 & 2751/2675 & -1499/928 & 1099/957 & -628/2021 \end{array}\right]^T.$$

The elements of the determined coefficient matrix are placed into Eq. (5.2) and thus we obtain the error function, approximately, as

$$e_{3,4}=-0.250752792284+1.02841121120e^{-x}-1.61530125599e^{-2x}+1.14838004752e^{-3x}-0.310737210440e^{-4x}.$$

Hence, the corrected approximate solution for $(N,M)=(3,4)$ is computed as

$$y_{3,4}(x)=y_3(x)+e_{3,4}(x)=1.49403734933-1.66362987469e^{-x}-0.465589509058e^{-2x}$$
$$+0.945919244855e^{-3x}-0.310737210440e^{-4x}.$$

In a similar way, we solve the problem by our method for $(N, M) = (3, 7), (6, 7), (6, 9), (15, 16), (15, 18)$. In Table 1, we tabulate the numerical results of the exact solution and the obtained approximate solutions (exponential solutions and corrected exponential solutions). Also, we give the actual absolute errors, the estimated absolute errors of the exponential solution, and the absolute errors of the corrected exponential solution for the same values of (N, M) in Table 2. Figures 1–3 show the comparisons of the absolute error functions (actual and estimated) for different values of (N, M). In Figures 4–6, we compare the absolute error function e_N and the corrected absolute error functions $E_{N,M}$ for some values of (N, M). We observe from Table 2 and Figures 1–3 that the estimated absolute errors are quite close to the actual absolute errors. Also, it is seen from Table 2 and Figures 4–6 that the corrected error function $E_{N,M}$ is better than the uncorrected error function e_N.

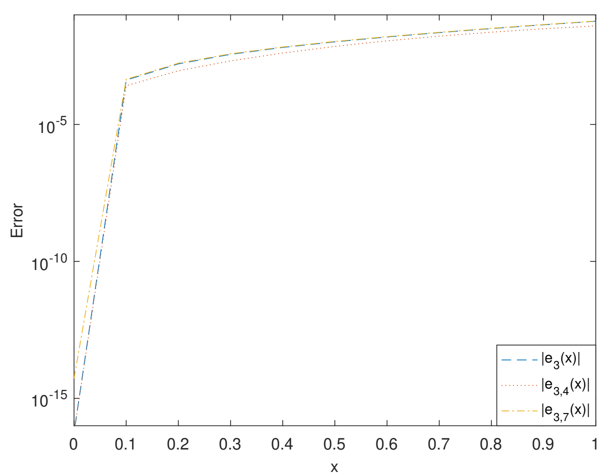

Figure 1. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the estimated absolute error functions $\left|e_{N,M}(x)\right|$ for $N = 3$ and $M = 4, 7$ of Eq. (5.1).

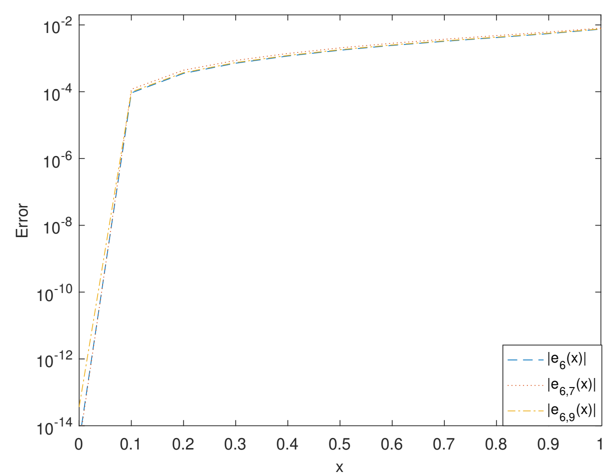

Figure 2. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the estimated absolute error functions $\left|e_{N,M}(x)\right|$ for $N = 6$ and $M = 7, 9$ of Eq. (5.1).

Example 5.2 We consider the Fredholm–Volterra integro-differential equation

$$y^{(3)}(x) - xy^{(1)}(x) + xy(x) + \int_0^1 \left[e^{-x-t}y^{(1)}(t) + e^{x-t}y^{(2)}(t)\right]dt - \int_0^x e^{-x+t}y(t)dt = (x-1)e^{-x} - \frac{(e^2-1)(e^{-x}-e^x)}{e^2}$$

under the initial conditions $y(0) = 1$, $y^{'}(0) = -1$, $y^{''}(0) = 1$. If this problem is solved by our method for $N = 3$ in the interval $0 \leq x, t \leq 1$, the approximate solution is obtained as $y(x) = e^{-x}$. We note that this solution is the exact solution of the problem at the same time.

Example 5.3 [27] Now let us solve the Fredholm integro-differential equation

$$y^{'}(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, 0 \leq x, t \leq 1 \tag{5.3}$$

under the initial conditions $y(0) = 1$. The exact solution of the problem is $y(x) = xe^x$. By applying the procedure in Section 3 for $(N, M) = (8, 8), (12, 12), (15, 15)$, we find the corrected exponential approximate

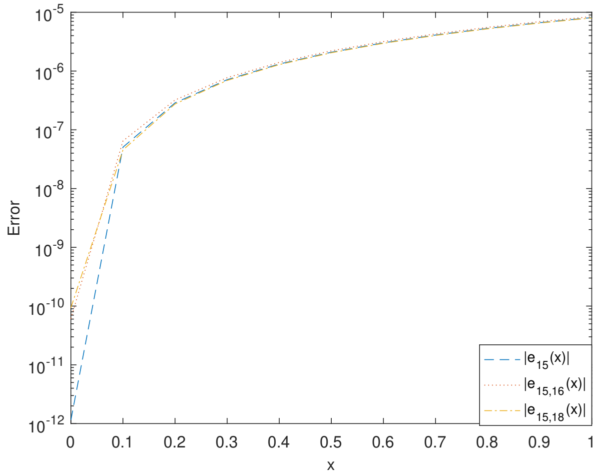

Figure 3. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the estimated absolute error functions $\left|e_{N,M}(x)\right|$ for $N = 15$ and $M = 16, 18$ of Eq. (5.1).

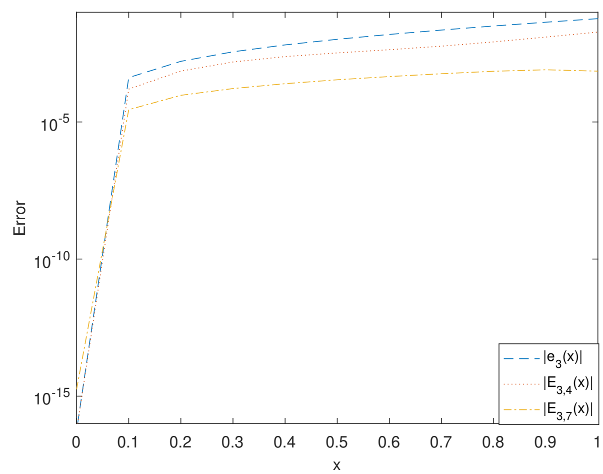

Figure 4. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the corrected absolute error functions $\left|E_{N,M}(x)\right| = \left|y(x) - y_{N,M}(x)\right|$ for $N = 3$ and $M = 4, 7$ of Eq. (5.1).

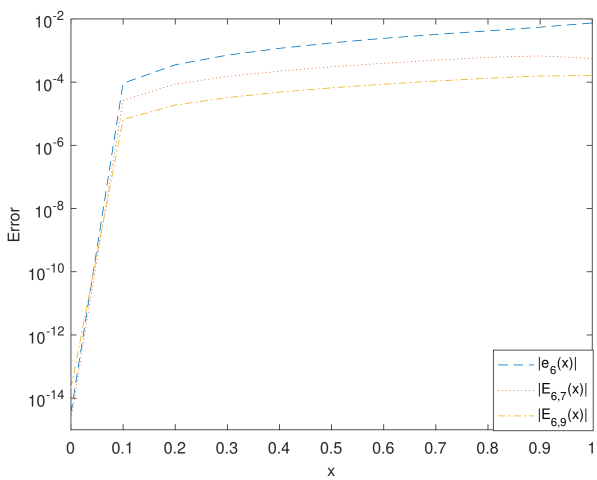

Figure 5. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the corrected absolute error functions $\left|E_{N,M}(x)\right| = \left|y(x) - y_{N,M}(x)\right|$ for $N = 6$ and $M = 7, 9$ of Eq. (5.1).

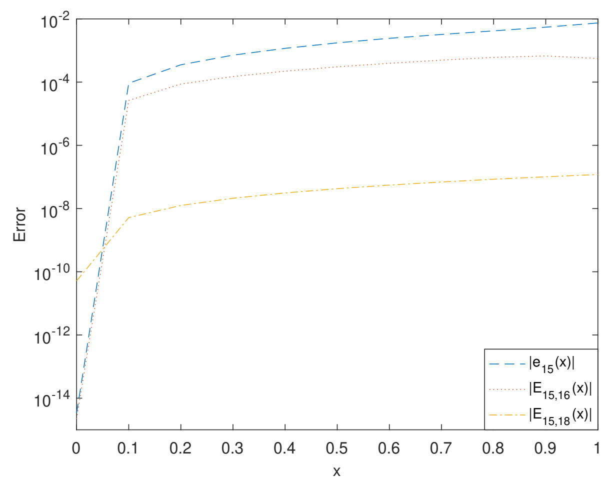

Figure 6. Comparison of the absolute error functions $\left|e_N(x)\right| = \left|y(x) - y_N(x)\right|$ and the corrected absolute error functions $\left|E_{N,M}(x)\right| = \left|y(x) - y_{N,M}(x)\right|$ for $N = 15$ and $M = 16, 18$ of Eq. (5.1).

solutions. In Table 3, the absolute errors of them are compared with the absolute errors of the other methods: the differential transformation method (DTM) [4], the CAS wavelet method (CASWM) [3], and the improved homotopy perturbation method (IHPM) [27]. It is seen from these comparisons that our results are quite good.

Table 1. Numerical results of the exact solution and approximate solutions for $(N, M) = (3,4), (3,7), (6,7), (6,9), (15,16), (15,18)$ of Eq. (5.1).

x_i	Exact solution	Exponential solution	Corrected exponential solution	
	$y(x_i) = sin(x_i)$	$y_3(x_i)$	$y_{3,4}(x_i)$	$y_{3,7}(x_i)$
0	0	-0.5551115e-16	0.55511151e-16	0.17763568e-14
0.2	0.198669330795	0.200295302828	0.199386687541	0.198575557070
0.4	0.389418342309	0.395879803375	0.391838684023	0.389169666815
0.6	0.564642473395	0.580186646791	0.568955006473	0.564189913337
0.8	0.717356090900	0.748934077896	0.725664884684	0.716649791192
1	0.841470984808	0.899960206415	0.860414628213	0.840758097706
x_i	$y(x_i) = sin(x_i)$	$y_6(x_i)$	$y_{6,7}(x_i)$	$y_{6,9}(x_i)$
0	0	0.33306690e-14	0.44408920e-14	0.31086244e-13
0.2	0.198669330795	0.199022602131	0.198582420244	0.198650549194
0.4	0.389418342309	0.390589998625	0.389196220499	0.389370124569
0.6	0.564642473395	0.567073285636	0.564247955627	0.564556443736
0.8	0.717356090900	0.721535078786	0.716750356012	0.717223752743
1	0.841470984808	0.848903039745	0.840911962707	0.841309490649
x_i	$y(x_i) = sin(x_i)$	$y_{15}(x_i)$	$y_{15,16}(x_i)$	$y_{15,18}(x_i)$
0	0	0.115107923e-11	-0.21827872e-10	0.513864506e-10
0.2	0.198669330795	0.198669618297	0.198669395902	0.198669343311
0.4	0.389418342309	0.389419659460	0.389418505380	0.389418373488
0.6	0.564642473395	0.564645494905	0.564642762600	0.564642528573
0.8	0.717356090900	0.717361404766	0.717356534287	0.717356175421
1	0.841470984808	0.841479102516	0.841471612391	0.841471103997

6. Conclusions

In this paper, we have presented an exponential collocation method for the solutions of the Fredholm–Volterra integro-differential equations under initial and boundary conditions. In addition, an error estimation technique was given for the approximate solutions and also the approximate solutions were improved by means of the residual correction method. It is observed from Table 2 and Figures 1–3 that the given error estimation scheme is very good. They are quite close to the actual absolute errors. This error estimation technique can be used to test the reliability of the solutions in the absence of the exact solutions of the problem. Also' by Also, it is seen from Figures 4-6 and Tables 1-2 that the corrected errors are better. In addition, the results of our method were compared with the results of other methods in Table 3. It was observed from these comparisons that our method gives more effective results than the other methods. In addition, if the exact solution of the problem exists and it is an exponential polynomial, such as in Example 2, then the exact solution of the problem can be computed by this method. This feature is seen from Example 2. Finally, we note that this method can be applied for other problems such as systems of integro-differential equations.

Table 2. Comparison of the absolute errors for $(N, M) = (3,4), (3,7), (6,7), (6,9), (15,16), (15,18)$ of Eq. (5.1).

x_i	Absolute errors for the exponential solution	Estimated absolute errors for exponential solution		Absolute errors for the corrected exponential solution	
	$\lvert e_3(x_i)\rvert = \lvert y(x_i) - y_3(x_i)\rvert$	$\lvert e_{3,4}(x_i)\rvert$	$\lvert e_{3,7}(x_i)\rvert$	$\lvert E_{3,4}(x_i)\rvert$	$\lvert E_{3,7}(x_i)\rvert$
0	5.5511e-017	5.5511e-017	5.3291e-015	5.5511e-017	1.7764e-015
0.2	1.6260e-003	9.0862e-004	1.7197e-003	7.1736e-004	9.3774e-005
0.4	6.4615e-003	4.0411e-003	6.7101e-003	2.4203e-003	2.4868e-004
0.6	1.5544e-002	1.1232e-002	1.5997e-002	4.3125e-003	4.5256e-004
0.8	3.1578e-002	2.3269e-002	3.2284e-002	8.3088e-003	7.0630e-004
1	5.8489e-002	3.9546e-002	5.9202e-002	1.8944e-002	7.1289e-004
x_i	$\lvert e_6(x_i)\rvert = \lvert y(x_i) - y_6(x_i)\rvert$	$\lvert e_{6,7}(x_i)\rvert$	$\lvert e_{6,9}(x_i)\rvert$	$\lvert E_{6,7}(x_i)\rvert$	$\lvert E_{6,9}(x_i)\rvert$
0	3.3307e-015	1.7764e-015	1.6875e-014	1.7764e-015	1.1546e-014
0.2	3.5327e-004	4.4018e-004	3.7205e-004	8.6911e-005	1.8782e-005
0.4	1.1717e-003	1.3938e-003	1.2199e-003	2.2212e-004	4.8218e-005
0.6	2.4308e-003	2.8253e-003	2.5168e-003	3.9452e-004	8.6030e-005
0.8	4.1790e-003	4.7847e-003	4.3113e-003	6.0573e-004	1.3234e-004
1	7.4321e-003	7.9911e-003	7.5935e-003	5.5902e-004	1.6149e-004
x_i	$\lvert e_{15}(x_i)\rvert = \lvert y(x_i) - y_{15}(x_i)\rvert$	$\lvert e_{15,16}(x_i)\rvert$	$\lvert e_{15,18}(x_i)\rvert$	$\lvert E_{15,16}(x_i)\rvert$	$\lvert E_{15,18}(x_i)\rvert$
0	1.1511e-012	1.0118e-011	9.3621e-011	2.1828e-011	5.1386e-011
0.2	2.8750e-007	2.2240e-007	2.7499e-007	6.5107e-008	1.2516e-008
0.4	1.3172e-006	1.1541e-006	1.2860e-006	1.6307e-007	3.1180e-008
0.6	3.0215e-006	2.7323e-006	2.9663e-006	2.8921e-007	5.5178e-008
0.8	5.3139e-006	4.8705e-006	5.2293e-006	4.4339e-007	8.4522e-008
1	8.1177e-006	7.4901e-006	7.9985e-006	6.2758e-007	1.1919e-007

Table 3. Comparison of the absolute errors obtained by the different methods of Eq. (5.3).

x_i	DTM [4]	CASWM [3]	IHPM [27]	Present method		
				$\lvert E_{8,8}(x_i)\rvert$	$\lvert E_{12,12}(x_i)\rvert$	$\lvert E_{15,15}(x_i)\rvert$
0.1	1.00118e-02	1.34917e-03	2.31481e-06	6.3466e-004	2.6105e-005	2.3870e-006
0.2	2.78651e-02	1.15960e-03	9.25925e-06	6.0386e-004	2.5064e-005	2.4019e-006
0.4	7.55356e-02	5.93105e-02	3.70370e-05	6.4560e-004	2.6888e-005	2.5650e-006
0.6	1.09551e-01	4.39287e-02	8.33333e-05	7.0993e-004	2.9752e-005	2.8409e-006
0.8	6.94512e-02	1.34514e-02	1.48148e-04	7.9955e-004	3.3750e-005	3.2270e-006
0.9	1.00034e-02	1.32045e-02	1.87500e-04	8.7135e-004	3.6310e-005	3.4609e-006

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