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Conformal Riemannian maps from almost Hermitian manifolds

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Abstract: Conformal Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, namely conformal invariant Riemannian maps, holomorphic conformal Riemannian maps, and conformal antiinvariant Riemannian maps, are introduced. We mainly focus on conformal antiinvariant Riemannian maps from Kaehlerian manifolds. We give proper examples of conformal antiinvariant Riemannian maps, obtain the integrability of certain distributions, and investigate the geometry of leaves of these distributions. We also obtain various conditions for such maps to be horizontally homothetic maps.

Key words: Riemannian submersion, Riemannian map, Kaehlerian manifold, conformal holomorphic Riemannian map, conformal antiinvariant Riemannian map

1. Introduction

As a generalization of the notions of isometric immersions and Riemannian submersions, Riemannian maps between Riemannian manifolds were introduced by Fischer [5]; see also [3, 4, 6, 7, 11, 20]. Let $\Phi : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rank\,\Phi \leq min\{m, n\}$, where $dim M_1 = m$ and $dim M_2 = n$. In that case, we state the kernel space of Φ_* by $ker\,\Phi_{*p_1}$ at $p_1 \in M_1$ and take into consideration the orthogonal complementary space $H = (ker\,\Phi_{*p_1})^\perp$ to $ker\,\Phi_{*p_1}$. Thus, the tangent space of M_1 at p_1 has the following decomposition:

$$T_{p_1}M_1 = H_{p_1} \oplus ker\,\Phi_{*p_1}.$$

Denote the range of Φ_{*p_1} by $range\,\Phi_{*p_1}$ and consider the orthogonal complementary space of $range\,\Phi_*$ by $(range\,\Phi_{*p_1})^\perp$ in $T_{\Phi(p_1)}M_2$. The tangent space $T_{\Phi(p_1)}M_2$ has the following decomposition:

$$T_{\Phi(p_1)}M_2 = (range\,\Phi_{*p_1}) \oplus (range\,\Phi_{*p_1})^\perp.$$

A smooth map $\Phi : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called a Riemannian map at $p_1 \in M_1$ if the horizontal restriction is a linear isometry between $((ker\,\Phi_{*p_1})^\perp,\ g_1(p_1)|_{(ker\,\Phi_{*p_1})^\perp})$, and $(range\,\Phi_{*p_1}, g_2(p_2)|_{range\,\Phi_{*p_1}}), p_2 = \Phi(p_1)$ [5]. It means that Φ_* satisfies

$$g_2(\Phi_*(X), \Phi_*(Y)) = g_1(X, Y) \tag{1.1}$$

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2010 *AMS Mathematics Subject Classification:* 53C15 for $X, Y \in H$. One can see that Riemannian maps with $ker\,\Phi_* = \{0\}$ (respectively, $(range\,\Phi_*)^\perp = \{0\}$) are isometric immersions (respectively, Riemannian submersions).

We note that there are many applications of conformal maps. Indeed, conformal maps have been used in medical imaging, computer vision, and geometric modeling [8, 18, 19].

In the literature, isometric immersions of almost Hermitian manifolds [3, 20] and Riemannian submersions from almost Hermitian manifolds [4, 12] have been studied widely. In this paper, as a generalization of antiinvariant submersions and holomorphic submersions, we introduce both conformal antiinvariant Riemannian maps and holomorphic conformal Riemannian maps from complex manifolds to Riemannian manifolds. We give some basic materials in Section 2. In Section 3, we introduce holomorphic conformal Riemannian maps and conformal invariant maps. Although the vertical distribution and the horizontal distribution of these map are invariant with respect to the complex structure of the total manifold, we show that they are different maps by supporting an example. In Section 4, conformal antiinvariant Riemannian maps are introduced. We investigate certain geometric structures of leaves under some conditions. In particular, we find new conditions in terms of pluriharmonic maps for a conformal Riemannian map to be a horizontally homothetic map. We also provide nontrivial examples for all these conformal maps.

2. Preliminaries

Let $\Phi : M \longrightarrow N$ be a smooth map between Riemannian manifolds $(M, g_{_M})$ and $(N, g_{_N})$. $Hom(TM, \Phi^{-1}TN)$ has the pullback connection ∇^Φ and a connection $\overset{M}{\nabla}$ induced from the Levi-Civita connection on M. The second fundamental form along Φ is defined as

$$(\nabla\Phi_*)(X, Y) = \nabla^\Phi_X \Phi_*(Y) - \Phi_*(\overset{M}{\nabla}_X Y) \tag{2.1}$$

for $X, Y \in \Gamma(TM)$ and it is symmetric. From now on, we denote both the Levi-Civita connection of (N, g_N) and its pullback along Φ by $\overset{N}{\nabla}$. From [9], we have $\nabla^{\Phi\perp}_X V$, which is the orthogonal projection of $\overset{N}{\nabla}_X V$ and a linear connection $\nabla^{\Phi\perp}$ on $(\Phi_*(TM))^\perp$, so $\nabla^{\Phi\perp} g_N = 0$ for $X \in \Gamma(TM)$ and $V \in \Gamma((range\,\Phi_*)^\perp)$. Here we give $\mathcal{S}_V$ as

$$\nabla^N_{\Phi_*(X)} V = -\mathcal{S}_{_V}\Phi_*(X) + \nabla^{\Phi\perp}_{_X} V, \tag{2.2}$$

where $\mathcal{S}_{_V}\Phi_*(X)$ is the tangential component of $\nabla^N_{\Phi_* X} V$. $\mathcal{S}_V\Phi_*(X)$ is bilinear with respect to V and both $\Phi_*(X)$ and $\mathcal{S}_V\Phi_*(X)$ depend only on V_p and $\Phi_{*p}(X_p)$ at p. Now we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_X Y = \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}X}\mathcal{H}Y, \mathcal{T}_X Y = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}X}\mathcal{H}Y, \tag{2.3}$$

for $X, Y \in \Gamma(TM)$. $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric and change the roles of distributions $\mathcal{V}$ and $\mathcal{H}$ for any $X \in \Gamma(TM)$ on $(\Gamma(TM), g)$. We can see that $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{\mathcal{V}X}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{\mathcal{H}X}$. In addition the tensor field $\mathcal{T}$ is symmetric on the vertical distribution. From (2.3), we have

$$\overset{M}{\nabla}_V X = \mathcal{H}\overset{M}{\nabla}_V X + \mathcal{T}_V X \quad , \quad \overset{M}{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.4}$$

$$\overset{M}{\nabla}_X Y = \mathcal{H}\overset{M}{\nabla}_X Y + \mathcal{A}_X Y \quad , \quad \overset{M}{\nabla}_X V = \mathcal{A}_X V + \mathcal{V}\overset{M}{\nabla}_X V, \tag{2.5}$$

where $\hat{\nabla}_V W = \mathcal{V}\overset{M}{\nabla}_V W$ for $X, Y \in \Gamma((ker\,\Phi_*)^\perp)$, $V, W \in \Gamma(ker\,\Phi_*)$. A vector field on M is called a projectable vector field if it is related to a vector field on N. Thus, we say a vector field is basic on M if it is both a horizontal and a projectable vector field. From now on, when we mention a horizontal vector field, we always consider a basic vector field.

We now recall conformal Riemannian maps.

Definition 1 *[13] Let $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a smooth map between Riemannian manifolds (M^m, g_M) and (N^n, g_N). Then Φ is a conformal Riemannian map at $p \in M$ if $0 < rank\,\Phi_{*p} \leq min\{m, n\}$ and Φ_{*p} maps $\mathcal{H}(p) = ((ker\,(\Phi_{*p})^\perp)$ conformally onto $range\,(\Phi_{*p})$, i.e. there exists a number $\lambda^2(p) \neq 0$ such that*

$$g_N(\Phi_{*p}(X), \Phi_{*p}(Y)) = \lambda^2(p) g_M(X, Y)$$

for $X, Y \in \mathcal{H}(p)$. Φ is called a conformal Riemannian map if Φ is a conformal Riemannian map at each point $p \in M$.

A conformal Riemannian map $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is proper if $\lambda \neq 1$ and $0 < rank\,\Phi < min\{m, n\}$. We now give the next result, which will be used in the rest of the paper.

Theorem 2.1 *[13] Let $\Phi : (M, g_M) \longrightarrow (N, g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then we get*

$$\begin{aligned}(\nabla\Phi_*)(X, Y)\mid_{range\,\Phi_*} &= X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) \\ &- g_M(X, Y)\Phi_*(grad(\ln\lambda)),\end{aligned} \tag{2.6}$$

where $X, Y \in \Gamma((ker\,\Phi_*)^\perp)$.

From (2.6), for $Y, Z \in \Gamma((ker\,\Phi_*)^\perp)$, we can write $\overset{N}{\nabla}{}^{\Phi}_Y \Phi_*(Z)$ as

$$\begin{aligned}\overset{N}{\nabla}{}^{\Phi}_Y \Phi_*(Z) &= \Phi_*(h\overset{M}{\nabla}_Y Z) + Y(\ln\lambda)\Phi_*(Z) + Z(\ln\lambda)\Phi_*(Y) \\ &- g_M(Y, Z)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^\perp(Y, Z)\end{aligned} \tag{2.7}$$

where $(\nabla\Phi_*)^\perp(Y, Z)$ is the component of $(\nabla\Phi_*)(Y, Z)$ on $(range\,\Phi_*)^\perp$.

3. Holomorphic conformal Riemannian maps

In this section, we introduce holomorphic conformal Riemannian maps and conformal invariant Riemannian maps. Since our aim is the study of conformal antiinvariant maps, we just provide examples and show that a conformal invariant Riemannian map may not be a holomorphic conformal Riemannian map.

Definition 2 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N, J')$ be a map between a Kaehlerian manifold (M, g_M, J) and an almost Hermitian manifold (N, g_N, J'). If*

i) $\Phi : (M, g_M, J) \longrightarrow (N, g_N, J')$ is a conformal Riemannian map,

ii) $J'\Phi_* = \Phi_* J$ on the horizontal distribution,

then Φ is called a holomorphic conformal Riemannian map.

The following examples show that holomorphic conformal Riemannian maps are natural generalizations of holomorphic submersions and holomorphic Riemannian maps.

Example 3.1 *Every holomorphic submersion (see [4]) is a holomorphic conformal Riemannian map with* $\lambda = 1$ *and* $(range\,\Phi_*)^{\perp} = \{0\}$.

Example 3.2 *Every holomorphic Riemannian map (see [16]) is a holomorphic conformal Riemannian map with* $\lambda = 1$.

We say that a holomorphic conformal Riemannian map is proper if $\lambda \neq 1$ and $(range\,\Phi_*)^{\perp} \neq \{0\}$. We now give an example of proper holomorphic conformal Riemannian maps. In the following $\mathbb{R}^{2m}$ denotes the Euclidean $2m$-space with the standard metric. We denote the compatible almost complex structure on $\mathbb{R}^{2m}$ by J, which is defined as

$$J(\overline{a}_1, ..., \overline{a}_{2m}) = (-\overline{a}_2, \overline{a}_1, ..., -\overline{a}_{2m}, \overline{a}_{2m-1}).$$

Example 3.3 *Let* $\Phi : (R^4, g_4, J_1) \longrightarrow (R^4, g_2, J_2)$ *be the map defined by*

$$(x_1, x_2, x_3, x_4) \longrightarrow (e^{x_1}\cos x_2, e^{x_1}\sin x_2, -e^{x_1}\cos x_2, -e^{x_1}\sin x_2)$$

for any point $x \in R^4$. *We obtain the horizontal distribution and the vertical distributions*

$$H = (ker\,\Phi_*)^{\perp} = \left\{ \begin{aligned} X_1 &= \left(e^{x_1}\cos x_2 \frac{\partial}{\partial x_1} - e^{x_1}\sin x_2 \frac{\partial}{\partial x_2}\right), \\ X_2 &= \left(e^{x_1}\sin x_2 \frac{\partial}{\partial x_1} + e^{x_1}\cos x_2 \frac{\partial}{\partial x_2}\right) \end{aligned} \right\},$$

and

$$V = (ker\,\Phi_*) = \left\{ U_1 = \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial x_4} \right\},$$

respectively. It follows that Φ *is a conformal Riemannian map at any point* $p \in R^4$ *with* $\lambda = e^{x_1}\sqrt{2}$. *On the other hand, by using the standard complex structure on* E^4, *one can see that* $J_2[\Phi_*(X_1)] = \Phi_*[J_1(X_1)]$ *and* $J_2[\Phi_*(X_2)] = \Phi_*[J_1(X_2)]$. *Thus,* Φ *is a proper holomorphic conformal Riemannian map.*

We also introduce the following notion as a generalization of invariant Riemannian submersion from almost Hermitian manifolds.

Definition 3 *Let* $\Phi : (M, g_M, J) \longrightarrow (N, g_N, J')$ *be a map between a Kaehlerian manifold* (M, g_M, J) *and an almost Hermitian manifold* (N, g_N, J'). *We say that* Φ *is an invariant conformal Riemannian map if* Φ *is a conformal Riemannian map and satisfies* $J(ker\,\Phi_*) \subset ker\,\Phi_*$ *for any point* $p \in M$.

We note that an invariant conformal Riemannian map may not be a holomorphic conformal Riemannian map. Indeed, we have the following example.

Example 3.4 *Let* $\Phi : (R^4, g_4, J_4) \longrightarrow (R^4, g_2, J_2)$ *be the map defined by*

$$(x_1, x_2, x_3, x_4) \longrightarrow (e^{x_2}\cos x_1, -e^{x_2}\sin x_1, e^{x_2}\sin x_1, -e^{x_2}\cos x_1)$$

for any point $x \in R^4$. Now we get the horizontal distribution

$$H = (ker\,\Phi_*)^\perp = \left\{ \begin{aligned} X_1 &= \left(-e^{x_2}\sin x_1 \frac{\partial}{\partial x_1} + e^{x_2}\cos x_1 \frac{\partial}{\partial x_2}\right), \\ X_2 &= \left(-e^{x_2}\cos x_1 \frac{\partial}{\partial x_1} - e^{x_2}\sin x_1 \frac{\partial}{\partial x_2}\right) \end{aligned} \right\}$$

and the vertical distribution

$$V = (ker\,\Phi_*) = \left\{ U_1 = \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial x_4} \right\}.$$

We can see that the vertical distribution is invariant because of $J_4(U_1) = U_2$. It follows that Φ is a conformal Riemannian map at any point $p \in R^4$ with $\lambda = e^{x_2}\sqrt{2}$. However, $J_2[\Phi_*(X_i)] \neq \Phi_*[J_4(X_i)], i = 1, 2$. Thus, Φ is not a holomorphic conformal Riemannian map.

4. Conformal antiinvariant Riemannian maps

In this section we study conformal antiinvariant Riemannian maps, provide examples, and investigate the geometry of leaves arising from such maps. We find new conditions for conformal Riemannian maps to be homothetic Riemannian maps by using the notion of pluriharmonic maps.

Definition 4 *Let $\Phi : (M, g_M, J_M) \longrightarrow (N, g_N)$ be a conformal Riemannian map between a Kaehlerian manifold (M, g_M, J_M) and a Riemannian manifold (N, g_N). If Φ satisfies the following condition:*

$$J_M(ker\,\Phi_*) \subset (ker\,\Phi_*)^\perp,$$

then Φ is called a conformal antiinvariant Riemannian map. In particular, if Φ satisfies the following condition:

$$J_M(ker\,\Phi_*) = (ker\,\Phi_*)^\perp,$$

then Φ is called a Lagrangian conformal Riemannian map.

Let Φ be a conformal antiinvariant Riemannian map. Then, for $X \in \Gamma((ker\,\Phi_*)^\perp)$, we write

$$JX = BX + CX \tag{4.1}$$

where $BX \in \Gamma(ker\,\Phi_*)$, $CX \in \Gamma(\mu)$. For $V \in \Gamma(ker\,\Phi_*)$ we have $0 = g_M(X, V)$. Hence, we get

$$g_M(CX, JV) = 0. \tag{4.2}$$

Thus, we get

$$(ker\,\Phi_*)^\perp = \mu \oplus J(ker\,\Phi_*).$$

One can see that μ is invariant with respect to J. We now give examples of conformal antiinvariant Riemannian maps.

Example 4.1 *Every antiinvariant Riemannian submersion [14] is a conformal antiinvariant Riemannian map with $\lambda = 1$ and $(range\,\Phi_*)^\perp = \{0\}$.*

Example 4.2 *Every Lagrangian submersion [14, 17] is a conformal Lagrangian Riemannian map with* $\lambda = 1$ *and* $(range\,\Phi_*)^\perp = \{0\}$.

Example 4.3 *Every conformal antiinvariant Riemannian submersion [1] is a conformal antiinvariant Riemannian map with* $(range\,\Phi_*)^\perp = \{0\}$.

Example 4.4 *Every antiinvariant Riemannian map [15] is a conformal antiinvariant Riemannian map with* $\lambda = 1$.

We say that a conformal antiinvariant Riemannian map is proper if $(range\,\Phi_*)^\perp \neq \{0\}$ and $\lambda \neq 1$. Here is an example of this type map.

Example 4.5 *Let* $\Phi : (R^4, g_4, J_4) \longrightarrow (R^5, g_5)$ *be the map from the Kaehlerian manifold* (R^4, g_4, J_4) *to the Riemannian manifold* (R^5, g_5) *defined by*

$$(-e^{x_1}\cos x_3, e^{x_1}\sin x_3, 0, e^{x_1}\cos x_3, -e^{x_1}\sin x_3).$$

Then we obtain horizontal distribution and vertical distribution

$$\begin{aligned} H = (ker\,\Phi_*)^\perp = \Big\{ X_1 &= \left(-e^{x_1}\cos x_3 \frac{\partial}{\partial x_1} + e^{x_1}\sin x_3 \frac{\partial}{\partial x_3}\right), \\ X_2 &= \left(e^{x_1}\sin x_3 \frac{\partial}{\partial x_1} + e^{x_1}\cos x_3 \frac{\partial}{\partial x_3}\right)\Big\}, \end{aligned}$$

and

$$\begin{aligned} V = (ker\,\Phi_*) = \Big\{ U_1 &= \left(e^{x_1}\cos x_3 \frac{\partial}{\partial x_2} - e^{x_1}\sin x_3 \frac{\partial}{\partial x_4}\right), \\ U_2 &= \left(-e^{x_1}\sin x_3 \frac{\partial}{\partial x_2} - e^{x_1}\cos x_3 \frac{\partial}{\partial x_4}\right)\Big\}, \end{aligned}$$

respectively. Hence, we get

$$\Phi_*(X_1) = e^{2x_1}\frac{\partial}{\partial x_1} - e^{2x_1}\frac{\partial}{\partial x_4}, \Phi_*(X_2) = e^{2x_1}\frac{\partial}{\partial x_2} - e^{2x_1}\frac{\partial}{\partial x_5},$$

which shows that Φ *is a conformal Riemannian map with* $\lambda = e^{x_1}\sqrt{2}$. *On the other hand, by direct computations we have*

$$\begin{aligned} J_4 U_1 &= -\sin 2x_3 X_1 - \cos 2x_3 X_2, \\ J_4 U_2 &= -\cos 2x_3 X_1 + \sin 2x_3 X_2, \end{aligned}$$

where we consider the complex structure J_4 *on* R^4 *acting as*

$$J_4 = (-a_4, a_3, -a_2, a_1).$$

Thus, we obtain $J_4(ker\,\Phi_*) \subset (ker\,\Phi_*)^\perp$, *so* Φ *is a conformal antiinvariant Riemannian map.*

Example 4.6 *Let* $\Phi : (R^4, g_4, J_4) \longrightarrow (R^3, g_3)$ *be the map from the Kaehlerian manifold* (R^4, g_4, J_4) *to the Riemannian manifold* (R^3, g_3) *defined by*

$$(e^{x_1}\cos x_3, e^{x_1}\sin x_3, 0).$$

Then we obtain horizontal distribution and vertical distribution

$$\begin{aligned} H = (ker\,\Phi_*)^{\perp} = \Big\{ X_1 &= \left(e^{x_1}\cos x_3 \frac{\partial}{\partial x_1} - e^{x_1}\sin x_3 \frac{\partial}{\partial x_3} \right), \\ X_2 &= \left(e^{x_1}\sin x_3 \frac{\partial}{\partial x_1} + e^{x_1}\cos x_3 \frac{\partial}{\partial x_3} \right) \Big\}, \end{aligned}$$

and

$$\begin{aligned} V = (ker\,\Phi_*) = \Big\{ U_1 &= \left(-e^{x_1}\cos x_3 \frac{\partial}{\partial x_2} - e^{x_1}\sin x_3 \frac{\partial}{\partial x_4} \right), \\ U_2 &= \left(e^{x_1}\sin x_3 \frac{\partial}{\partial x_2} - e^{x_1}\cos x_3 \frac{\partial}{\partial x_4} \right) \Big\}, \end{aligned}$$

respectively. Hence, we get

$$\Phi_*(X_1) = e^{2x_1}\frac{\partial}{\partial x_1}, \Phi_*(X_2) = e^{2x_1}\frac{\partial}{\partial x_2},$$

which shows that Φ *is a conformal Riemannian map with* $\lambda = e^{x_1}$. *On the other hand, by direct computations we have*

$$\begin{aligned} J_4 U_1 &= (e^{x_1}\sin x_3 \frac{\partial}{\partial x_1} + e^{x_1}\cos x_3 \frac{\partial}{\partial x_3}) = X_2, \\ J_4 U_2 &= (e^{x_1}\cos x_3 \frac{\partial}{\partial x_1} - e^{x_1}\sin x_3 \frac{\partial}{\partial x_3}) = X_1, \end{aligned}$$

where we consider the complex structure J_4 *on* R^4 *acting as*

$$J_4 = (-a_4, a_3, -a_2, a_1).$$

Thus, we obtain $J_4(ker\,\Phi_*) = (ker\,\Phi_*)^{\perp}$, *so* Φ *is a Lagrangian conformal Riemannian map.*

We now examine the geometry of certain distributions by assuming the existence of conformal antiinvariant Riemannian maps.

Theorem 4.1 *Let* $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal antiinvariant Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *Then the invariant distribution* μ *is integrable if and only if*

$$A_X JY - A_Y JX = 0,$$

for $X, Y \in \Gamma(\mu)$.

Proof Since M is a Kaehlerian manifold for $X,Y \in \Gamma(\mu)$, we have

$$A_X JY + h\overset{M}{\nabla}_X JY = JA_X Y + Bh\overset{M}{\nabla}_X Y + Ch\overset{M}{\nabla}_X Y. \tag{4.3}$$

If we change the roles of X and Y in (4.3), we have

$$A_Y JX + h\overset{M}{\nabla}_Y JX \quad = \quad JA_Y X + Bh\overset{M}{\nabla}_Y X + Ch\overset{M}{\nabla}_Y X. \tag{4.4}$$

Thus, if we take the vertical parts of (4.3) and (4.4), we get

$$Bh[X,Y] = A_X JY - A_Y JX.$$

If $A_X JY - A_Y JX = 0$, then $Bh[X,Y] = 0$, and we obtain $h[X,Y] \in \Gamma(\mu)$. Conversely, if μ is integrable, then $Bh[X,Y] = 0$. Then the proof is complete. □

Theorem 4.2 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i) *$ker\,\Phi_*$ defines a totally geodesic foliation on M,*

ii) $\overset{N}{\nabla}{}^{\Phi}_{JV}\Phi_*(JW) = \Phi_*(J[JV,W]) - g_M(JV,JW)\Phi_*(grad(\ln\lambda))$

$$+(\nabla\Phi_*)^{\perp}(JV,JW), V, W \in \Gamma(ker\Phi_*),$$

iii) $grad(\ln\lambda) \in \mu$.

Proof From (2.1), (2.6), and (2.7), we have

$$\begin{aligned}
\Phi_*(\overset{M}{\nabla}_{JV}JW) \quad &= \quad -(\nabla\Phi_*)(JV,JW) + \overset{N}{\nabla}_{JV}\Phi_*(JW) \\
&= \quad -JV(\ln\lambda)\Phi_*(JW) - JW(\ln\lambda)\Phi_*(JV) \\
&+ \quad g_M(JV,JW)\Phi_*(grad(\ln\lambda)) - (\nabla\Phi_*)^{\perp}(JV,JW) \\
&+ \quad \overset{N}{\nabla}_{JV}\Phi_*(JW),
\end{aligned} \tag{4.5}$$

for $V,W \in \Gamma(ker\,\Phi_*)$. Now, in a similar way, we derive

$$\begin{aligned}
\Phi_*(\overset{M}{\nabla}_{JV}JW) \quad &= \quad \Phi_*(J[JV,W] + J\overset{M}{\nabla}_W JV) \\
&= \quad \Phi_*(J[JV,W]) - \Phi_*(\overset{M}{\nabla}_W V).
\end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned}
\Phi_*(\overset{M}{\nabla}_W V) \quad &= \quad \Phi_*(J[JV,W]) + JV(\ln\lambda)\Phi_*(JW) \\
&+ \quad JW(\ln\lambda)\Phi_*(JV) - g_M(JV,JW)\Phi_*(grad(\ln\lambda)) \\
&+ \quad (\nabla\Phi_*)^{\perp}(JV,JW) - \overset{N}{\nabla}_{JV}\Phi_*(JW).
\end{aligned} \tag{4.7}$$

Suppose that (i) and (ii) are satisfied for $V,W \in \Gamma(ker\,\Phi_*)$ in (4.7). Then, for $V=W$, we obtain

$$g_M(grad(\ln\lambda), JV)\Phi_*(JV)=0,$$

which shows that $grad(\ln\lambda)\in\Gamma(\mu)$. If (ii) and (iii) are satisfied, by (4.7) we obtain $\Phi_*(\overset{M}{\nabla}_W V)=0$. Thus, $ker\,\Phi_*$ defines a totally geodesic foliation on M for $V,W\in\Gamma(ker\,\Phi_*)$. Supposing that (i) and (iii) are satisfied for $V,W\in\Gamma(ker\,\Phi_*)$ in (4.7), then we get (ii). □

We now recall the notion of a pluriharmonic map from [10].

Definition 5 *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a map from a complex manifold (M,g_M,J) to a Riemannian manifold (N,g_N). Then Φ is called a pluriharmonic map if Φ satisfies the following equation:*

$$(\nabla\Phi_*)(X,Y)+(\nabla\Phi_*)(JX,JY)=0$$

for $X,Y\in\Gamma(TM)$.

We introduce the following notion by considering the above definition. We say that a conformal antiinvariant Riemannian map Φ from a complex manifold (M,g_M,J) to a Riemannian manifold (N,g_N) is a $(ker\,\Phi_*)^\perp$-pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(X,Y)+(\nabla\Phi_*)(JX,JY)=0$$

for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$. By using this notion, we have the next result.

Theorem 4.3 *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M,g_M,J) to a Riemannian manifold (N,g_N). If Φ is a $(ker\,\Phi_*)^\perp$-pluriharmonic map, the distribution $(ker\,\Phi_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{BX}BY) &= X(\ln\lambda)\Phi_*(Y)+Y(\ln\lambda)\Phi_*(X)\\ &+ CX(\ln\lambda)\Phi_*(CY)+CY(\ln\lambda)\Phi_*(CX)\\ &- \Phi_*(A_{CY}BX+A_{CX}BY)\\ &- \Phi_*(grad(\ln\lambda))[g_M(X,Y)+g_M(CX,CY)],\end{aligned}$$

for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$.

Proof Since Φ is a $(ker\,\Phi_*)^\perp$-pluriharmonic map, we obtain

$$(\nabla\Phi_*)(X,Y)+(\nabla\Phi_*)(JX,JY)=0$$

for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$. Then using (4.1) we get

$$\begin{aligned}0 &= (\nabla\Phi_*)(X,Y)+(\nabla\Phi_*)(CX,CY)\\ &+ (\nabla\Phi_*)(BX,CY)+(\nabla\Phi_*)(CX,BY)\\ &+ (\nabla\Phi_*)(BX,BY).\end{aligned}$$

Then taking $range\,\Phi_*$ and $(range\,\Phi_*)^\perp$ components of second fundamental forms we get

$$\begin{aligned} 0 &= (\nabla\Phi_*)^\perp(X,Y) + (\nabla\Phi_*)^\top(X,Y) \\ &+ (\nabla\Phi_*)^\perp(CX,CY) + (\nabla\Phi_*)^\top(CX,CY) \\ &+ (\nabla\Phi_*)(BX,CY) + (\nabla\Phi_*)(CX,BY) \\ &+ (\nabla\Phi_*)(BX,BY). \end{aligned}$$

Since the second fundamental form of Φ is symmetric and from (2.1), (2.5), and (2.6) we find

$$\begin{aligned} 0 &= (\nabla\Phi_*)^\perp(X,Y) + X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) \\ &- g_M(X,Y)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^\perp(CX,CY) + CX(\ln\lambda)\Phi_*(CY) \\ &+ CY(\ln\lambda)\Phi_*(CX) - g_M(CX,CY)\Phi_*(grad(\ln\lambda)) \\ &- \Phi_*(A_{CY}BX) - \Phi_*(A_{CX}BY) - \Phi_*(\overset{M}{\nabla}_{BX}BY), \end{aligned}$$

then, taking $range\,\Phi_*$ components, we obtain

$$\begin{aligned} \Phi_*(\overset{M}{\nabla}_{BX}BY) &= X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) \\ &- g_M(X,Y)\Phi_*(grad(\ln\lambda)) \\ &+ CX(\ln\lambda)\Phi_*(CY) + CY(\ln\lambda)\Phi_*(CX) \\ &- g_M(CX,CY)\Phi_*(grad(\ln\lambda)) \\ &- \Phi_*(A_{CY}BX) - \Phi_*(A_{CX}BY). \end{aligned}$$

Then the proof is complete. □

The notion of a $(ker\,\Phi_*)^\perp$-pluriharmonic map is also useful to characterize the distribution $(ker\,\Phi_*)$.

Theorem 4.4 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If Φ is a $(ker\,\Phi_*)^\perp$-pluriharmonic map, then any two assertions below imply the third assertion:*

i) The distribution $(ker\,\Phi_*)$ defines a totally geodesic foliation,

ii) $grad(\ln\lambda) \in \Gamma(J(ker\,\Phi_*))$,

iii)

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}{}_X\Phi_*(Y) &= \Phi_*(A_{CX}BY) - \Phi_*(JA_XCY) - \Phi_*(Ch\overset{M}{\nabla}_XCY) \\ &- \Phi_*(CA_XBY) - \Phi_*(Jv\overset{M}{\nabla}_XBY) - (\nabla\Phi_*)^\perp(CX,CY) \\ &- g_M(CX,CY)\Phi_*(grad(\ln\lambda)), \end{aligned}$$

for $X, Y \in \Gamma((ker\,\Phi_*)^\perp)$.

Proof Since Φ is a $(ker\,\Phi_*)^\perp$-pluriharmonic map for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$ we have

$$(\nabla\Phi_*)(X,Y)+(\nabla\Phi_*)(JX,JY)=0.$$

Thus, we get

$$\begin{aligned}(\nabla\Phi_*)(X,Y) &+ (\nabla\Phi_*)(JX,JY)\\ &= \overset{N}{\nabla^\Phi}{}_X\Phi_*(Y)+\Phi_*(J\overset{M}{\nabla}_X JY)+\overset{N}{\nabla^\Phi}{}_{CX}\Phi_*(CY)\\ &- \Phi_*(h\overset{M}{\nabla}_{BX}CY)-\Phi_*(A_{CX}BY)-\Phi_*(\overset{M}{\nabla}_{CX}CY)\\ &- \Phi_*(\overset{M}{\nabla}_{BX}BY)\\ &= \overset{N}{\nabla^\Phi}{}_X\Phi_*(Y)+\overset{N}{\nabla^\Phi}{}_{CX}\Phi_*(CY)-\Phi_*(h\overset{M}{\nabla}_{BX}CY)\\ &- \Phi_*(A_{CX}BY)-\Phi_*(\overset{M}{\nabla}_{CX}CY)+\Phi_*(JA_XCY)\\ &+ \Phi_*(CA_XBY)+\Phi_*(Jv\overset{M}{\nabla}_XBY)\\ &+ \Phi_*(Ch\overset{M}{\nabla}_XCY)-\Phi_*(\overset{M}{\nabla}_{BX}BY).\end{aligned}\tag{4.8}$$

From (4.8), (2.6), and (2.7) we get

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{BX}BY) &= \overset{N}{\nabla^\Phi}{}_X\Phi_*(Y)-\Phi_*(h\overset{M}{\nabla}_{BX}CY)-\Phi_*(A_{CX}BY)\\ &+ \Phi_*(JA_XCY)+\Phi_*(Ch\overset{M}{\nabla}_XCY)+\Phi_*(Jv\overset{M}{\nabla}_XBY)\\ &+ \Phi_*(CA_XBY)+(\nabla\Phi_*)^\perp(CX,CY)+CX(\ln\lambda)\Phi_*(CY)\\ &+ CY(\ln\lambda)\Phi_*(CX)-g_M(CX,CY)\Phi_*(grad(\ln\lambda)).\end{aligned}\tag{4.9}$$

Suppose that (i) and (ii) are satisfied in (4.9). Thus, we have $\Phi_*(\overset{M}{\nabla}_{BX}BY)=0$ and $CX(\ln\lambda)\Phi_*(CY)+CY(\ln\lambda)\Phi_*(CX)=0$ for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$. Then, by (4.8), we get (iii). If (ii) and (iii) are satisfied for $X,Y\in\Gamma((ker\,\Phi_*)^\perp)$, we get $\Phi_*(\overset{M}{\nabla}_{BX}BY)=0$. Since $\Phi_*(\overset{M}{\nabla}_{BX}BY)=0$, the distribution $(ker\,\Phi_*)$ defines a totally geodesic foliation. Suppose that (i) and (iii) are satisfied. Putting in (4.9) $X=Y$ we have

$$g_M(CX,grad(\ln\lambda))\Phi_*(CX)=0,$$

which shows that $grad(\ln\lambda)\in\Gamma(J(ker\,\Phi_*))$. □

We now introduce the following notion by adaption from [2, Definition 2.4.18].

Definition 6 *Let $\Phi:M\longrightarrow N$ be a conformal Riemannian map. Then Φ is a horizontally homothetic map if $\mathcal{H}(grad\lambda)=0$.*

We have the next result for the distribution $(ker\,\Phi_*)$.

Theorem 4.5 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i) The distribution $J(ker\,\Phi_*)$ defines a totally geodesic foliation on M,

ii) Φ is a horizontally homothetic map,

iii) $\overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(JV) = (\nabla\Phi_*)^{\perp}(X, JV) + \Phi_*([X, JV]) - \Phi_*(J\overset{M}{\nabla}_{JV} CX)$

for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$.

Proof From (2.1), we obtain

$$\Phi_*(\overset{M}{\nabla}_X JV) = -(\nabla\Phi_*)(X, JV) + \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(JV),$$

for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$. Thus, from (2.6) and (2.7) we have

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_X JV) &= -(\nabla\Phi_*)^{\perp}(X, JV) + \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(JV) + X(\ln\lambda)\Phi_*(JV) \\ &+ JV(\ln\lambda)\Phi_*(X) - g_M(X, JV)\Phi_*(grad(\ln\lambda))\end{aligned}$$

and

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JV} X) &= -\Phi_*([X, JV]) - (\nabla\Phi_*)^{\perp}(X, JV) + \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(JV) \\ &+ X(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(X) \\ &- g_M(X, JV)\Phi_*(grad(\ln\lambda)).\end{aligned}$$

Hence, we get

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JV} JBX) &= -\Phi_*(J\overset{M}{\nabla}_{JV} CX) + \Phi_*([X, JV]) + (\nabla\Phi_*)^{\perp}(X, JV) \\ &- \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(JV) - X(\ln\lambda)\Phi_*(JV) - JV(\ln\lambda)\Phi_*(X) \\ &+ g_M(X, JV)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{4.10}$$

Suppose that (i) and (ii) are satisfied in (4.10). Then we have $\Phi_*(\overset{M}{\nabla}_{JV} JBX) = 0$ and $X(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(X) - g_M(X, JV)\Phi_*(grad(\ln\lambda)) = 0$ for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$, respectively. Thus, by (4.10) we obtain (iii). If (ii) and (iii) are satisfied, we get $\Phi_*(\overset{M}{\nabla}_{JV} JBX) = 0$, so the distribution $J(ker\,\Phi_*)$ defines a totally geodesic foliation on M. Suppose that (i) and (iii) are satisfied. We have $X(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(X) - g_M(X, JV)\Phi_*(grad(\ln\lambda)) = 0$ for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$. Hence, the gradient of $\ln\lambda$ is a vertical vector field. Then Φ is a horizontally homothetic map. □

For the distribution μ, we have the following theorem.

Theorem 4.6 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i) The distribution μ defines a totally geodesic foliation on M,

ii) $grad(\ln\lambda) \in \Gamma(J(ker\,\Phi_*))$,

iii) $\overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y) = (\nabla\Phi_*)^{\perp}(X, Y) - g_M(X, Y)\Phi_*(grad(\ln\lambda))$ for $X, Y \in \Gamma(\mu)$.

Proof From (2.6), (2.7), and (2.1), we obtain

$$\begin{aligned}
\Phi_*(\overset{M}{\nabla}_X Y) &= \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y) + g_M(X, Y)\Phi_*(grad(\ln\lambda)) \\
&- X(\ln\lambda)\Phi_*(Y) - Y(\ln\lambda)\Phi_*(X) \\
&- (\nabla\Phi_*)^{\perp}(X, Y). \qquad (4.11)
\end{aligned}$$

Suppose that (i) and (ii) are satisfied in (4.11). Then we have $\Phi_*(\overset{M}{\nabla}_X Y) = 0$ and $X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) = 0$ for $X, Y \in \Gamma(\mu)$. Then we obtain (iii). If (ii) and (iii) are satisfied, we get $\Phi_*(\overset{M}{\nabla}_X Y) = 0$, so the distribution μ defines a totally geodesic foliation on M. Supposing that (i) and (iii) are satisfied, we get $X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) = 0$ for $X, Y \in \Gamma(\mu)$. Then, for $X = Y$, we have

$$g_M(X, grad(\ln\lambda))\Phi_*(X) = 0,$$

which shows that $grad(\ln\lambda) \in \Gamma(J(ker\,\Phi_*))$. □

We say that a conformal antiinvariant Riemannian map Φ from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a *mixed*-pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(X, V) + (\nabla\Phi_*)(JX, JV) = 0$$

for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$. We get the following result by using this notion.

Theorem 4.7 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If Φ is a mixed-pluriharmonic map, then the following two conditions are satisfied:*

i) $(\nabla\Phi_*)^{\perp}(CX, JV) = 0$,

ii) Φ is a horizontally homothetic map $\Leftrightarrow$ $A_X V = -h\overset{M}{\nabla}_{BX} JV$,

for $X \in \Gamma((ker\,\Phi_*)^{\perp})$, $V \in \Gamma(ker\,\Phi_*)$.

Proof Since Φ is a mixed-pluriharmonic map, from (2.1), (2.6), and (2.7) we obtain

$$\begin{aligned}
0 &= (\nabla\Phi_*)^{\perp}(CX, JV) - \Phi_*(A_X V) - \Phi_*(h\overset{M}{\nabla}_{BX} JV) \\
&+ CX(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(CX), \qquad (4.12)
\end{aligned}$$

for $X \in \Gamma((ker\,\Phi_*)^\perp)$, $V \in \Gamma(ker\,\Phi_*)$. Then, considering the $(range\,\Phi_*)$ and the $(range\,\Phi_*)^\perp$ components, we obtain $(\nabla\Phi_*)^\perp(CX, JV) = 0$ and

$$0 = -\Phi_*(A_X V) - \Phi_*(h\overset{M}{\nabla}_{BX} JV) + CX(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(CX).$$

If Φ is a horizontally homothetic map, we get

$$CX(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(CX) = 0.$$

Thus, we obtain $A_X V = -h\overset{M}{\nabla}_{BX} JV$. Suppose we have $A_X V = -h\overset{M}{\nabla}_{BX} JV$. Then we get

$$0 = CX(\ln\lambda)\Phi_*(JV) + JV(\ln\lambda)\Phi_*(CX). \tag{4.13}$$

We get $JV(\ln\lambda) = 0$ in (4.13) for $JV \in \Gamma(J(ker\,\Phi_*))$, which shows that $J(ker\,\Phi_*)(grad\ln\lambda) = 0$. In a similar way, we derive $CX(\ln\lambda) = 0$ in (4.13) for $CX \in \Gamma(\mu)$, which shows that $\mu(grad\ln\lambda) = 0$. Hence, we obtain $\mathcal{H}(grad\ln\lambda) = 0$. Thus, the proof is complete. □

We say that a conformal antiinvariant Riemannian map Φ from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a $(ker\,\Phi_*)$-pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(V, W) + (\nabla\Phi_*)(JV, JW) = 0$$

for $V, W \in \Gamma(ker\,\Phi_*)$. By using this notion we have the next result, which shows that Φ is horizontally homothetic under some conditions.

Theorem 4.8 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If Φ is a $(ker\,\Phi_*)$-pluriharmonic map, then the following two conditions are satisfied:*

i) $(\nabla\Phi_*)^\perp(JV, JW) = 0$*, for any* $V, W \in \Gamma(ker\,\Phi_*)$*,*

ii) $T_V JV = 0$ *and* $T_V W \in \Gamma(J(ker\,\Phi_*))$*, for any* $V, W \in \Gamma(ker\,\Phi_*)$ *if and only if* Φ *is a horizontally homothetic Riemannian map.*

Proof Since Φ is a $(ker\,\Phi_*)$-pluriharmonic map, from (2.1), (2.6), and (2.7) we get

$$\begin{aligned} 0 &= (\nabla\Phi_*)^\perp(JV, JW) - \Phi_*(T_V W) + JV(\ln\lambda)\Phi_*(JW) \\ &+ JW(\ln\lambda)\Phi_*(JV) - g_M(JV, JW)\Phi_*(grad(\ln\lambda)) \end{aligned} \tag{4.14}$$

for $V, W \in \Gamma(ker\,\Phi_*)$. (i) is now clear. Since Φ is conformal we have

$$0 = \lambda^2 JW(\ln\lambda)g_M(V, V) + \lambda^2 g_M(T_V JV, W) \tag{4.15}$$

for $JV \in \Gamma(J(ker\,\Phi_*))$. If $grad(\ln\lambda) \in \Gamma(\mu) \Leftrightarrow T_V JV = 0$ in (4.15). Now, for $X \in \Gamma(\mu)$, we get

$$0 = \lambda^2 g_M(V, W)g_M(X, grad(\ln\lambda)) + \lambda^2 g_M(T_V W, X). \tag{4.16}$$

If $grad(\ln\lambda) \in \Gamma(J(ker\,\Phi_*)) \Leftrightarrow T_V W \in \Gamma(J(ker\,\Phi_*))$ in (4.16). Thus, we have $T_V JV = 0$, $T_V W \in \Gamma(J(ker\,\Phi_*))$. Conversely, if $T_V JV = 0$ and $T_V W \in \Gamma(J(ker\,\Phi_*))$, first from (4.15) we get $\lambda^2 JW(\ln\lambda) g_M(V,V) = 0$, which implies that $Jker\,\Phi_*(grad(\ln\lambda)) = 0$. On the other hand, from (4.16), we get $\lambda^2 g_M(V,W) g_M(X, grad(\ln\lambda)) = 0$, which implies that $\mu(grad(\ln\lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln\lambda)) = 0$. This shows that Φ is a horizontally homothetic Riemannian map. □

Theorem 4.9 *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal antiinvariant Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is a horizontally homothetic Riemannian map if and only if*

$$\begin{aligned} 0 &= g_N(\overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y), \Phi_*(Z)) + \lambda^2 g_M(A_X BZ, CY) \\ &+ \lambda^2 g_M(CA_X BY + Ch\overset{M}{\nabla}_X CY + Jv\overset{M}{\nabla}_X BY, Z) \end{aligned} \tag{4.17}$$

for $X, Y, Z \in \Gamma((ker\,\Phi_*)^\perp)$.

Proof From (2.1), we have

$$(\nabla\Phi_*)(X,Y) = \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y) - \Phi_*(\overset{M}{\nabla}_X Y).$$

Using (2.6) and (2.7) we obtain

$$\begin{aligned} 0 &= \overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y) + \Phi_*(CA_X BY) + \Phi_*(Jv\overset{M}{\nabla}_X BY) - (\nabla\Phi_*)^\perp(X,Y) \\ &+ \Phi_*(Ch\overset{M}{\nabla}_X CY) + \Phi_*(JA_X CY) + g_M(X,Y)\Phi_*(grad(\ln\lambda)) \\ &- Y(\ln\lambda)\Phi_*(X) - X(\ln\lambda)\Phi_*(Y). \end{aligned} \tag{4.18}$$

Now we find

$$g_M(JA_X CY, Z) = -g_M(A_X CY, JZ) = g_M(A_X BZ, CY).$$

Using this in (4.18), we get

$$\begin{aligned} 0 &= g_N(\overset{N}{\nabla}{}^{\Phi}{}_X \Phi_*(Y), \Phi_*(Z)) + \lambda^2 g_M(CA_X BY, Z) + \lambda^2 g_M(Ch\overset{M}{\nabla}_X CY, Z) \\ &+ \lambda^2 g_M(Jv\overset{M}{\nabla}_X BY, Z) + \lambda^2 g_M(A_X BZ, CY) - \lambda^2 X(\ln\lambda) g_M(Y,Z) \\ &- \lambda^2 Y(\ln\lambda) g_M(X,Z) + \lambda^2 Z(\ln\lambda) g_M(Y,X) \end{aligned} \tag{4.19}$$

for $Z \in \Gamma((ker\,\Phi_*)^\perp)$. If (4.17) is satisfied we derive

$$-\lambda^2 X(\ln\lambda) g_M(Y,Z) - \lambda^2 Y(\ln\lambda) g_M(X,Z) + \lambda^2 Z(\ln\lambda) g_M(Y,X) = 0.$$

Taking $Y = Z$, we get $-\lambda^2 X(\ln\lambda) g_M(Y,Y) = 0$, which shows that $X(\ln\lambda) = 0$, i.e. Φ is a horizontally homothetic Riemannian map. The converse is clear. □

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