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BANU GÜNTÜRK

BAHAETTİN CENGİZ

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Banu GÜNTÜRK[*], **Bahaettin CENGİZ**
Faculty of Engineering, Başkent University, Ankara, Turkey

Abstract: This paper is devoted to hyperstonean spaces that are precisely the Stone spaces of measure algebras, or the Stone spaces of the Boolean algebras of L^p-projections of Banach spaces for $1 \leq p < \infty$, $p \neq 2$. Several new results that have been achieved recently are discussed. Among these, in our opinion, the most significant one is that which states that any Bochner L^p space is the p-direct sum of Bochner L^p-spaces of perfect regular Borel measures on Stonean spaces for $1 \leq p < \infty$. Overall, we try to shed some light on the inner structure of these spaces, about which very little is known.

Key words: Stonean space, perfect measure, equivalent measures, Bochner space, p-direct sum

1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach space of all continuous scalar-valued functions on X with the usual supremum norm. On the one hand, it is a rare event in analysis that $C(X)$ is the dual of a Banach space, but, on the other hand, as we shall see soon, it is not so rare at all. Clearly finite spaces fall into this category, and the Stone–Čech compactifications of infinite discrete spaces are trivial examples of infinite spaces of this kind, as, for any discrete space D, $l^\infty(D) \simeq C(\beta D)$ is the dual of $l^1(D)$, where βD denotes the Stone–Čech compactification of D. (For two normed spaces E and F, $E \simeq F$ means that they are linearly isometric.)

In this article we shall discuss several equivalent descriptions of these spaces, which are made mostly in term of concepts of analysis, and some new properties we obtained recently. We shall also try to shed some light on their topological structure, about which our knowledge is very limited.

Let us recall that a compact Hausdorff space is called *extremally disconnected* if the closure of every open set is open. These spaces are also called Stonean. They are precisely the Stone spaces of complete Boolean algebras [9].

Following Behrends [1], we call an extended real-valued positive Borel measure on a Stonean space perfect if:

(i) the measure of every nonempty open set is strictly positive,

(ii) every nonempty open set contains a clopen (closed and open) set with nonzero finite measure,

*Correspondence: bgunturk@baskent.edu.tr

(iii) the measure of every nowhere dense Borel set is zero (equivalently, the measure of every closed set with empty interior is zero).

A Stonean space with a perfect measure on it is called *hyperstonean* or *a hyperstonean measure space.* The Stone–Čech compactifications of discrete spaces are trivial examples of hyperstonean spaces; for example, $\beta\mathbb{N}$, the Stone–Čech compactification of the discrete space $\mathbb{N}$ of positive integers, is hyperstonean, for the counting measure m on $\mathbb{N}$ can be extended to a perfect measure $\overline{m}$ on $\beta\mathbb{N}$ defined by the equation

$$\overline{m}(B) = m(B \cap \mathbb{N})$$

for every Borel subset B of $\beta\mathbb{N}$.

(The perfectness of $\overline{m}$ follows from the fact that $\beta\mathbb{N} \setminus \mathbb{N}$ is a closed subset of $\beta\mathbb{N}$ with empty interior. More generally, every locally compact Hausdorff space X is an open subset of βX, and therefore $\beta X \setminus X$ is a closed subset of βX with empty interior.)

In [2], Cengiz proved that any arbitrary positive measure is equivalent to a perfect Borel measure on a Stonean space in the sense that for every number $1 \le p < \infty$, their corresponding L^p spaces are linearly isometric. Clearly equivalent measures are equivalent to the same perfect measure. This result shows that the class of all hyperstonean spaces is huge indeed.

It is known that there is essentially one perfect measure on a hyperstonean space, meaning that all perfect measures on the same space are equivalent to one another. It is also a fact that in a hyperstonean measure space, any Borel set differs from a clopen set by a null set (see [1,5]).

Given an arbitrary positive measure space $(X, \mathcal{A}, \nu)$, let $(\Omega, \mathcal{B}, \mu)$ be its hyperstonean equivalent. Examples show that, in general, the scalar $L^\infty(\nu)$ need not be the dual $L^1(\nu)^*$ of $L^1(\nu)$; there may not be enough functions in $L^\infty(\nu)$ to represent (via integral) all the members of $L^1(\nu)^*$ (see [6, p. 349] and [7, p. 249]).

In the above mentioned paper, Cengiz also proved that

$$L^1(\nu)^* \simeq L^1(\mu)^* \simeq L^\infty(\Omega, \mu) \simeq C(\Omega),$$

which shows that the dual of an L^1 space is always an L^∞ space.

Let us recall that a regular Borel measure on a compact Hausdorff space is called *normal* if it vanishes at nowhere dense Borel sets.

Let Ω be a Stonean space, and let N denote the Banach space of all normal measures on Ω and S the union of the supports of all positive normal measures on Ω. (The support of a positive regular Borel measure is the complement of the largest open set of measure zero.)

Dixmier [3] proved that $C(\Omega)$ is the dual of a Banach space if and only if S is dense in Ω. Grothendieck [4] showed that if $C(\Omega)$ is the dual of a Banach space E then $E \simeq N$. Finally, Behrends [1] proved that S is dense in Ω if and only if Ω is hyperstonean. Thus, summing up, we have the following theorem.

Theorem 1 *For a Stonean space Ω the following are equivalent:*

i) Ω is hyperstonean,

ii) $C(\Omega)$ is a dual space,

iii) $C(\Omega)$ is the dual space of N,

iv) S is dense in Ω.

Remark 1 *It should be noted that all of the equivalent descriptions of hyperstonean spaces are given in terms of concepts from functional analysis. As far as we know, there is still no definition made entirely in pure topological terms. Therefore, the problem of topological characterization of hyperstonean spaces still stands open.*

Remark 2 *Although there are some important applications of hyperstonean spaces, it is still mostly an unexploited area in analysis. Our expectation is that it might turn out to be a promising field of research.*

2. The results

The fact that an arbitrary positive measure is equivalent to a perfect one reduces integration in Lebesgue sense (with respect to an arbitrary positive measure) to integration with respect to a perfect Borel measure on a Stonean space. In the new setting, measurable sets may be replaced by clopen sets and measurable simple functions by simple functions of the form $\sum_i^n e_i \chi_{U_i}$, where U_is are clopen sets and e_is are vectors from the range space of the measurable functions under consideration. Consequently, each measurable function (essentially bounded measurable function) is the a.e. limit (a.e. uniform limit) of a sequence of continuous simple functions.

By definition, a regular Borel measure on a compact space is finite while a perfect measure need not be so. The relationship between regularity and perfectness of a Borel measure on a Stonean space is as follows:

Theorem 2 *A perfect Borel measure on a Stonean space is regular if and only if it is finite.*

Proof   Let Z be a Stonean space and μ a perfect measure on it. Since by definition a regular Borel measure on a compact Hausdorff space is finite, if μ is regular, then it has to be finite.

Conversely, we assume that μ is finite, and we show that it is regular.

Since μ is finite, the mapping

$$f \longrightarrow \int_Z f d\mu, \ f \in C(Z)$$

is a bounded linear functional on $C(Z)$, and therefore by the well-known Riesz representation theorem there exists a regular Borel measure ι on Z such that

$$\int_Z f d\mu = \int_Z f d\iota \text{ for all } f \in C(Z).$$

For each clopen subset G of Z, the characteristic function χ_G is continuous, and therefore

$$\mu(G) = \iota(G). \tag{1}$$

Next we show that (1) also holds for open sets.

Let $U \subset Z$ be an open subset. Since μ is perfect and $\overline{U}$ is clopen,

$$\mu(U) = \mu(\overline{U}) = \iota(\overline{U}). \tag{2}$$

On the other hand, each compact set contained in U has a clopen neighborhood also contained in U. By the regularity of ι we can find a sequence of clopen sets $K_1 \subset K_2 \subset \ldots \subset U$ such that

$$\iota(\overline{K}) = \mu(\overline{K}) = \mu(K) = \lim_n \mu(K_n) = \lim_n \iota(K_n) = \iota(K) = \iota(U) \tag{3}$$

where $K = \bigcup K_n$.

Since $0 = \iota(U \setminus K) \geq \iota(U \setminus \overline{K})$, and $\iota \geq 0$, $\iota(U \setminus \overline{K}) = 0$, and since ι coincides with μ at clopen sets and μ is perfect, we conclude that the open set $U \setminus \overline{K}$ must be empty or else (1) will be violated, which implies that $U \subset \overline{K}$.

Thus, it follows that

$$\mu(\overline{K}) = \iota(\overline{K}) = \iota(K) = \mu(K) \leq \mu(U) \leq \mu(\overline{K}),$$

so we have equalities throughout. From this and (3) one obtains

$$\mu(U) = \mu(K) = \iota(K) = \iota(U),$$

which proves that (1) also holds for all open sets, and therefore for closed sets as well.

This last result implies that μ and ν coincide at G_δ and F_σ sets. Again using the regularity of ι, for each Borel set B one can find G_δ and F_σ sets such that $G_\delta \supset B \supset F_\sigma$ and

$$\mu(G_\delta) = \iota(G_\delta) = \iota(B) = \iota(F_\sigma) = \mu(F_\sigma)$$

and consequently $\mu(B) = \iota(B)$. □

(Recall that a G_δ set is the intersection of a sequence of open sets, and an F_σ set is a countable union of closed sets.)

We shall fix a hyperstonean measure space $(Z, \mathcal{B}, \mu)$ (i.e. Z is Stonean, $\mathcal{B}$ is the Borel algebra, and μ is perfect), and it will remain fixed throughout the rest of the paper.

As an application of the preceding theorem, we obtain the following important corollary:

Corollary 1 *Any L^p space with $1 \leq p < \infty$ is isometric to a p-direct sum of L^p spaces of perfect regular Borel measures.*

Proof Given any measure space $(X, \mathcal{A}, \mu)$, let $(\Omega, \mathcal{B}, \mu)$ be its hyperstonean equivalent, and let $\{\Omega_i : i \in I\}$ be any disjoint maximal family of clopen subsets of Ω with strictly positive finite measure. (In view of Zorn's lemma, such families exist.) Then for any $1 \leq p < \infty$,

$$L^p(\mu) \simeq L^p(\Omega, \mu) \simeq \sum_i \oplus L^p(\Omega_i, \mu_i) \ (p\text{-direct sum}),$$

where $\mu_i = \mu \mid_{\Omega_i} \ \forall i \in I$. □

(Recall that the p-direct sum $\sum_i \oplus E_i$ of Banach spaces E_i consists of generalized sequences $x = \{x_i : x_i \in E_i, \ i \in I\}$ such that $\|x\| = [\sum_i \|x_i\|^p]^{\frac{1}{p}} < \infty$.)

We call a maximal family $\{B_i : i \in I\}$ of measurable sets a μ-*decomposition* of the measure (or of the measure space) if $0 < \mu(B_i) < \infty$, and $\mu(B_i \cap B_j) = 0 \ \forall i \neq j$ in I.

Theorem 3 *Every maximal family of disjoint clopen sets with strictly positive finite measure is a μ-decomposition.*

Proof First we recall that each Borel subset F of Z is μ-equivalent to a clopen set; that is, there exists clopen set V such that $\mu(F \setminus V) + \mu(V \setminus F) = 0$.

Let $\{\Omega_i : i \in I\}$ be a maximal family of disjoint clopen sets with strictly positive finite measure.

To prove the theorem it suffices to show that if F is a Borel set of finite measure such that $\mu(F \cap \Omega_i) = 0$ for all $i \in I$ then $\mu(F) = 0$.

Now let us consider a Borel set F satisfying these conditions and let V be a clopen set equivalent to F. Since $V = (F \cap V) \cup (V \setminus F)$, we have

$$\mu(V \cap \Omega_i) = \mu(F \cap V \cap \Omega_i) + \mu((V \setminus F) \cap \Omega_i)$$
$$= 0$$

for all $i \in I$, and since the family $\{\Omega_i : i \in I\}$ is maximal we conclude that $V = \emptyset$. Hence, $\mu(F) = 0$. □

Corollary 2 *μ is σ-finite if and only if each clopen μ-decomposition is countable.*

Proof Let $\mathcal{G} = \{\Omega_i : i \in I\}$ be a clopen μ-decomposition (i.e. it is a maximal family of disjoint clopen sets with strictly positive finite measure).

Let F be a Borel set with finite measure. Since $Z \setminus \bigcup_i \Omega_i$ has measure zero we may assume that $F \subset \bigcup_i \Omega_i$. Clearly

$$\sum_i \mu(F \cap \Omega_i) \leq \mu(F) < \infty,$$

which implies that only countably many terms of the series on the left-hand side can be different from zero.

Let $J = \{i \in I : \mu(F \cap \Omega_i) > 0\}$ and let $F_0 = \bigcup_{j \in J} (F \cap \Omega_j)$. Then the set $F \setminus F_0$ has finite measure and as in the proof of the preceding theorem its intersection with each Ω_i has measure zero, which means that is F is contained μ a.e. in the union of the countable subfamily $\{\Omega_j : j \in I\}$ of $\mathcal{G}$.

Now suppose that $Z = \bigcup_{k \geq 1} F_k$ where F_ks are mutually disjoint and have finite measure.

By the above discussion, for each k, F_k is contained almost everywhere in the union of countable subfamily $\mathcal{G}_k$ of $\mathcal{G}$, which implies that Z is contained a.e. in the union of the countable subfamily $\bigcup_k \mathcal{G}_k$ of $\mathcal{G}$. Thus, $\mathcal{G} = \bigcup_{k \geq 1} \mathcal{G}_k$ is countable.

The converse is trivial, for if $\mathcal{G} = \{\Omega_k : k = 1, 2, 3, \ldots\}$ is a clopen μ decomposition of μ then $\mathcal{G} \cup \{Z \setminus \bigcup \Omega_k\}$ is a countable partition of Z of sets with finite measure. □

The following theorem establishes a relation between the σ-finiteness of μ and the existence of a finite perfect measure.

Theorem 4 *μ is σ-finite if and only if there exists a perfect regular Borel measure on Z.*

Proof Suppose that μ is σ-finite and let $\mathcal{F} = \{\Omega_n : n \in \mathbb{N}\}$ be a clopen μ-decomposition of μ and define ν on the Borel algebra $\mathcal{B}$ as follows:

$$\nu(B) = \sum_{n \geq 1} \frac{1}{2^n \mu(\Omega_n)} \mu(B \cap \Omega_n),\ B \in \mathcal{B}.$$

The set $Z_0 = Z \setminus \bigcup_n \Omega_n$ is null and

$$\nu(Z) \leq \sum_{n \geq 1} \frac{1}{2^n} \leq 1.$$

Let $V \neq \emptyset$ be any open set. By the maximality of the family $\mathcal{F}$, we must have $V \cap \Omega_{n_0} \neq \emptyset$ for some n_0. $V \cap \Omega_{n_0}$ is a nonempty open set, and so it contains a clopen set W with strictly positive finite μ-measure. From this we obtain

$$\nu(W) = \sum_{n \geq 1} \frac{1}{2^n \mu(\Omega_n)} \mu(W \cap \Omega_n) = \frac{1}{2^{n_0} \mu(\Omega_{n_0})} \mu(W) > 0.$$

Thus, we have shown that every nonempty open set contains a clopen set with strictly positive finite ν-measure.

Now let F be a closed set with empty interior. Then $\forall n$, $F \cap \Omega_n$ is a closed set with empty interior, and so $\mu(F \cap \Omega_n) = 0$, $\forall n$, which implies that $\nu(F) = 0$. Hence, we have shown that ν is a finite perfect measure, and therefore, by Theorem 2, it is regular.

Conversely, assume that there exists a perfect regular Borel measure ν on Z and show that μ is σ-finite.

Let $f_0 = \frac{d\nu}{d\mu}$, (the Radon–Nikodým derivative) and fix any clopen μ-decomposition $\mathcal{F} = \{\Omega_j : j \in J\}$. Since ν is finite, f_0 is μ-integrable, which implies that the support S of f_0 is μ-σ-finite. Therefore, as in the proof of Corollary 2, S is contained μ-a.e. in the union of a countable subfamily $\{\Omega_j : j \in J\}$ of the family $\mathcal{F}$, and so for each $i \in I \setminus J$, $\nu(\Omega_i) = 0$, but since ν is perfect this last observation implies that $J = I$, proving that the measure μ is σ-finite. □

Example 1 *If D is an uncountable discrete space then none of the perfect measures on βD are σ-finite, but if D is countable then all of them are σ-finite. In particular, all of the perfect measures on $\beta \mathbb{N}$ are σ-finite. A perfect measure μ on $\beta \mathbb{N}$ has the following form:*

$$\mu = \sum_{n \geq 1}^{\infty} a_n \delta_n,$$

where each $\alpha_n > 0$, and δ_n is the unit point mass at n. (This is because of the facts that each singleton $\{n\}$ is open, so $\alpha_n = \delta_n(\{n\}) > 0$, and that $\mu(\beta \mathbb{N} \setminus \mathbb{N}) = 0$.)

More generally, each perfect measure μ on βD, with D discrete, has the form

$$\mu = \sum_{x \in D} \alpha_x \delta_x,\ \alpha_{x > 0}\ \forall x \in D.$$

Remark 3 *Unlike the previous ones, the next result is purely topological. It is known that the closure of every open subset of a Stonean space Z is its Stone–Čech compactification [8, p. 109]; however if Z is hyperstonean, the proof becomes very simple and brief.*

Proof Let U be an open set and $W = \overline{U}$. We shall try to prove that every bounded, continuous scalar-valued function on U can be extended to a continuous function on W.

W is a clopen subset of Z, so it is a hyperstonean space with the perfect measure $\mu \mid_W$ on it. Since $\mu(W \setminus U) = 0$ (for $W \setminus U$ is a closed set with empty interior), we have

$$L^1(W, \mu) \simeq L^1(U, \mu) \text{ (the isometry here is } f \longrightarrow f \mid_U \text{)}.$$

We know that

$$L^1(W, \mu)^* \simeq L^\infty(W, \mu) \simeq C(W) \text{ ([1])}.$$

Let g be a bounded continuous scalar function on U and define g_0 on W as follows:

$$g_0 = g \text{ on } U \text{ and } g_0 = 0 \text{ on } W \setminus U.$$

Since $g_0 \in L^\infty(\mu)$, there exists a function h in $C(W)$ such that $g_0 = h$ μ a.e., which implies that $g_0 = h$ on U; that is, h is a continuous extension of g to W. □

Remark 4 *This result is a partial answer to the problem of characterizing hyperstonean spaces in terms of purely topological terms. It is necessary but not sufficient. There are Stonean spaces that are not hyperstonean; for instance, the Dedekind completion of $C([0,1], \mathbb{R})$ is an M-space with a strong unit whose Kakutani space is Stonean but not hyperstonean ([8, p. 123]).*

Problem 1 *To the best of our knowledge, it is not known whether or not the condition "the closure of every open subset of a Stonean space is its Stone–Čech compactification" makes the space hyperstonean.*

Corollary 3 *Either all or none of the perfect measures on a hyperstonean space must be σ-finite.*

Theorem 5 *Every Stonean space is a topological direct sum of a hyperstonean space and a Stonean space on which there is no nonzero normal Borel measure.*

Proof Let Y be a Stonean space and ν be a normal measure on Y. By definition, the support of ν is the complement of the largest open set U with ν-measure zero. (This definition is valid for regular Borel measures.)

Since $\overline{U} \setminus U$ is a closed set with empty interior and ν is normal, the measure of $\overline{U} \setminus U$ is zero, and so, by the maximality of U, we conclude that U is clopen. Consequently, the support of ν is clopen.

Now let W denote the closure of the supports of all the normal measures on Y, and let $V \subset Y$ be any open set. $V \cap W$ is an open subset of W (as a subspace). Since $\overline{V}$ is also clopen, $\overline{V} \cap W$, which is the closure of $V \cap W$ in the subspace W, is a clopen subset of W. This discussion shows that W is a Stonean space and the supports of all the normal measures on W are dense in W. Hence, W is a hyperstonean space, and clearly the largest such subspace of Y. Since W is clopen in Y, then so is its complement Δ. Hence, $Y = W \oplus \Delta$, and clearly there is no nonzero normal measure on Δ. □

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