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AHMET TURAN GÜRKANLI

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Inclusions and the approximate identities of the generalized grand Lebesgue spaces

A. Turan GÜRKANLI* 

Department of Mathematics and Computer Science, Faculty of Science and Letters, İstanbul Arel University, Tepekent-Büyükçekmece İstanbul, Turkey

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Abstract: Let (Ω, Σ, μ) and (Ω, Σ, ν) be two finite measure spaces and let $L^{p,\theta}(\mu)$ and $L^{q,\theta}(\nu)$ be two generalized grand Lebesgue spaces [9, 10], where $1 < p, q < \infty$ and $\theta \geq 0$. In Section 2 we discuss the inclusion properties of these spaces and investigate under what conditions $L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ for two different measures μ and ν . Let Ω be a bounded subset of \mathbb{R}^n . We know that the Lebesgue space $L^p(\mu)$ admits an approximate identity, bounded in $L^1(\mu)$, [5, 8]. In Section 3 we investigate the approximate identities of $L^{p,\theta}(\mu)$ and show that it does not admit such an approximate identity. Later we discuss approximate identities of the space $[L^p]_{p,\theta}$, the closure of $C_c^\infty(\Omega)$ in $L^{p,\theta}(\mu)$, where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable complex-valued functions with compact support on \mathbb{R}^n .

Key words: Lebesgue space, grand Lebesgue space, generalized grand Lebesgue space

1. Introduction

Let (Ω, Σ, μ) be a measure space. It is well known that $\ell^p(\Omega) \subseteq \ell^q(\Omega)$ whenever $0 < p \leq q \leq \infty$. Subramanian [19] investigated all positive measures μ on Ω for which $L^p(\mu) \subseteq L^q(\mu)$ whenever $0 < p \leq q \leq \infty$. Romero [17] improved and completed some results of Subramanian. Miamee [13] considered the more general inclusion $L^p(\mu) \subseteq L^q(\nu)$, where μ and ν are two measures. Gürkanlı [10] generalized these results to the Lorentz spaces.

Let Ω be a nonempty set, Σ a σ -algebra of subsets of Ω and μ a positive finite measure on the measurable space (Ω, Σ) . The grand Lebesgue space $L^p(\mu)$ was introduced in [11]. This is a Banach space defined by the norm

$$\|f\|_p = \sup_{0 < \varepsilon \leq p-1} \left(\varepsilon \int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} ;$$

where $1 < p < \infty$. For $0 < \varepsilon \leq p-1$, $L^p(\mu) \subset L^p(\mu) \subset L^{p-\varepsilon}(\mu)$ hold. For some properties and applications of $L^p(\mu)$ spaces we refer to papers [1–4, 6, 11]. A generalization of the grand Lebesgue spaces are the spaces

*Correspondence: turangurkanli@arel.edu.tr

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$L^{p),\theta}(\mu)$, $\theta \geq 0$, defined by the norm (see [1, 11])

$$\|f\|_{p),\theta,\mu} = \|f\|_{p),\theta} = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} < \infty;$$

when $\theta = 0$ the space $L^{p),0}(\mu)$ reduces to the Lebesgue space $L^p(\mu)$ and when $\theta = 1$ the space $L^{p),1}(\mu)$ reduces to the grand Lebesgue space $L^p(\mu)$. More precisely, we have for all $1 < p < \infty$ and $0 < \varepsilon \leq p-1$

$$L^p(\mu) \subset L^{p),\theta}(\mu) \subset L^{p-\varepsilon}(\mu).$$

Different properties and applications of these spaces were discussed in [1, 2, 6, 7, 9].

If μ and ν are two measures on a σ -algebra Σ of subsets of Ω , we say that ν is absolutely continuous with respect to μ if $\nu(E) = 0$ for every $E \in \Sigma$ such that $\mu(E) = 0$. We denote it by the symbol $\nu \ll \mu$. If μ and ν are absolutely continuous with respect to each other (i.e. $\nu \ll \mu$ and $\mu \ll \nu$) then we denote it by the symbol $\mu \approx \nu$.

Let A be a Banach algebra. A Banach space $(B, \|\cdot\|_B)$ is called Banach module over $(A, \|\cdot\|_A)$ if B is a module over A in the algebraic sense for some multiplication, $(a, b) \rightarrow a.b$, and satisfies

$$\|a.b\|_B \leq \|a\|_A \|b\|_B.$$

An approximate identity in a Banach algebra A is a net $(e_\alpha)_{\alpha \in I} \subset A$ such that for every $f \in A$,

$$\lim_{\alpha} \|f e_\alpha - f\| = 0.$$

For two Banach modules B_1 and B_2 over a Banach algebra A , we write $M_A(B_1, B_2)$ or $Hom_A(B_1, B_2)$ for the space of all bounded linear operators T from B_1 into B_2 satisfying $T(ab) = aT(b)$ for all $a \in A, b \in B_1$. These operators are called multipliers (right) or module homomorphism from B_1 into B_2 , [12, 14–16]. By Corollary 2.13 in [15],

$$Hom_A(B_1, B_2^*) \cong (B_1 \otimes_A B_2)^*,$$

where B_2^* is the dual of B and \otimes_A is the A -module tensor product.

2. Inclusions of generalized grand Lebesgue spaces

In this section we will accept that $1 < p, q < \infty$, $\theta \geq 0$, and (Ω, Σ) is a measurable space and all measures are defined on the σ -algebra Σ .

Lemma 1 *Let (Ω, Σ, μ) and (Ω, Σ, ν) be two finite measure spaces. Then the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$ holds in the sense of equivalence classes if and only if μ and ν are absolutely continuous with respect to each other (i.e. $\mu \approx \nu$) and $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$ in the sense of individual functions.*

Proof Suppose that $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$ in the sense of equivalence classes. Let $f \in L^{p),\theta}(\mu)$ be any individual function. Then $f \in L^{p),\theta}(\mu)$ in the sense of equivalence classes. By assumption, $f \in L^{q),\theta}(\nu)$ in the sense of equivalence classes. This implies $f \in L^{q),\theta}(\nu)$ in the sense of individual functions. Then $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$

in the sense of individual functions. To show $\nu \ll \mu$, take any set $E \in \Sigma$ with $\mu(E) = 0$. Then $\chi_E = 0$, $\mu - a.e.$, and it is in the equivalence classes of $0 \in L^p(\mu)$, where χ_E is the characteristic function of E . By the inclusion $L^p(\mu) \subseteq L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ in the sense of equivalence classes, we have $0 \in L^{q,\theta}(\nu)$. Then

$$\sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} [\nu(E)]^{\frac{1}{q-\varepsilon}} = \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \|\chi_E\|_{q-\varepsilon} = \|\chi_E\|_{q,\theta} = 0. \tag{1}$$

Since $L^{q,\theta}(\nu) \subset L^{q-\varepsilon}(\nu)$, there exists a constant $C > 0$ such that

$$\|\chi_E\|_{p-\varepsilon} \leq C \|\chi_E\|_{q,\theta}.$$

Then by (1) we have $\chi_E = 0$, $\nu - a.e.$ Thus, $\nu(E) = 0$ and so $\nu \ll \mu$. Similarly, one can prove that $\mu \ll \nu$. The proof of the other direction is clear. □

Theorem 1 *Let (Ω, Σ, μ) and (Ω, Σ, ν) be two finite measure spaces. Then $L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists a constant $C(p, q) > 0$ such that*

$$\|f\|_{q,\theta,\nu} \leq C(p, q) \|f\|_{p,\theta,\mu} \tag{2}$$

for all $f \in L^{p,\theta}(\mu)$.

Proof Assume that the inequality (2) is satisfied and $\mu \approx \nu$. By the inequality (2) the inclusion $L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ holds in the sense of individual functions. Then by Lemma 1, the inclusion $L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ holds in the sense of equivalence classes.

Conversely, assume that $L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\nu)$ holds in the sense of equivalence classes. The grand Lebesgue space $L^{p,\theta}(\mu)$ is a Banach space with the sum norm

$$\|f\| = \|f\|_{p,\theta,\mu} + \|f\|_{q,\theta,\nu}.$$

Indeed, if we get any Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in the normed space $(L^{p,\theta}(\mu), \|\cdot\|)$, it is also a Cauchy sequence in the spaces $(L^{p,\theta}(\mu), \|\cdot\|_{p,\theta,\mu})$ and $(L^{q,\theta}(\nu), \|\cdot\|_{q,\theta,\nu})$. Then $(f_n)_{n \in \mathbb{N}}$ converges to functions f and g in spaces $L^{p,\theta}(\mu)$ and $L^{q,\theta}(\nu)$, respectively. Thus, one can find a subsequence (f_{n_i}) of (f_n) such that $f_{n_i} \rightarrow f$, $\mu - a.e.$ and $f_{n_i} \rightarrow g$, $\nu - a.e.$ Since ν is absolutely continuous with respect to μ , then $f_{n_i} \rightarrow f$, $\nu - a.e.$ Thus, $f = g$, $\nu - a.e.$ Then (f_n) converges to f in the normed space $(L^{p,\theta}(\mu), \|\cdot\|)$. Then the norms $\|\cdot\|$ and $\|\cdot\|_{p,\theta,\mu}$ are equivalent (see proposition 11, in [18]), and so there exists a constant $C(p, q) > 0$ such that

$$\|f\| \leq C(p, q) \|f\|_{p,\theta,\mu}$$

for all $f \in L^{p,\theta}(\mu)$. This implies

$$\|f\|_{q,\theta,\nu} \leq \|f\| \leq C(p, q) \|f\|_{p,\theta,\mu}$$

for all $f \in L^{p,\theta}(\mu)$. On the other hand, by Lemma 1, μ and ν are absolutely continuous with respect to each other. This completes the proof. □

Theorem 2 Let (Ω, Σ, μ) and (Ω, Σ, ν) be two finite measure spaces. Then the following statements are equivalent.

1. We have $L^{p,\theta}(\mu) \subseteq L^{p,\theta}(\nu)$ for $p > 1$ and for all $\theta \geq 0$.
2. $\mu \approx \nu$ and there exists a constant $C(p, \theta) > 0$ such that

$$\sup_{0 < \varepsilon \leq p-1} (\nu(E))^{\frac{1}{p-\varepsilon}} \leq C(p, \theta) \sup_{0 < \varepsilon \leq p-1} (\mu(E))^{\frac{1}{p-\varepsilon}}$$

for all $E \in \Sigma$.

3. $L^1(\mu) \subseteq L^1(\nu)$.
4. $L^{p,\theta}(\mu) \subseteq L^{p,\theta}(\nu)$ for $p > 1$ and for all $\theta \geq 0$.

Proof (1) \implies (2) : By Theorem 1, $\mu \approx \nu$ and there exists $C(p, \theta) > 0$ such that

$$\|f\|_{p,\theta,\nu} \leq C(p, \theta) \|f\|_{p,\theta,\mu} \tag{3}$$

for all $f \in L^{p,\theta}(\mu)$. If $E \in \Sigma$, then $\chi_E \in L^p(\mu)$. Since $L^p(\mu) \subset L^{p,\theta}(\mu) \subset L^{p,\theta}(\nu)$, then $\chi_E \in L^{p,\theta}(\mu) \subset L^{p,\theta}(\nu)$ and by (3) we have

$$\|\chi_E\|_{p,\theta,\nu} \leq C(p, \theta) \|\chi_E\|_{p,\theta,\mu}. \tag{4}$$

Thus,

$$\sup_{0 < \varepsilon \leq p-1} (\varepsilon^\theta \nu(E))^{\frac{1}{p-\varepsilon}} \leq C(p, \theta) \sup_{0 < \varepsilon \leq p-1} (\varepsilon^\theta \mu(E))^{\frac{1}{p-\varepsilon}}. \tag{5}$$

(2) \implies (3) : Since when $\theta = 0$, the space $L^{p,\theta}(\mu)$ reduces to the Lebesgue space $L^p(\mu)$, by (5),

$$(\nu(E))^{\frac{1}{p}} \leq C(p, 0) (\mu(E))^{\frac{1}{p}} = C(p) (\mu(E))^{\frac{1}{p}}.$$

This implies

$$\nu(E) \leq M \mu(E), \tag{6}$$

where $M = C(p)^p$. Then by Proposition 1 in [13], we have $L^1(\mu) \subseteq L^1(\nu)$.

(3) \implies (4) : By the inclusion $L^1(\mu) \subseteq L^1(\nu)$ there exists $C_1 > 0$ such that

$$\|g\|_{1,\nu} \leq C_1 \|g\|_{1,\mu} \tag{7}$$

for all $g \in L^1(\mu)$. Let $f \in L^{p,\theta}(\mu)$. Then

$$\|f\|_{p,\theta,\mu} = \sup_{0 < \varepsilon \leq p-1} \left(\varepsilon^\theta \int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} < M$$

for some $M > 0$. This implies $|f|^{p-\varepsilon} \in L^1(\mu)$ for all $\varepsilon \in (0, p-1]$. Since $L^1(\mu) \subseteq L^1(\nu)$, then $|f|^{p-\varepsilon} \in L^1(\nu)$. By (7) we have

$$\int_{\Omega} |f|^{p-\varepsilon} d\nu \leq C_1 \int_{\Omega} |f|^{p-\varepsilon} d\mu.$$

Thus, we obtain

$$\left(\int_{\Omega} |f|^{p-\varepsilon} dv \right)^{\frac{1}{p-\varepsilon}} \leq C \left(\int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}},$$

where $C = C_1^{\frac{1}{p-\varepsilon}}$. If we get the supremum in both sides, we have

$$\sup_{0 < \varepsilon \leq p-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{p-\varepsilon} dv \right)^{\frac{1}{p-\varepsilon}} \leq C \sup_{0 < \varepsilon \leq p-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}},$$

for all $\theta \geq 0$. Then

$$\|f\|_{p,\theta,v} \leq C \|f\|_{p,\theta,\mu} < CM < \infty$$

for all $f \in L^{p,\theta}(\mu)$. Finally, we have $L^{p,\theta}(\mu) \subseteq L^{p,\theta}(v)$ for all $\theta \geq 0$.

(4) \implies (1) : This is easy. □

Theorem 3 Let (Ω, Σ, μ) be a finite measure space and let p and q be any two positive real numbers. Then

$$L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\mu) \tag{8}$$

whenever $1 < q < p$, and for all $\theta \geq 0$.

Proof Since for every $0 < \varepsilon \leq q - 1$, we have $q - \varepsilon < p - \varepsilon$, then $L^{p-\varepsilon}(\mu) \subset L^{q-\varepsilon}(\mu)$. Thus, there exists $C > 0$ such that

$$\|f\|_{q-\varepsilon} \leq C \|f\|_{p-\varepsilon}$$

for all $f \in L^{p,\theta}(\mu)$. Let $f \in L^{p,\theta}(\mu)$. We have

$$\begin{aligned} \|f\|_{q,\theta,\mu} &= \sup_{0 < \varepsilon \leq q-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{q-\varepsilon} d\mu \right)^{\frac{1}{q-\varepsilon}} = \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \|f\|_{q-\varepsilon} \\ &\leq C \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \|f\|_{p-\varepsilon} = C \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \varepsilon^{\frac{-\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\ &= C \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\ &\leq C \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\ &\leq C_0 \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = C_0 \|f\|_{p,\theta,\mu}, \end{aligned}$$

where $C_0 = C \sup_{0 < \varepsilon \leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}}$. Since $q < p$, C_0 is finite and thus $f \in L^{q,\theta}(\mu)$. Hence,

$$L^{p,\theta}(\mu) \subseteq L^{q,\theta}(\mu)$$

whenever $p < q$, and for all $\theta \geq 0$. □

3. Approximate identities and consequences

In this section we will assume that Ω is a bounded subset of \mathbb{R}^n and $1 < p, q < \infty, \theta \geq 0$.

We know that $C_c^\infty(\Omega)$ is not dense in $L^{p,\theta}(\mu)$, where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable complex-valued functions with compact support on Ω [9]. Its closure $[L^p]_{p,\theta}$ consists of functions $f \in L^{p,\theta}(\mu)$ such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = 0.$$

It is known that the Lebesgue space $L^p(\mu)$ admits an approximate identity bounded in $L^1(\mu)$ [5, 8]. The following theorem shows that this property is not true for generalized grand Lebesgue space.

Theorem 4 *The generalized grand Lebesgue space $L^{p,\theta}(\mu)$ does not admit an approximate identity, bounded in $L^1(\mu)$.*

Proof Assume that $(e_\alpha)_{\alpha \in I}$ is an approximate identity in $L^{p,\theta}(\mu)$ bounded in $L^1(\mu)$. Then there exists a constant $M > 0$ such that $\|e_\alpha\|_1 < M$ for all $\alpha \in I$. Take any function $f \in L^{p,\theta}(\mu) - [L^p]_{p,\theta}$ (for example the function $f(t) = x^{-\frac{1}{p}}, 1 < p < \infty$). Then $e_\alpha * f \rightarrow f$ in $L^{p,\theta}(\mu)$. Since

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(\varepsilon^\theta \int_{\Omega} |e_\alpha * f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|e_\alpha * f\|_{p-\varepsilon} \\ &\leq \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|e_\alpha\|_1 \|f\|_{p-\varepsilon} \\ &\leq M \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = 0, \end{aligned}$$

then $e_\alpha * f \in [L^p]_{p,\theta}$ for each $\alpha \in I$. This implies $f \in [L^p]_{p,\theta}$. This contradicts the assumption $f \in L^{p,\theta}(\mu) - [L^p]_{p,\theta}$. Then $L^{p,\theta}(\mu)$ does not admit an approximate identity bounded in $L^1(\mu)$. \square

Theorem 5 a. *The generalized grand Lebesgue space $L^{p,\theta}(\mu)$ is a Banach convolution module over $L^1(\mu)$.*

b. The space $[L^p]_{p,\theta}$ is a Banach convolution module over $L^1(\mu)$.

Proof a. We know that $L^{p,\theta}(\mu)$ is a Banach space [9], and $L^p(\mu)$ is a Banach $L^1(\mu)$ -module. Let $f \in L^1(\mu)$ and $g \in L^{p,\theta}(\mu)$. Then

$$\begin{aligned} \|f * g\|_{p,\theta} &= \sup_{0 < \varepsilon \leq p-1} \left(\varepsilon^\theta \int_{\Omega} |f * g|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} \\ &= \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f * g\|_{p-\varepsilon} \leq \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_1 \|g\|_{p-\varepsilon} \\ &= \|f\|_1 \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|g\|_{p-\varepsilon} = \|f\|_1 \|g\|_{p,\theta}. \end{aligned} \tag{9}$$

It is easy to prove the other conditions for $L^{p,\theta}(\mu)$ to be a Banach convolution module over $L^1(\mu)$.

b. It is easy to see that $[L^p]_{p,\theta}$ is a vector space. Since $[L^p]_{p,\theta} \subset L^{p,\theta}(\mu)$ is closed in $L^{p,\theta}(\mu)$, and $L^{p,\theta}(\mu)$ is a Banach space, then $[L^p]_{p,\theta}$ is a Banach space. The inequality (9) is satisfied for all $f \in L^1(\mu)$ and $g \in [L^p]_{p,\theta}$. Then $[L^p]_{p,\theta}$ is a Banach $L^1(\mu)$ – module. \square

Theorem 6 a. *The space $[L^p]_{p,\theta}$ admits an approximate identity bounded in $L^1(\mu)$.*

b. *$[L^p]_{p,\theta}$ admits an approximate identity bounded in $L^1(\mu)$ and with compactly supported Fourier transforms.*

Proof First we shall prove that the closure of $L^p(\mu)$ in $L^{p,\theta}(\mu)$ is $[L^p]_{p,\theta}$. Let $h \in L^p(\mu)$ be given. Since $L^p(\mu) \subset L^{p,\theta}(\mu) \subset L^{p-\varepsilon}(\mu)$, then

$$\lim_{\varepsilon \rightarrow 0} \left(\varepsilon^\theta \int_{\Omega} |h|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|h\|_{p-\varepsilon} = 0.$$

Hence, $h \in [L^p]_{p,\theta}$. This implies

$$L^p(\mu) \subset [L^p]_{p,\theta}.$$

Since

$$C_c^\infty(\mathbb{R}^n) \subset L^p(\mu) \subset [L^p]_{p,\theta}, \tag{10}$$

we have

$$[L^p]_{p,\theta} = \overline{C_c^\infty(\mathbb{R}^n)} \subset \overline{L^p(\mu)} \subset [L^p]_{p,\theta},$$

where the closures are in the norm $\|\cdot\|_{p,\theta,\mu}$. Then

$$\overline{L^p(\mu)} = \overline{C_c^\infty(\mathbb{R}^n)} = [L^p]_{p,\theta}. \tag{11}$$

It is known by Lemma 1.12 in [8] that $L^p(\mu)$ admits an approximate identity $(e)_{\alpha \in I}$, bounded in $L^1(\mu)$. Then there exists a constant $M > 1$, such that $\|e_\alpha\|_1 \leq M$ for all $\alpha \in I$. Also, given any $u \in L^p(\mu)$ and $\delta > 0$, there exists $\alpha_0 \in I$ such that

$$\|e_\alpha * u - u\|_p \leq \frac{\delta}{3} \tag{12}$$

for all $\alpha \geq \alpha_0$. We shall show that $(e)_{\alpha \in I}$ is also an approximate identity in $[L^p]_{p,\theta}$. Let $f \in [L^p]_{p,\theta}$ be given. Since $L^p(\mu)$ is dense in $[L^p]_{p,\theta}$, in the norm $\|\cdot\|_{p,\theta}$, there exists $g \in L^p(\mu)$ such that

$$\|f - g\|_{p,\theta} \leq \frac{\delta}{3M}. \tag{13}$$

Then

$$\begin{aligned} \|e_\alpha * f - f\|_{p,\theta} &= \|e_\alpha * f - f - e_\alpha * g + e_\alpha * g + g - g\|_{p,\theta} \\ &\leq \|e_\alpha * f - e_\alpha * g\|_{p,\theta} + \|e_\alpha * g - g\|_{p,\theta} + \|g - f\|_{p,\theta}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \|e_\alpha * f - e_\alpha * g\|_{p,\theta} &= \|e_\alpha * (f - g)\|_{p,\theta} \\ &\leq \|e_\alpha\|_1 \|f - g\|_{p,\theta} \leq M \|f - g\|_{p,\theta} \leq M \frac{\delta}{3M} = \frac{\delta}{3}. \end{aligned} \tag{15}$$

Since $M > 1$, combining (12) (13), (14), and (15), we obtain

$$\|e_\alpha * f - f\|_{p,\theta} \leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3M} < \delta.$$

This completes the proof of part (a). The proof of part (b) is obvious. □

As an application of the approximate identities we will give the following theorem.

Theorem 7 a) *The space of multipliers $M(L^1(\mu), ([L^p]_{p,\theta})^*)$ is isometrically isomorphic to dual space $([L^p]_{p,\theta})^*$ (dual of $[L^p]_{p,\theta}$).*

b) *The space of multipliers $M(L^1(\mu), (L^{p,\theta}(\mu))^*)$ is isometrically isomorphic to the dual space $(L^1(\mu) * L^{p,\theta}(\mu))^*$. If f is an element in the space of multipliers $M(L^1(\mu), (L^{p,\theta}(\mu))^*)$, then there is an extension F of f to a continuous linear form on $L^{p,\theta}(\mu)$ so that*

$$\|F | (L^{p,\theta}(\mu))^*\| = \|f | (L^1(\mu) * L^{p,\theta}(\mu))^*\|,$$

where $\|F | (L^{p,\theta}(\mu))^*\|$ and $\|f | (L^1(\mu) * L^{p,\theta}(\mu))^*\|$ denote the norms on the spaces $(L^{p,\theta}(\mu))^*$ and $(L^1(\mu) * L^{p,\theta}(\mu))^*$, respectively.

Proof a) We know by Theorem 5 that $[L^p]_{p,\theta}$ is a Banach $L^1(\mu)$ -module. Also, by Theorem 6, $L^1(\mu) * [L^p]_{p,\theta}$ is dense in $[L^p]_{p,\theta}$ in the $\|\cdot\|_{p,\theta,\mu}$ norm. Then by the module factorization theorem [20], we have

$$L^1(\mu) * [L^p]_{p,\theta} = [L^p]_{p,\theta}. \tag{16}$$

Thus, $[L^p]_{p,\theta}$ is an essential Banach module over $L^1(\mu)$. Then by Corollary 2.13 in [15], and by (16) we obtain

$$M(L^1(\mu), ([L^p]_{p,\theta})^*) = (L^1(\mu) * [L^p]_{p,\theta})^* = ([L^p]_{p,\theta})^*.$$

b) Again by Corollary 2.13 in [15],

$$M(L^1(\mu), (L^{p,\theta}(\mu))^*) = (L^1(\mu) * L^{p,\theta}(\mu))^*.$$

On the other hand, by Theorem 5, $L^{p,\theta}(\mu)$ is a Banach $L^1(\mu)$ -convolution module. Thus, $L^1(\mu) * L^{p,\theta}(\mu) \subset L^{p,\theta}(\mu)$. Then if $f \in M(L^1(\mu), (L^{p,\theta}(\mu))^*)$, by the Hahn-Banach extension theorem, there is an extension F of f to a continuous linear form on $L^{p,\theta}(\mu)$ so that $\|F | (L^{p,\theta}(\mu))^*\| = \|f | (L^1(\mu) * L^{p,\theta}(\mu))^*\|$. This completes the proof. □

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