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Inequalities for submanifolds of Sasaki-like statistical manifolds

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Abstract: We consider statistical submanifolds in Sasaki-like statistical manifolds. We give some examples of invariant and antiinvariant submanifolds of Sasaki-like statistical manifolds. We prove Chen-like inequality involving scalar curvature and Chen–Ricci inequality for these kinds of submanifolds.

Key words: Sasaki-like statistical manifold, Chen–Ricci inequality, Ricci curvature, scalar curvature

1. Introduction

Statistical manifolds have arisen from the study of a statistical distribution. In 1985 Amari [2] introduced a differential geometric approach for a statistical model of discrete probability distribution. Statistical manifolds have many applications in information geometry, which is a branch of mathematics that applies the techniques of differential geometry to the field of probability theory. Some of these applications are statistical inference, linear systems, time series, neural networks, nonlinear systems, linear programming, convex analysis and completely integrable dynamical systems, quantum information geometry, and geometric modeling (for more details see [1]).

Let (M, g) be a Riemannian manifold given by a pair of torsion-free affine connections ∇ and ∇^*. A pair of (∇, g) is called a *statistical structure* on M if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0 \tag{1.1}$$

holds for $X, Y, Z \in TM$ [2]. If a Riemannian manifold (M, g) with its statistical structure satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

then it is called a *statistical manifold* and denoted by (M, g, ∇, ∇^*) (see [2] and [22]).

Any torsion-free affine connection ∇ always has a dual connection ∇^* given by

$$\nabla + \nabla^* = 2\nabla^0, \tag{1.2}$$

where ∇^0 is the Levi-Civita connection of M [2].

The study to find some inequalities between the extrinsic and intrinsic invariants of a submanifold was started by Chen in 1993 [8]. He established some inequalities in a real space form and now they are well

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known as Chen inequalities. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants are the scalar curvature and the Ricci curvature. A relation between the Ricci curvature and the main extrinsic invariant squared mean curvature for a submanifold in a real space form was given in [10] by Chen and is now known as the Chen–Ricci inequality. In [14] Mihai and in [19] Matsumoto and Mihai found relations between Ricci curvature and the squared mean curvature for submanifolds in Sasakian space forms. Since then, many geometers have studied similar problems for different submanifolds in various ambient spaces; for example, see [3, 9, 10, 15, 17, 18]. For the collections of the results related to Chen inequalities see also [11] and the references therein.

Furthermore, in [4], Aydın et al. found relations between the extrinsic and intrinsic invariants for submanifolds in statistical manifolds of constant curvature. In [16], Mihai and Mihai studied statistical submanifolds of Hessian manifolds of constant Hessian curvature. As generalizations of the results given in [4], the present authors studied the same problems for submanifolds in statistical manifolds of quasiconstant curvature [5].

Motivated by the studies of the above authors, in the present paper, we define invariant and antiinvariant submanifolds of Sasaki-like statistical manifolds and give some examples of these submanifolds. Furthermore, we obtain Chen-like inequality involving scalar curvature and Chen–Ricci inequality for these types of submanifolds.

2. Preliminaries

Let M be an odd-dimensional manifold and ϕ, ξ, η a tensor field of type $(1,1)$, a vector field, and a 1-form on M, respectively. If ϕ, ξ, and η satisfy the following conditions,

$$\eta(\xi) = 1, \qquad \phi^2 X = -X + \eta(X)\xi \tag{2.1}$$

for $X \in TM$, then M is said to have an *almost contact structure* (ϕ, ξ, η) and is called an *almost contact manifold*.

In [21], Takano considered a semi-Riemannian manifold (M, g) with the almost contact structure (ϕ, ξ, η), which has another tensor field ϕ^* of type $(1,1)$ satisfying

$$g(\phi X, Y) + g(X, \phi^* Y) = 0 \tag{2.2}$$

for vector fields X and Y on (M, g). Then (M, g, ϕ, ξ, η) is called an *almost contact metric manifold of certain kind* [20]. Obviously, we find $(\phi^*)^2 X = -X + \eta(X)\xi$ and

$$g(\phi X, \phi^* Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

Because of (2.2), the tensor field ϕ is not symmetric with respect to g. This means that $\phi + \phi^*$ does not vanish everywhere. Equations $\phi\xi = 0$ and $\eta(\phi X) = 0$ hold on the almost contact manifold. We obtain $\phi^*\xi = 0$ and $\eta(\phi^* X) = 0$ on the almost contact metric manifold of certain kind. In [21], Takano defined a statistical manifold on the almost contact metric manifold of certain kind. If

$$\nabla_X \xi = -\phi X, \qquad (\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X, \tag{2.4}$$

then $(M, \nabla, g, \phi, \xi, \eta)$ is called a *Sasaki-like statistical manifold* and considered the curvature tensor R with respect to ∇ such that

$$\begin{aligned}R(X,Y)Z = &\frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + \{g(X,\phi Y) - g(\phi X,Y)\}\,\phi Z], \end{aligned} \tag{2.5}$$

where c is a constant. Changing ϕ for ϕ^* in (2.5), we get the curvature tensor R^* [21].

Denote by R and R^* the curvature tensor fields of ∇ and ∇^*, respectively. Then R and R^* satisfy

$$g\left(R^*\left(X,Y\right)Z,W\right) = -g\left(Z,R\left(X,Y\right)W\right) \tag{2.6}$$

(see [12]).

Let (M,g,∇,∇^*) and $\left(\widetilde{M},\widetilde{g},\widetilde{\nabla},\widetilde{\nabla}^*\right)$ be two statistical manifolds. An immersion $f: M \longrightarrow \widetilde{M}$ is called a *statistical immersion* if (∇,g) coincides with the induced statistical structure, i.e. if (1.1) holds [12]. If there is a statistical immersion between two statistical manifolds (M,g,∇,∇^*) and $\left(\widetilde{M},\widetilde{g},\widetilde{\nabla},\widetilde{\nabla}^*\right)$, then M is called a *statistical submanifold* of $\widetilde{M}$. (For the definition of affine immersions of statistical manifolds into $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$ see also [13].)

Denote the normal bundle on M by $T^{\perp}M$. In the present study, we use the ambient space $\widetilde{M}$ as a statistical manifold $\left(\widetilde{M},\widetilde{g},\widetilde{\nabla},\widetilde{\nabla}^*\right)$.

Let M be a statistical submanifold of a statistical manifold $\widetilde{M}$. Then the Gauss formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\widetilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y),$$

where the normal valued tensor fields h and h^* are symmetric and bilinear, called *the embedding curvature tensors* of M in $\widetilde{M}$ for $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. ∇ and ∇^* are called the *induced connections* of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. Since h and h^* are symmetric and bilinear, we have the linear transformations A_ξ and A^*_ξ defined by

$$g\left(A_\xi X,Y\right) = \widetilde{g}\left(h\left(X,Y\right),\xi\right) \tag{2.7}$$

and

$$g\left(A^*_\xi X,Y\right) = \widetilde{g}\left(h^*\left(X,Y\right),\xi\right) \tag{2.8}$$

for any unit $\xi \in T^{\perp}M$ and $X,Y \in TM$ [22]. The corresponding Weingarten formulas are as follows:

$$\widetilde{\nabla}_X \xi = -A^*_\xi X + \nabla^{\perp}_X \xi$$

and

$$\widetilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi.$$

If we use the Levi-Civita connection, it is known that h and A_ξ are called the *second fundamental form* and the *shape operator* with respect to the unit $\xi \in T^{\perp}M$, respectively [7]. Let $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ be affine and dual connections on $\widetilde{M}$. We denote the induced connections ∇ and ∇^* of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively, on M. Let $\widetilde{R}, \widetilde{R}^*, R$, and R^* be the Riemannian curvature tensors of $\widetilde{\nabla}, \widetilde{\nabla}^*, \nabla$, and ∇^*, respectively. Then the Gauss equations are given by

$$\widetilde{g}\left(\widetilde{R}(X,Y)Z,W\right) = g(R(X,Y)Z,W) + \widetilde{g}(h(X,Z),h^*(Y,W)) - \widetilde{g}(h^*(X,W),h(Y,Z)) \tag{2.9}$$

and

$$\widetilde{g}\left(\widetilde{R}^*(X,Y)Z,W\right) = g(R^*(X,Y)Z,W) + \widetilde{g}(h^*(X,Z),h(Y,W)) - \widetilde{g}(h(X,W),h^*(Y,Z)),$$

where $X,Y,Z,W \in TM$ [22].

3. Statistical submanifolds in Sasaki-like statistical manifolds

In this section, we give some examples of invariant and antiinvariant submanifolds of Sasaki-like statistical manifolds. We find some properties for these kinds of submanifolds.

Similar to the classical definition of the invariant or antiinvariant submanifold of a Sasakian manifold (see [23]), we give the following definition:

Definition 3.1 *Let $\widetilde{M}$ be a Sasaki-like statistical manifold and M a submanifold of $\widetilde{M}$. For $X \in TM$, if $\phi X \in T^{\perp}M$, then M is called an antiinvariant submanifold of $\widetilde{M}$. On the other hand, for a submanifold M, if $\phi X \in TM$, then M is called an invariant submanifold of $\widetilde{M}$.*

Example 3.2 *[21] Let $\mathbb{R}_n^{2n+1}$ be a $(2n+1)$-dimensional affine space with the standard coordinates $\{x_1,...,x_n,y_1,...,y_n,z\}$. We define a semi-Riemannian metric g, the affine connection ∇, ϕ, ξ, and η on $\mathbb{R}_n^{2n+1}$ respectively by*

$$g = \begin{pmatrix} 2\delta_{ij} + y_i y_j & 0 & -y_i \\ 0 & -\delta_{ij} & 0 \\ -y_j & 0 & 1 \end{pmatrix},$$

$$\nabla_{\partial x_i}\partial x_j = -y_j\partial y_i - y_i\partial y_j,$$

$$\nabla_{\partial x_i}\partial y_j = \nabla_{\partial y_j}\partial x_i = y_i\partial x_j + (y_i y_j - 2\delta_{ij})\partial z,$$

$$\nabla_{\partial x_i}\partial z = \nabla_{\partial z}\partial x_i = \partial y_i,$$

$$\nabla_{\partial y_i}\partial z = \nabla_{\partial z}\partial y_i = -\partial x_i - y_i\partial z,$$

$$\nabla_{\partial y_i}\partial y_i = \nabla_{\partial z}\partial z = 0, \text{ where } \partial x_i = \frac{\partial}{\partial x_i}, \partial y_i = \frac{\partial}{\partial y_i} \text{ and } \partial z = \frac{\partial}{\partial z}.$$

$$\phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_i & 0 \end{pmatrix}, \qquad \xi = \partial_z = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad \eta = (-y_1, 0, -y_2, ..., -y_n, 0, 1).$$

Then $\left(\mathbb{R}_n^{2n+1}, \nabla, g, \phi, \xi, \eta\right)$ *is a Sasaki-like statistical manifold such that the curvature tensor of* $\mathbb{R}_n^{2n+1}$ *satisfies equation* (2.5) *with* $c = -3$. *From here, it can be easily found that*

$$\phi^* = \frac{1}{2}\left(\begin{array}{ccc} 0 & -\delta_{ij} & 0 \\ 4\delta_{ij} & 0 & 0 \\ 0 & -y_i & 0 \end{array}\right).$$

Similar to the examples given in [6], now we present the following examples in $\mathbb{R}^5$ and $\mathbb{R}^9$ with the Sasaki-like structure given in Example 3.2:

Example 3.3 *Let* M *be a submanifold of dimension* 3 *such that*

$$x(u, v, t) = (u, 0, v, 0, t).$$

For any $U \in TM$, *it is easy to see that* $\phi U \in TM$ *and* $\phi^* U \in TM$, *so* M *is an invariant submanifold of Sasaki-like manifold* $\mathbb{R}^5$ *with the structure* $(\nabla, g, \phi, \xi, \eta)$.

Example 3.4 *Let* M *be a submanifold of dimension* 3 *such that*

$$x(u, v, t) = (0, v, u, 0, t).$$

For any $U \in TM$, *it is easy to see that* $\phi U \in T^\perp M$ *and* $\phi^* U \in T^\perp M$, *so* M *is an antiinvariant submanifold of Sasaki-like manifold* $\mathbb{R}^5$ *with the structure* $(\nabla, g, \phi, \xi, \eta)$ *and* ξ *is tangent to* M.

Example 3.5 *Let* M *be a submanifold of dimension* 4 *such that*

$$x(u, v, w, s) = (0, 0, 0, 0, u, v, w, s, 0).$$

For any $U \in TM$, *it is easy to see that* $\phi U \in T^\perp M$ *and* $\phi^* U \in T^\perp M$, *so* M *is an antiinvariant submanifold of Sasaki-like manifold* $\mathbb{R}^9$ *with the structure* $(\nabla, g, \phi, \xi, \eta)$ *and* ξ *is normal to* M.

For $X \in TM$, we put

$$\phi X = PX + FX,$$

where PX and FX are the tangential and normal components of ϕX, respectively. Similarly, we can write

$$\phi^* X = P^* X + F^* X,$$

where $P^* X$ and $F^* X$ are the tangential and normal components of $\phi^* X$, respectively. We define

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\phi e_i, e_j),$$

and

$$\lambda = trP.$$

From the Gauss equation and (2.5), for the curvature tensor R with respect to induced connection ∇, we obtain

$$
\begin{aligned}
g\left(R\left(X,Y\right)Z,W\right) = & \frac{c+3}{4}\left[g\left(Y,Z\right)g\left(X,W\right)-g\left(X,Z\right)g\left(Y,W\right)\right] \\
& +\frac{c-1}{4}\left[\eta(X)\eta(Z)g\left(Y,W\right)-\eta(Y)\eta(Z)g\left(X,W\right)\right. \\
& +g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)+g(X,\phi Z)g\left(\phi Y,W\right) \\
& \left.-g(Y,\phi Z)g\left(\phi X,W\right)+\left\{g(X,\phi Y)-g(\phi X,Y)\right\}g\left(\phi Z,W\right)\right] \\
& +\widetilde{g}\left(h^{*}\left(X,W\right),h\left(Y,Z\right)\right)-\widetilde{g}\left(h\left(X,Z\right),h^{*}\left(Y,W\right)\right),
\end{aligned} \tag{3.1}
$$

where $X,Y,Z,W\in TM$.

Let M be an n-dimensional statistical submanifold of a $(2m+1)$-dimensional Sasaki-like statistical manifold $\widetilde{M}$ and $\{e_1,...,e_n\}$, $\{e_{n+1},...,e_{2m+1}\}$ orthonormal tangent and normal frames on M, respectively. The mean curvature vector fields are given by

$$
H=\frac{1}{n}\sum_{i=1}^{n}h\left(e_i,e_i\right)=\frac{1}{n}\sum_{\alpha=1}^{2m-n+1}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)e_{n+\alpha}\quad,\quad h_{ij}^{\alpha}=\widetilde{g}\left(h\left(e_i,e_j\right),e_{n+\alpha}\right),
$$

and

$$
H^{*}=\frac{1}{n}\sum_{i=1}^{n}h^{*}\left(e_i,e_i\right)=\frac{1}{n}\sum_{\alpha=1}^{2m-n+1}\left(\sum_{i=1}^{n}h_{ii}^{*\alpha}\right)e_{n+\alpha}\quad,\quad h_{ij}^{*\alpha}=\widetilde{g}\left(h^{*}\left(e_i,e_j\right),e_{n+\alpha}\right).
$$

Now, we compute Ricci tensor S and dual Ricci tensor S^{*} with respect to induced connections ∇ and ∇^{*}. Denote by R the Riemannian curvature tensor of M with respect to ∇. Then we write

$$
S(X,Y)=\sum_{j=1}^{n}g(R(e_j,X)Y,e_j),
$$

and by using equation (3.1), we have

$$
\begin{aligned}
S(X,Y) = & \sum_{j=1}^{n}\Big(\frac{c+3}{4}\left\{g(X,Y)g(e_j,e_j)-g(e_j,Y)g(X,e_j)\right\} \\
& +\frac{c-1}{4}\left\{g(X,e_j)\eta(Y)\eta(e_j)-g(e_j,e_j)\eta(X)\eta(Y)\right. \\
& +g(e_j,Y)\eta(X)\eta(e_j)-g(X,Y)\eta(e_j)\eta(e_j)-g\left(X,\phi Y\right)g\left(\phi e_j,e_j\right) \\
& \left.+g\left(e_j,\phi Y\right)g\left(e_j,\phi X\right)+\left[g\left(e_j,\phi X\right)-g\left(\phi e_j,X\right)\right]g\left(e_j,\phi Y\right)\right\} \\
& +\widetilde{g}(h^{*}(e_j,e_j),h(X,Y))-\widetilde{g}(h^{*}(X,e_j),h(e_j,Y))\Big),
\end{aligned} \tag{3.2}
$$

which gives us

$$
\begin{aligned}
S\left(X,Y\right) = & \frac{c+3}{4}\left(n-1\right)g\left(X,Y\right)+\frac{c-1}{4}\left\{(2-n)\eta(X)\eta(Y)\right. \\
& \left.-g\left(X,Y\right)\left\|\xi^{T}\right\|^{2}-\lambda g\left(X,PY\right)+2g\left(PX,PY\right)+g\left(P^{*}X,PY\right)\right\} \\
& +\sum_{i=n+1}^{2m-n+1}\left\{g\left(A_{e_{n+i}}X,Y\right)trA_{e_{n+i}}^{*}-g\left(A_{e_{n+i}}^{*}X,A_{e_{n+i}}Y\right)\right\}.
\end{aligned} \tag{3.3}
$$

In a similar way, for dual Ricci tensor S^*, we obtain

$$\begin{aligned} S^*(X,Y) =& \frac{c+3}{4}(n-1)\,g(X,Y) + \frac{c-1}{4}\{(2-n)\eta(X)\eta(Y) \\ & - g(X,Y)\left\|\xi^T\right\|^2 - \lambda g(PX,Y) + 2g(P^*X,P^*Y) + g(PX,P^*Y)\} \\ & + \sum_{i=n+1}^{2m-n+1}\left\{g\left(A^*_{e_{n+i}}X,Y\right) trA_{e_{n+i}} - g\left(A^*_{e_{n+i}}Y, A_{e_{n+i}}X\right)\right\}. \end{aligned} \tag{3.4}$$

We have the following propositions:

Proposition 3.6 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(i) *Assume that ξ is tangent to M.*

(a) *If M is invariant, then*

$$\begin{aligned} S(X,Y) =& \frac{c+3}{4}(n-1)\,g(X,Y) \\ & + \frac{c-1}{4}\{2g(PX,PY) - (n-1)\eta(X)\eta(Y) - \lambda g(X,PY)\} \\ & + \sum_{i=n+1}^{2m-n+1}\left\{g\left(A_{e_{n+i}}X,Y\right) trA^*_{e_{n+i}} - g\left(A^*_{e_{n+i}}X, A_{e_{n+i}}Y\right)\right\}. \end{aligned} \tag{3.5}$$

(b) *If M is antiinvariant, then*

$$\begin{aligned} S(X,Y) =& \frac{c+3}{4}(n-1)\,g(X,Y) \\ & - \frac{c-1}{4}\{(n-2)\eta(X)\eta(Y) + g(X,Y)\} \\ & + \sum_{i=n+1}^{2m-n+1}\left\{g\left(A_{e_{n+i}}X,Y\right) trA^*_{e_{n+i}} - g\left(A^*_{e_{n+i}}X, A_{e_{n+i}}Y\right)\right\}. \end{aligned} \tag{3.6}$$

(ii) *If ξ is normal to M (which means that M is antiinvariant), then*

$$\begin{aligned} S(X,Y) =& \frac{c+3}{4}(n-1)\,g(X,Y) \\ & + \sum_{i=n+1}^{2m-n+1}\left\{g\left(A_{e_{n+i}}X,Y\right) trA^*_{e_{n+i}} - g\left(A^*_{e_{n+i}}X, A_{e_{n+i}}Y\right)\right\}. \end{aligned}$$

Proposition 3.7 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(i) *Assume that ξ is tangent to M.*

(*a*) *If* M *is invariant, then*

$$\begin{aligned}S^{*}(X,Y)=&\frac{c+3}{4}(n-1)g(X,Y)\\&+\frac{c-1}{4}\{2g(P^{*}X,P^{*}Y)-(n-1)\eta(X)\eta(Y)-\lambda g(PX,Y)\}\\&+\sum_{i=n+1}^{2m-n+1}\left\{g\left(A^{*}_{e_{n+i}}X,Y\right)trA_{e_{n+i}}-g\left(A^{*}_{e_{n+i}}Y,A_{e_{n+i}}X\right)\right\}.\end{aligned}$$

(*b*) *If* M *is antiinvariant, then*

$$\begin{aligned}S^{*}(X,Y)=&\frac{c+3}{4}(n-1)g(X,Y)\\&-\frac{c-1}{4}\{(n-2)\eta(X)\eta(Y)+g(X,Y)\}\\&+\sum_{i=n+1}^{2m-n+1}\left\{g\left(A^{*}_{e_{n+i}}X,Y\right)trA_{e_{n+i}}-g\left(A^{*}_{e_{n+i}}Y,A_{e_{n+i}}X\right)\right\}.\end{aligned}$$

(*ii*) *If* ξ *is normal to* M *(which means that* M *is antiinvariant), then*

$$\begin{aligned}S^{*}(X,Y)=&\frac{c+3}{4}(n-1)g(X,Y)\\&+\sum_{i=n+1}^{2m-n+1}\left\{g\left(A^{*}_{e_{n+i}}X,Y\right)trA_{e_{n+i}}-g\left(A^{*}_{e_{n+i}}Y,A_{e_{n+i}}X\right)\right\}.\end{aligned}$$

Theorem 3.8 *Let* $\widetilde{M}$ *be a* $(2m+1)$*-dimensional Sasaki-like statistical manifold and* M *an* n*-dimensional statistical submanifold of* $\widetilde{M}$*. Then*

$$\begin{aligned}2\tau\geq&\frac{c+3}{4}\left(n^{2}-n\right)+\frac{c-1}{4}\left\{2\|P\|^{2}-(n-2)\left\|\xi^{T}\right\|^{2}-\lambda^{2}+\sum_{i=1}^{n}g(Pe_i,P^{*}e_i)\right\}\\&+n^{2}\widetilde{g}(H,H^{*})-\|h\|\,\|h^{*}\|,\end{aligned}\tag{3.7}$$

where $\tau=\sum_{1\leq i<j\leq n}g(R(e_i,e_j)e_j,e_i)$ *is the scalar curvature of* (M,g,∇,∇^{*}) *and* $\lambda=trP$*. Moreover, the equality holds if and only if* $h\parallel h^{*}$*.*

Proof We denote by $\|h\|^{2}=\sum_{\alpha=n+1}^{2m-n+1}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^{2}$ and similarly $\|h^{*}\|^{2}$.

From (3.1), taking $X = W = e_i$ and $Y = Z = e_j$, we can write

$$\begin{aligned}\sum_{i,j=1}^{n} g\left(R\left(e_i, e_j\right) e_j, e_i\right) = \sum_{i,j=1}^{n} [\frac{c+3}{4} & \left\{g\left(e_j, e_j\right) g\left(e_i, e_i\right) - g\left(e_i, e_j\right) g\left(e_i, e_j\right)\right\} \\ & + \frac{c-1}{4}\left\{g\left(e_i, e_j\right) \eta(e_j)\eta(e_i) - \eta\left(e_j\right) \eta\left(e_j\right) g\left(e_i, e_i\right)\right. \\ & + g\left(e_i, e_j\right) \eta(e_j)\eta(e_i) - g\left(e_j, e_j\right) \eta(e_i)\eta(e_i) \\ & + g\left(e_i, \phi e_j\right) g\left(e_i, \phi e_j\right) - g\left(e_j, \phi e_j\right) g\left(\phi e_i, e_i\right) \\ & \left.[g\left(e_i, \phi e_j\right) - g\left(\phi e_i, e_j\right)] g\left(e_i, \phi e_j\right)\right\} \\ & + \widetilde{g}\left(h^*\left(e_i, e_i\right), h\left(e_j, e_j\right)\right) - \widetilde{g}\left(h\left(e_i, e_j\right), h^*\left(e_i, e_j\right)\right)] . \end{aligned} \tag{3.8}$$

We obtain

$$\begin{aligned} 2\tau = & \frac{c+3}{4}\left(n^2 - n\right) + \frac{c-1}{4}\left\{2\left\|P\right\|^2 - (n-2)\left\|\xi^T\right\|^2 - \lambda^2 + \sum_{i=1}^{n} g\left(Pe_i, P^* e_i\right)\right\} \\ & + n^2 \widetilde{g}\left(H, H^*\right) - \sum_{\alpha=n+1}^{2m-n+1} \sum_{1 \le i,j \le n} h_{ij}^{\alpha} h_{ij}^{*\alpha} \\ & \ge \frac{c+3}{4}\left(n^2 - n\right) + \frac{c-1}{4}\left\{2\left\|P\right\|^2 - (n-2)\left\|\xi^T\right\|^2 - \lambda^2 + \sum_{i=1}^{n} g\left(Pe_i, P^* e_i\right)\right\} \\ & + n^2 \widetilde{g}\left(H, H^*\right) - \left\|h\right\| \left\|h^*\right\| . \end{aligned} \tag{3.9}$$

From (3.9), it is easy to see that the equality holds if and only if $h \parallel h^*$. Hence, we finish the proof. □

4. Chen–Ricci inequality

In the present section, we prove the Chen–Ricci inequality for statistical submanifolds in Sasaki-like statistical manifolds.

Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold and M an n-dimensional statistical submanifold of $\widetilde{M}$. Then from (3.1), we obtain

$$\begin{aligned} 2\tau = & \frac{c+3}{4}\left(n^2 - n\right) + \frac{c-1}{4}\left\{2\left\|P\right\|^2 - (n-2)\left\|\xi^T\right\|^2 - \lambda^2 + \sum_{i=1}^{n} g\left(Pe_i, P^* e_i\right)\right\} \\ & + n^2 \widetilde{g}(H, H^*) - \sum_{i,j=1}^{n} \widetilde{g}\left(h\left(e_i, e_j\right), h^*\left(e_i, e_j\right)\right), \end{aligned}$$

where H and H^* are the mean curvature vector fields. Then it follows that

$$
\begin{aligned}
2\tau =& \frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\|P\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n} g\left(Pe_i, P^*e_i\right)\right\} \\
&+\frac{n^2}{2}\{2\widetilde{g}(H,H^*)+\widetilde{g}(H,H)+\widetilde{g}(H^*,H^*)-\widetilde{g}(H,H)-\widetilde{g}(H^*,H^*)\} \\
&-\frac{1}{2}\{\sum_{i,j=1}^{n} 2\widetilde{g}\left(h\left(e_i,e_j\right), h^*\left(e_i,e_j\right)\right)+\widetilde{g}\left(h\left(e_i,e_j\right), h\left(e_i,e_j\right)\right) \\
&+\widetilde{g}\left(h^*\left(e_i,e_j\right), h^*\left(e_i,e_j\right)\right)-\widetilde{g}\left(h\left(e_i,e_j\right), h\left(e_i,e_j\right)\right)-\widetilde{g}\left(h^*\left(e_i,e_j\right), h^*\left(e_i,e_j\right)\right) \\
=& \frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\|P\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n} g\left(Pe_i, P^*e_i\right)\right\} \\
&+\frac{n^2}{2}\{\widetilde{g}(H+H^*,H^*+H)-\widetilde{g}(H,H)-\widetilde{g}(H^*,H^*)\} \\
&-\frac{1}{2}\{\sum_{i,j=1}^{n} \widetilde{g}\left(h\left(e_i,e_j\right)+h^*\left(e_i,e_j\right), h^*\left(e_i,e_j\right)+h\left(e_i,e_j\right)\right) \\
&-\widetilde{g}\left(h\left(e_i,e_j\right), h\left(e_i,e_j\right)\right)-\widetilde{g}\left(h^*\left(e_i,e_j\right), h^*\left(e_i,e_j\right)\right)\}.
\end{aligned}
$$

From (1.2), since $2H^0 = H + H^*$, we have

$$
\begin{aligned}
2\tau =& \frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\|P\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n} g\left(Pe_i, P^*e_i\right)\right\} \\
&+2n^2\widetilde{g}(H^0,H^0)-\frac{n^2}{2}\widetilde{g}(H,H)-\frac{n^2}{2}\widetilde{g}(H^*,H^*)-2\sum_{i,j=1}^{n}\widetilde{g}\left(h^0\left(e_i,e_j\right), h^0\left(e_i,e_j\right)\right) \\
&+\frac{1}{2}\sum_{i,j=1}^{n}\widetilde{g}\left(h\left(e_i,e_j\right), h\left(e_i,e_j\right)\right)+\widetilde{g}\left(h^*\left(e_i,e_j\right), h^*\left(e_i,e_j\right)\right)
\end{aligned}
$$

and then

$$
\begin{aligned}
2\tau =& \frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\|P\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n} g\left(Pe_i, P^*e_i\right)\right\} \\
&+2n^2\left\|H^0\right\|^2-\frac{n^2}{2}\|H\|^2-\frac{n^2}{2}\|H^*\|^2-2\left\|h^0\right\|^2+\frac{1}{2}(\|h\|^2+\|h^*\|^2).
\end{aligned} \tag{4.1}
$$

On the other hand, we can write

$$
\begin{aligned}
\|h\|^2 &= \sum_{\alpha=n+1}^{2m-n+1}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 \\
&= \sum_{\alpha=n+1}^{2m-n+1}\left\{(h_{11}^{\alpha})^2+(h_{12}^{\alpha})^2+...+(h_{1n}^{\alpha})^2+(h_{21}^{\alpha})^2+(h_{22}^{\alpha})^2\right. \\
&\quad \left.+...+(h_{11}^{\alpha})^2+(h_{n1}^{\alpha})^2+...+(h_{nn}^{\alpha})^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\alpha=n+1}^{2m-n+1}\left[(h_{11}^{\alpha})^2+(h_{22}^{\alpha}+...+h_{nn}^{\alpha})^2\right] \\
&\quad -\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n} h_{ii}^{\alpha}h_{jj}^{\alpha}+2\sum_{\alpha=n+1}^{2m-n+1}\sum_{1\leq i<j\leq n}(h_{ij}^{\alpha})^2 \\
&= \frac{1}{2}\sum_{\alpha=n+1}^{2m-n+1}\{(h_{11}^{\alpha}+h_{22}^{\alpha}+...+h_{nn}^{\alpha})^2+(h_{11}^{\alpha}-h_{22}^{\alpha}-...-h_{nn}^{\alpha})^2\} \\
&\quad +2\sum_{\alpha=n+1}^{2m-n+1}\sum_{1\leq i<j\leq n}(h_{ij}^{\alpha})^2-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha} \\
&\geq \frac{1}{2}n^2\left\|H\right\|^2-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2].
\end{aligned}
$$

Similarly, we have

$$
\left\|h^{*}\right\|^2\geq\frac{1}{2}n^2\left\|H^{*}\right\|^2-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-(h_{ij}^{*\alpha})^2].
$$

The summation of the last two inequalities gives us

$$
\left\|h\right\|^2+\left\|h^{*}\right\|^2\geq\frac{1}{2}n^2\left\|H\right\|^2+\frac{1}{2}n^2\left\|H^{*}\right\|^2
$$

$$
-\sum_{\alpha=1}^{n+1}\sum_{2\leq i\neq j\leq n}[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2]-\sum_{\alpha=1}^{n+1}\sum_{2\leq i\neq j\leq n}[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-(h_{ij}^{*\alpha})^2].
$$

Hence, we have

$$
\left\|h\right\|^2+\left\|h^{*}\right\|^2\geq\frac{1}{2}n^2\left\|H\right\|^2+\frac{1}{2}n^2\left\|H^{*}\right\|^2
$$

$$
-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}(h_{ii}^{\alpha}+h_{ii}^{*\alpha})(h_{jj}^{\alpha}+h_{jj}^{*\alpha})
$$

$$
+2\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{*\alpha}+\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left((h_{ij}^{\alpha})^2+(h_{ij}^{*\alpha})^2\right). \tag{4.2}
$$

Using (4.2) and (4.1), we find

$$\begin{aligned}
&\frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\left\|P\right\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n}g\left(Pe_i,P^*e_i\right)\right\}\\
&\leq 2\tau-2n^2\left\|H^0\right\|^2+\frac{n^2}{2}\left\|H\right\|^2+\frac{n^2}{2}\left\|H^*\right\|^2+2\left\|h^0\right\|^2-\frac{1}{4}n^2\left\|H\right\|^2\\
&-\frac{1}{4}n^2\left\|H^*\right\|^2+\frac{1}{2}\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}(h_{ii}^{\alpha}+h_{ii}^{*\alpha})(h_{jj}^{\alpha}+h_{jj}^{*\alpha})-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{*\alpha}\\
&-\frac{1}{2}\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left((h_{ij}^{\alpha})^2+(h_{ij}^{*\alpha})^2\right)\\
&=2\tau-2n^2\left\|H^0\right\|^2+\frac{n^2}{4}\left\|H\right\|^2+\frac{n^2}{4}\left\|H^*\right\|^2+2\left\|h^0\right\|^2\\
&+2\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{0\alpha}h_{jj}^{0\alpha}-\sum_{\alpha=1}^{n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{*\alpha}\\
&-\frac{1}{2}\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left((h_{ij}^{\alpha})^2+(h_{ij}^{*\alpha})^2\right).
\end{aligned}$$

The last inequality can be written as

$$\begin{aligned}
&\frac{c+3}{4}\left(n^2-n\right)+\frac{c-1}{4}\left\{2\left\|P\right\|^2-(n-2)\left\|\xi^T\right\|^2-\lambda^2+\sum_{i=1}^{n}g\left(Pe_i,P^*e_i\right)\right\}\\
&\leq 2\tau-2n^2\left\|H^0\right\|^2+\frac{n^2}{4}\left\|H\right\|^2+\frac{n^2}{4}\left\|H^*\right\|^2+2\left\|h^0\right\|^2+2\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{0\alpha}h_{jj}^{0\alpha}\\
&-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left(h_{ii}^{\alpha}h_{jj}^{*\alpha}-h_{ij}^{\alpha}h_{ij}^{*\alpha}\right)-\frac{1}{2}\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}(h_{ij}^{\alpha}+h_{ij}^{*\alpha})^2.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{2\leq i\neq j\leq n}g\left(R(e_i,e_j)e_j,e_i\right)=&\frac{c+3}{4}(n-1)(n-2)+\frac{c-1}{4}\{2\left\|P\right\|^2\\
&-(n-4)\left\|\xi^T\right\|^2-\lambda^2+2(n-2)\eta(e_1)^2+2\lambda g\left(Pe_1,e_1\right)\\
&-2g\left(Pe_1,P^*e_1\right)-2g\left(P^*e_1,P^*e_1\right)-2g\left(Pe_1,Pe_1\right)\\
&+\sum_{i=1}^{n}g\left(Pe_i,P^*e_i\right)\}+\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left(h_{ii}^{\alpha}h_{jj}^{*\alpha}-h_{ij}^{\alpha}h_{ij}^{*\alpha}\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{c+3}{2}(n-1)-\frac{c-1}{4}\{2\left\|\xi^{T}\right\|^{2}+2(n-2)\eta(e_1)^2+2\lambda g(Pe_1,e_1)\\
&-2g(Pe_1,P^*e_1)-2g(P^*e_1,P^*e_1)-2g(Pe_1,Pe_1)\}\\
&\leq 2\tau-2n^2\left\|H^0\right\|^2+\frac{n^2}{4}\|H\|^2+\frac{n^2}{4}\|H^*\|^2+2\left\|h^0\right\|^2\\
&+2\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}h_{ii}^{0\alpha}h_{jj}^{0\alpha}-\sum_{2\leq i\neq j\leq n}g(R(e_i,e_j)e_j,e_i)\\
&-2\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}(h_{ij}^{0\alpha})^2.
\end{aligned}$$

Hence, we find

$$\begin{aligned}
Ric(X)\geq&\frac{c+3}{4}(n-1)-\frac{c-1}{4}\{\left\|\xi^{T}\right\|^{2}+(n-2)\eta(e_1)^2\\
&+\lambda g(Pe_1,e_1)-g(Pe_1,P^*e_1)-g(P^*e_1,P^*e_1)\\
&-g(Pe_1,Pe_1)\}+n^2\left\|H^0\right\|^2-\frac{n^2}{8}\|H\|^2-\frac{n^2}{8}\|H^*\|^2-\left\|h^0\right\|^2\\
&-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left[h_{ii}^{0\alpha}h_{jj}^{0\alpha}-\left(h_{ij}^{0\alpha}\right)^2\right]. \qquad (4.3)
\end{aligned}$$

By the Gauss equation with respect to the Levi-Civita connection, we have

$$\begin{aligned}
\sum_{1\leq i\neq j\leq n}\widetilde{R}^0(e_i,e_j,e_j,e_i)=&\sum_{1\leq i\neq j\leq n}\{R^0(e_i,e_j,e_j,e_i)\\
&+\widetilde{g}(h^0(e_i,e_j),h^0(e_i,e_j))-\widetilde{g}(h^0(e_i,e_i),h^0(e_j,e_j))\}\\
&=2\tau^0-n^2\widetilde{g}(H^0,H^0)+\left\|h^0\right\|^2. \qquad (4.4)
\end{aligned}$$

Furthermore, by the Gauss equation, we can write

$$\begin{aligned}
\sum_{2\leq i\neq j\leq n}\widetilde{R}^0(e_i,e_j,e_j,e_i) \quad = \quad &\sum_{2\leq i\neq j\leq n}R^0(e_i,e_j,e_j,e_i)\\
&-\sum_{\alpha=n+1}^{2m-n+1}\sum_{2\leq i\neq j\leq n}\left[h_{ii}^{0\alpha}h_{jj}^{0\alpha}-\left(h_{ij}^{0\alpha}\right)^2\right]. \qquad (4.5)
\end{aligned}$$

Using (4.4) and (4.5) in (4.3), we obtain

$$\begin{aligned}
Ric(X)\geq&2Ric^0(X)+\frac{c+3}{4}(n-1)-\frac{c-1}{4}\{\left\|\xi^{T}\right\|^{2}\\
&+(n-2)\eta(X)^2+\lambda g(PX,X)-g(PX,P^*X)-g(P^*X,P^*X)\\
&-g(PX,PX)\}-\frac{n^2}{8}\|H\|^2-\frac{n^2}{8}\|H^*\|^2-2\sum_{i=2}^{n}\widetilde{K}^0(X\wedge e_i),
\end{aligned}$$

where $\widetilde{K}^0(X \wedge .)$ is the sectional curvature of $\widetilde{M}$ with respect to $\widetilde{\nabla}$ restricted to 2-plane sections of the tangent space T_pM, which are tangent to X.

The vector field $X = e_1$ satisfies the above equality if and only if

$$h_{11}^{\alpha} = h_{22}^{\alpha} + ... + h_{nn}^{\alpha}, \quad h_{1j}^{\alpha} = 0, \quad 2 \leq j \leq n \text{ and } n+1 \leq \alpha \leq 2m+1,$$
$$h_{11}^{*\alpha} = h_{22}^{*\alpha} + ... + h_{nn}^{*\alpha}, \quad h_{1j}^{*\alpha} = 0, \quad 2 \leq j \leq n \text{ and } n+1 \leq \alpha \leq 2m+1,$$

or, equivalently,

$$2h(X,X) = nH(p), \qquad h(X,Y) = 0, \quad \forall Y \in T_pM \text{ orthogonal to } X,$$
$$2h^*(X,X) = nH^*(p), \quad h^*(X,Y) = 0, \quad \forall Y \in T_pM \text{ orthogonal to } X.$$

Thus, we can state the following theorem:

Theorem 4.1 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(i) Assume that ξ is tangent to M.

(a) If M is invariant, then

$$\begin{aligned} Ric(X) \geq & 2Ric^0(X) + \frac{c+3}{4}(n-1) - \frac{c-1}{4}\{1 + \lambda g(PX,X) \\ & + (n-1)\eta(X)^2 - \|X\|^2 - g(P^*X, P^*X) - g(PX,PX)\} \\ & - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 - 2\sum_{i=2}^{n} \widetilde{K}^0(X \wedge e_i). \end{aligned}$$

(b) If M is antiinvariant, then

$$\begin{aligned} Ric(X) \geq & 2Ric^0(X) + \frac{c+3}{4}(n-1) - \frac{c-1}{4}\left\{1 + (n-2)\eta(X)^2\right\} \\ & - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 - 2\sum_{i=2}^{n} \widetilde{K}^0(X \wedge e_i). \end{aligned}$$

(ii) If ξ is normal to M (which means that M is antiinvariant), then

$$\begin{aligned} Ric(X) \geq & 2Ric^0(X) + \frac{c+3}{4}(n-1) \\ & - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 - 2\sum_{i=2}^{n} \widetilde{K}^0(X \wedge e_i). \end{aligned}$$

Moreover, one of the equality holds in all cases if and only if

$$2h(X,X) = nH(p), \qquad h(X,Y) = 0, \quad \forall Y \in T_pM \text{ orthogonal to } X,$$
$$2h^*(X,X) = nH^*(p), \quad h^*(X,Y) = 0, \quad \forall Y \in T_pM \text{ orthogonal to } X,$$

where $\widetilde{K}^0(X \wedge .)$ is the sectional curvature of $\widetilde{M}$ with respect to $\widetilde{\nabla}$ restricted to 2-plane sections of the tangent space T_pM, which are tangent to X.

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