Turkish Journal of Mathematics

Volume 42 | Number 6 Article 20

1-1-2018

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Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2018) 42: 3078 – 3089
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doi:10.3906/mat-1804-32

Some characterizations of right c-regularity and (b,c)-inverse

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Received: 09.04.2018 • **Accepted/Published Online:** 03.10.2018 • **Final Version:** 27.11.2018

Abstract: Let R be a ring and $a,b,c\in R$. We give a novel characterization of group inverses (resp. EP elements) by the properties of right (resp. left) c-regular inverses of a and discuss the relation among the strongly left (b,c)-invertibility of a, the right ca-regularity of b, and the (b,c)-invertibility of a. Finally, we investigate the sufficient and necessary condition for a ring to be a strongly left min-Abel ring by means of the (b,c)-inverse of a.

Key words: Right c-regular element, (b,c)-inverse, group inverse, EP element, left min-Abel ring

1. Introduction

Let S be a semigroup and $a,b,c\in S$. Then a is said to be (b,c)-invertible [4] if there exists $y\in bSy\cap ySc$ such that $yab=b$ and $cay=c$. Such an y is called a (b,c)-inverse of a, which is always unique if it exists, denoted by $a^{||(b,c)}$.

In [5], Drazin considered the following problem: in any semigroup S (or any associative ring) with unit element 1, and for any given $a\in S$, the properties $1\in Sa$ $(1\in aS)$ of left (right) invertibility are often useful as weaker versions of ordinary two-sided invertibility, and it is natural to seek corresponding one-sided versions for at least some types of generalized invertibility. Hence, Drazin in [5] introduced the left (b,c)-inverse as follows: let S be any semigroup and let $a,b,c\in S$. Then a is said to be left (b,c)-invertible if $b\in Scab$, or equivalently if there exists $x\in Sc$ such that $xab=b$, in which case any such x will be called a left (b,c)-inverse of a. The left (b,c)-inverse of a is not unique [5, Example 3.4]. Dually, a is said to be right (b,c)-invertible if $c\in cabS$, or equivalently if there exists $z\in bS$ such that $caz=c$, and any such z will be called a right (b,c)-inverse of a. Related studies of the one-sided (b,c)-inverse can be found in [7] and [12]. The main purpose of this article is to do some further research on the left (right) (b,c)-inverse of a. Therefore, the following concepts need to be introduced.

Let R be a ring and $a,c\in R$. If there exists $b\in R$ such that $a=abca$ $(a=acba)$, then we say that a is right (left) c-regular and b is a right (left) c-regular inverse of a. We denote by a_c^- $({}_ca^-)$ the set of all right (left) c-regular inverses of a.

In [1], an element a of a ring R is said to be group invertible if there is $a^\#\in R$ such that

$$aa^\#a=a,\ a^\#aa^\#=a^\#,\ aa^\#=a^\#a.$$

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2010 *AMS Mathematics Subject Classification:* 15A09, 16U99, 16W10

Denote by $R^{\#}$ the set of all group invertible elements of R. An element $a \in R$ is group invertible if and only if $a \in a^2R \cap Ra^2$ [3, 6]. Clearly, a ring R is strongly regular if and only if $R = R^{\#}$.

An involution $a \longmapsto a^*$ in a ring R is an antiisomorphism of degree 2; that is,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*.$$

A ring R with an involution $*$ is called a $*$-ring. An element $p \in R$ is called a projection if $p^2 = p = p^*$.

An element $a^\dagger$ in a $*$-ring R is called the Moore–Penrose inverse (or MP-inverse) [9] of a, if

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ aa^\dagger = (aa^\dagger)^*,\ a^\dagger a = (a^\dagger a)^*.$$

In this case, we say a is MP-invertible in R. The set of all MP-invertible elements of R is denoted by $R^\dagger$.

In [2], an element a of a $*$-ring R is said to be EP if $a \in R^\dagger$ and $a^\dagger a = aa^\dagger$, which is equivalent to $a \in R^{\#} \cap R^\dagger$ and $a^{\#} = a^\dagger$. Denote by R^{EP} the set of all EP-invertible elements of R.

An idempotent $e \in R$ is called a left minimal idempotent if Re is a minimal left ideal of R. We denote by $ME_l(R)$ the set of all left minimal idempotents of R, and e is said to be left (right) semicentral if $ae = eae$ ($ea = eae$) for each $a \in R$. A ring R is said to be (strongly) left min-Abel [10] if either $ME_l(R) = \emptyset$ or every element e of $ME_l(R)$ is (right) left semicentral.

In this paper, we first study the right (left) c-regular elements by means of left and right (b,c)-inverses of a. Next, with the help of right (left) c-regular elements, we characterize group invertible elements, MP-invertible elements, and EP elements. Finally, we give some new characterizations of directly finite rings, left min-Abel rings, and strongly left min-Abel rings.

2. c-Regular inverses

Recall that an element a of a ring R is said to be regular if there exists $b \in R$ such that $a = aba$. Such a b is called an inner inverse of a. Clearly, if b is an inner inverse of a, then so is bab. We denote by a^- the set of all inner inverses of a.

Let R be a ring. For any $a, c \in R$, if there exists $b \in R$ such that $a = abca$ ($a = acba$), then we say that a is right (left) c-regular and b is right (left) c-regular inverse of a. Obviously, if a is right c-regular, then a is regular, but the converse is not true from the following example.

Example 2.1 *Let* $R = T_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in \mathbb{Z}_2 \right\}$. *It is easy to check that* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *is regular. Take* $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* $CA = 0$. *Consequently, we obtain that* $ABCA \neq A$, *for any* $B \in R$. *That is,* A *is not right* C*-regular.*

In order to study the (b,c)-inverse of a in the next section, we first discuss right (left) c-regular inverses of a in this section.

Remark 2.2 *Let* R *be a ring. For each* $a, b, c \in R$, *if* b *is a right* c*-regular inverse of* a, *so is* $bcab$. *In fact,* $a(bcab)ca = (abca)bca = abca = a$. *If* a *is right (left)* c*-regular, then we denote by* a_c^- ($_ca^-$) *the set of all right (left)* c*-regular inverses of* a.

Example 2.3 *Let a be a regular element of a ring R. If $d \in a^-$, then a is right ad-regular and left da-regular. In fact, $a = ada = ad(ad)a = a(da)da$, which implies $d \in a^-_{ad}$ and $d \in_{da} a^-$.*

If a is regular and $b \in a^-$, then $b \in a^-_{ab} \cap {}_{ba}a^-$. Conversely, if a is regular and $b \in R$ satisfying $b \in a^-_{ab} \cap {}_{ba}a^-$, then $b \in a^-$?

From the following example, we know that the above question is not true.

Example 2.4 *Let $R = T_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in \mathbb{Z}_3 \right\}$. Write $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in R$. It is easy to check that $ABA = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \neq A$ and $ABABA = A$. Therefore, $B \in A^-_{AB} \cap {}_{BA}A^-$, but $B \notin A^-$.*

Proposition 2.5 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) ab is right c-regular and $Rb = Rab$;
(2) ab is right c-regular and $Rb = Rcab$;
(3) cab is regular and $Rb = Rcab$.

Proof (1) $\Rightarrow$ (2) Since ab is right c-regular, we get $ab = ab(ab)^-_c cab$. This clearly forces $Rb = Rab = Rab(ab)^-_c cab \subseteq Rcab \subseteq Rab$. That is, $Rb = Rcab$.

(2) $\Rightarrow$ (3) Since ab is right c-regular, we have $ab = ab(ab)^-_c cab$. Premultiplying by c, we have $cab = cab(ab)^-_c cab$. Hence, cab is regular.

(3) $\Rightarrow$ (1) Since $Rb = Rcab$, $b = vcab$ for some $v \in R$. From the hypothesis that cab is regular, we have $b = vcab(cab)^- cab = b(cab)^- cab$. Premultiplying by a, we get $ab = ab(cab)^- cab$. Therefore, ab is right c-regular, and $(cab)^- \subseteq (ab)^-_c$. □

Corollary 2.6 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) ab is right c-regular, and $Rb = Rab$;
(2) $b \in bRcab$;
(3) b is right ca-regular.

Proof (1) $\Rightarrow$ (2) Write $b = vab$. We deduce that

$$b = vab = vab(ab)^-_c cab = b(ab)^-_c cab \in bRcab.$$

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Since $b = bb^-_{ca} cab$, we obtain that $ab = abb^-_{ca} cab$. Hence, ab is right c-regular and $b^-_{ca} \subseteq (ab)^-_c$. Moreover, we have $Rb = Rbb^-_{ca} cab \subseteq Rab \subseteq Rb$. That is, $Rb = Rab$. □

Proposition 2.7 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) $b \in bRcab$;
(2) $r(ca) \cap bR = 0$, and b is right ca-regular;
(3) $r(ab) = r(b)$, and ab is right c-regular.

Proof (1) $\Rightarrow$ (2) Set $b = bvcab$. Then b is right ca-regular. Assume that $t \in r(ca) \cap bR$. Writing $t = bs$, we get $cabs = cat = 0$. Moreover, we get $bs = bvcabs = 0$. This means that $t = 0$.

(2) $\Rightarrow$ (3) For any $y \in r(ab)$, we have $aby = 0$. Premultiplying by c, we get $caby = 0$. It follows that $by \in r(ca) \cap bR = 0$. Thus, $y \in r(b)$. This gives $r(b) \supseteq r(ab)$. However, $r(b) \subseteq r(ab)$ is clear. Hence, $r(b) = r(ab)$. Moreover, we get that ab is right c-regular, because $b = bb^-_{ca}cab$.

(3) $\Rightarrow$ (1) Since $ab = ab(ab)^-_c cab$, we obtain that $1-(ab)^-_c cab \in r(ab) = r(b)$. Therefore, $b = b(ab)^-_c cab \in bRcab$. □

Next, we give some characterizations of group invertible elements, MP-invertible elements, and EP-elements with c-regular inverses.

Proposition 2.8 *Let R be a ring and $a \in R^{\#}$. Then $a^-_{a^{\#}} = \{x \in R | a^{\#}a = axa^{\#}\}$.*

Proof Since $a \in R^{\#}$, $a^{\#}$ exists and $a = a(a^{\#}a)a^{\#}a$. It follows that a is right $a^{\#}$-regular and $a^{\#}a \in a^-_{a^{\#}}$. Thus, $a^-_{a^{\#}}$ is not empty. For any $x \in a^-_{a^{\#}}$, we have $a = axa^{\#}a$. This gives $aa^{\#} = axa^{\#}aa^{\#} = axa^{\#}$. That is, $x \in \{x \in R | a^{\#}a = axa^{\#}\}$. Conversely, if $x \in \{x \in R | a^{\#}a = axa^{\#}\}$, then $a = a^{\#}a^2 = axa^{\#}a$. Therefore, $x \in a^-_{a^{\#}}$. □

Proposition 2.9 *Let R be a ring and a be a regular element of R. Then $a \in R^{\#}$ if and only if there exists $b \in R$ such that $b \in a^-_{ba} \cap {}_{ab}a^-$.*

Proof Assume that $a \in R^{\#}$. Then $a^{\#}$ exists. Write $b = a^{\#} \in R$. Then we have

$$ab(ba)a = aa^{\#}(a^{\#}a^2) = aa^{\#}a = a,$$
$$a(ab)ba = a^2a^{\#}a^{\#}a = aa^{\#}a = a,$$

which imply $b \in a^-_{ba} \cap {}_{ab}a^-$.

Conversely, since $b \in a^-_{ba} \cap {}_{ab}a^-$, we get $ab(ba)a = a = a(ab)ba$, which yields $a \in a^2R \cap Ra^2$. Therefore, $a \in R^{\#}$. □

Proposition 2.10 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{\#}$;
(2) there exist $x \in R$ and $d \in {}_xa^-$, such that ${}_xa^- = a^-_x$ is not empty and $dxa = axd$.

Proof (1) $\Rightarrow$ (2) Assume that $a \in R^{\#}$. Then $a^{\#}$ exists and $a^{\#}a \in a^-_{a^{\#}} \cap {}_{a^{\#}}a^-$. Thus, $a^-_{a^{\#}}$ and ${}_{a^{\#}}a^-$ are not empty. Set $y \in {}_{a^{\#}}a^-$. We get $a = aa^{\#}ya$. Premultiplying by a, we have $a^2 = a^2a^{\#}ya = aya$. We conclude from the above equality that $a^{\#}a = aa^{\#} = a^2(a^{\#})^2 = aya(a^{\#})^2 = aya^{\#}$, which gives $y \in a^-_{a^{\#}}$, and hence that ${}_{a^{\#}}a^- \subseteq a^-_{a^{\#}}$. In the same manner we can see that $a^-_{a^{\#}} \subseteq {}_{a^{\#}}a^-$, and so ${}_{a^{\#}}a^- = a^-_{a^{\#}}$. Since $a^{\#}a \in {}_{a^{\#}}a^-$, we have $(a^{\#}a)a^{\#}a = a^{\#}a = aa^{\#} = aa^{\#}(aa^{\#}) = aa^{\#}(a^{\#}a)$. Thus, the conclusion is proved by writing $x = a^{\#}$ and $d = a^{\#}a$.

(2) $\Rightarrow$ (1) Let $x \in R$ satisfy ${}_xa^- = a^-_x$, which is not empty, and let $d \in {}_xa^-$ satisfy $dxa = axd$. Then $a = axda = adxa$. Write $y = dxaxd$. We get

$$aya = adxaxda = axda = a,$$
$$yay = dxaxdadxaxd = dxadxaxd = dxaxd = y,$$
$$ya = dxaxda = dxa = axd = adxaxd = ay.$$

Consequently, $a \in R^{\#}$ and $a^{\#} = y = dxaxd$. □

Proposition 2.11 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{\dagger}$;

(2) there exists $x \in a^{-}_{ax}$ such that ax and xa are projections.

Proof (1) ⇒ (2) From the hypothesis that $a \in R^{\dagger}$, $a^{\dagger}$ exists. Write $x = a^{\dagger}$. It is easy to check that the element x satisfies condition (2).

(2) ⇒ (1) Assume that there exists $x \in a^{-}_{ax}$ such that ax and xa are projections. Then we get $ax(ax)a = a$, $ax = axax = (ax)^*$, and $xa = xaxa = (xa)^*$. Thus, $axa = (axax)a = a$. Take $b = xax$. Then we obtain

$$ab = axax = ax = (ax)^* = (ab)^*,$$
$$ba = xaxa = xa = (xa)^* = (ba)^*,$$
$$aba = axa = a,\ bab = (xax)(ax) = xax = b.$$

Consequently, $a \in R^{\dagger}$ and $a^{\dagger} = b = xax$. □

Proposition 2.12 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{EP}$;

(2) $a \in R^{\dagger}$, ${}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$, and there exists $d \in {}_{a^{\dagger}}a^{-}$, such that $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$.

Proof (1) ⇒ (2) Suppose that $a \in R^{EP}$. Then $a \in R^{\#} \cap R^{\dagger}$. From the proof of Proposition 2.10, we know that ${}_{a^{\#}}a^{-} = a^{-}_{a^{\#}}$ and there exists $d \in {}_{a^{\#}}a^{-}$ such that $da^{\#}a = aa^{\#}d = aa^{\#}$. Accordingly, we have $d \in {}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$, which satisfies $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$.

(2) ⇒ (1) Let $d \in {}_{a^{\dagger}}a^{-}$ satisfy $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$. Then $a = aa^{\dagger}da = ada^{\dagger}a$ follows from $d \in {}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$. Write $x = da^{\dagger}d$. Then we get

$$axa = ada^{\dagger}da = ada^{\dagger}aa^{\dagger}da = aa^{\dagger}da = a,$$
$$xax = da^{\dagger}dada^{\dagger}d = d(a^{\dagger}aa^{\dagger})dada^{\dagger}d = da^{\dagger}(aa^{\dagger}da)da^{\dagger}d = da^{\dagger}ada^{\dagger}d = da^{\dagger}ad(a^{\dagger}aa^{\dagger})d = da^{\dagger}(ada^{\dagger}a)a^{\dagger}d =$$
$$da^{\dagger}aa^{\dagger}d = da^{\dagger}d = x,$$
$$ax = ada^{\dagger}d = ad(a^{\dagger}aa^{\dagger})d = aa^{\dagger}d = da^{\dagger}a = da^{\dagger}(aa^{\dagger}da) = d(a^{\dagger}aa^{\dagger})da = da^{\dagger}da = xa.$$

Thus, we deduce that $a \in R^{\#}$ and $a^{\#} = x = da^{\dagger}d$. Premultiplying by a, we obtain that $aa^{\#} = ada^{\dagger}d = aa^{\dagger}d = aa^{\dagger}$. That is, $a \in R^{EP}$ by [8, Theorem 7.3]. □

Recall that a ring R is quasinormal [11] if $eR(1-e)Re = 0$ for each $e^2 = e \in R$. The following theorem gives a new characterization of quasinormal rings. At the end of this section, we study the quasinormal rings and the directly finite rings by means of c-regular inverses.

Theorem 2.13 *Let R be a ring and e be an idempotent of R. Then the following conditions are equivalent:*

(1) R is a quasinormal ring;

(2) if there exists an idempotent $g \in R$ such that $e^-_{eg} \neq \emptyset$, then $e^-_{eg} = e^-_{ge}$.

Proof $\Rightarrow$ Assume that R is quasinormal and $e^2 = e, g^2 = g \in R$ with $e^-_{eg} \neq \emptyset$. Choose $x \in e^-_{eg}$. Then $e = exege$. Note that R is quasinormal. Then $ex(1-e)ge \in eR(1-e)Re = 0$, and it follows that $exge = exege$. Hence, $e = exge = ex(ge)e$, which implies that $x \in e^-_{ge}$, so $e^-_{eg} \subseteq e^-_{ge}$. Conversely, assume that $y \in e^-_{ge}$, and then $e = ey(ge)e = eyge$. Since R is quasinormal, $eyge = eyege = ey(eg)e$, one obtains that $y \in e^-_{eg}$. Hence, $e^-_{ge} \subseteq e^-_{eg}$.

$\Leftarrow$ Assume that $e^2 = e \in R$. For any $a, b \in R$, write $g = e + (1-e)ae, f = e + eb(1-e)$. Then $eg = e = fe, ge = g, ef = f, g^2 = g$, and $f^2 = f$. Note that $e = ef(eg)e$. Then $f \in e^-_{eg}$, by hypothesis, and we have $e^-_{eg} = e^-_{ge}$. Hence, $f \in e^-_{ge}$; that is, $e = ef(ge)e = fg = e + eb(1-e)ae$, and we have $eb(1-e)ae = 0$ for any $a, b \in R$. Therefore, $eR(1-e)Re = 0$, and so R is quasinormal. □

Proposition 2.14 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a directly finite ring;

(2) if $ab = 1$ for $a, b \in R$, then $a^-_b = \{1\}$.

Proof (1) $\Rightarrow$ (2) Assume that $ab = 1$. Then we get $a = a(ba)ba$. That is, $ba \in a^-_b$. Since R is a directly finite ring, we see that $ba = 1$. It follows that a and b are invertible and $1 \in a^-_b$. For any $x \in a^-_b$, we conclude that $a = axba = ax$. Thus, $x = 1$. Hence, $a^-_b = \{1\}$.

(2) $\Rightarrow$ (1) Let $a, b \in R$ satisfy $ab = 1$. By the hypothesis, we know $a^-_b = \{1\}$. As $ba \in a^-_b$, we have $ba = 1$. Consequently, R is a directly finite ring. □

Proposition 2.15 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a directly finite ring;

(2) if $ab = 1$ for $a, b \in R$, then $a^-_b = b^-_a$.

Proof (1) $\Rightarrow$ (2) Suppose that R is a directly finite ring and $ab = 1$. Then we could find $a^-_b = \{1\}$ by Proposition 2.14. Since $ba = 1$, we have $b^-_a = \{1\}$ by Proposition 2.14. Hence, $a^-_b = b^-_a$.

(2) $\Rightarrow$ (1) Let $a, b \in R$ satisfy $ab = 1$. Then $a^-_b = b^-_a$ follows from the hypothesis. We have $ba \in a^-_b = b^-_a$, because $a = a(ba)ba$. That is, $b = b(ba)ab = b^2a$. This clearly forces $1 = ab = ab^2a = (ab)(ba) = ba$. Therefore, R is a directly finite ring. □

3. Characterizations of the (b, c)-inverse of a

Let R be a ring. For each $a, b, c \in R$, a is said to be strongly left (b, c)-invertible if there exists $x \in bRc$ such that $b = xab$. Such an x is called a strongly left (b, c)-inverse of a. Clearly, if x is a strongly left (b, c)-inverse of a, then so is xax. Denote by $a_l^{s\|(b,c)}$ the set of all strongly left (b, c)-inverses of a.

In this section, we will consider the relation among the strongly left (b,c)-invertibility of a, the right ca-regularity of b, and the (b,c)-invertibility of a.

In the following, we give an example in which the strongly left (b,c)-inverse of a is not unique.

Example 3.1 *Let $R = M_2(\mathbb{Z}_2)$. Write $a = x_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $b = x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is obvious that $x_1 = buc \in bRc$, $x_2 = bvc \in bRc$, and $x_1ab = b = x_2ab$. This gives $x_1, x_2 \in a_l^{s\|(b,c)}$, but $x_1 \neq x_2$.*

Proposition 3.2 *Let R be a ring and $a,b,c \in R$. If a is strongly left (b,c)-invertible and $x \in a_l^{s\|(b,c)}$, then we have:*

(1) $x \in bRx \cap xRc$;
(2) $xax = x$;
(3) cax is left ab-regular;
(4) $xR = bR$;
(5) $r(c) \subseteq r(x)$.

Proof It follows from $x \in a_l^{s\|(b,c)}$ that $x \in bRc$ and $b = xab$. Write $x = bvc$. Then we get $xax = xabvc = bvc = x$. This gives $bvcax = xax = x = bvc = xabvc$. Thus, $x \in bRx \cap xRc$. Furthermore, we have

$$cax = caxax = cabvcax = ca(xab)vcax = cax(ab)vcax.$$

Hence, cax is left ab-regular. We have $xR = bR$, because $xR = bvcR \subseteq bR = xabR \subseteq xR$. Finally, for any $d \in r(c)$, we have $cd = 0$. Premultiplying by bv, we get $xd = bvcd = 0$. That is, $d \in r(x)$. □

We first give some equivalent conditions for an element to be strongly left (b,c)-invertible.

Corollary 3.3 *Let R be a ring and $a,b,c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;

(2) there exists $x \in R$, such that $xax = x$, $l(x) = l(b)$, $Rx \subseteq Rc$, and $xR \subseteq bR$. In this case, $x \in a_l^{s\|(b,c)}$.

Proof (1) $\Rightarrow$ (2) Fix $x \in a_l^{s\|(b,c)}$. It follows from Proposition 3.2 that

$$xax = x,\ xR = bR,\ Rx \subseteq Rc, \text{ and } l(x) = l(b).$$

(2) $\Rightarrow$ (1) Since $1 - xa \in l(x) = l(b)$, it follows that $b = xab$. Write $x = vc = bs$. Then we obtain $x = xax = (bs)a(vc) \in bRc$. Hence, a is strongly left (b,c)-invertible. This means that $x \in a_l^{s\|(b,c)}$. □

Corollary 3.4 *Let R be a ring and $a,b,c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;
(2) there exists $x \in R$ such that $xax = x$, $xR = bR$, and $Rx \subseteq Rc$.

Proof (1) $\Rightarrow$ (2) Let $x \in a_l^{s\|(b,c)}$. Then $b = xab$ and $x \in bRc$. This gives that $bR = xR$ and $Rx \subseteq Rc$. Again, by Proposition 3.2, we have that $x = xax$.

(2) $\Rightarrow$ (1) Since $xR = bR$ and $Rx \subseteq Rc$, one has that $x = xax \in bRc$. By $1 - xa \in l(x) = l(b)$, we get that $b = xab$. Thus, a is strongly left (b,c)-invertible, and $x \in a_l^{s\|(b,c)}$. □

Corollary 3.5 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;
(2) $b \in bRcab$.

Proof (1) $\Rightarrow$ (2) It is clear from the definition of strongly left (b,c)-invertibility.

(2) $\Rightarrow$ (1) Set $b = bvcab$ and $x = bvc$. Then we get $x \in bRc$ and $b = xab$. That is, a is strongly left (b,c)-invertible. □

Next, we discuss when a strongly left (b,c)-invertible element actually becomes a (b,c)-invertible element.

Proposition 3.6 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $caa_l^{s\|(b,c)} = c$. In this case, $a^{\|(b,c)} \in a_l^{s\|(b,c)}$.

Proof (1) $\Rightarrow$ (2) Set $y = a^{\|(b,c)}$. It is straightforward that

$$y \in bRy \cap yRc,\ y = yay,\ yab = b, \text{ and } cay = c.$$

Thus, $y = yay \in (bRy)a(yRc) \subseteq bRc$. Therefore, a is strongly left (b,c)-invertible, $y \in a_l^{s\|(b,c)}$, and $cay = c$. Now, for each $x \in a_l^{s\|(b,c)}$, we get $l(x) = l(b) = l(y)$ by Corollary 3.3. We conclude from $1 - xa \in l(x) = l(y)$ that $y = xay$, and hence that $c = cay = caxay$ and finally that $1 - axay \in r(c) \subseteq r(x)$ by Corollary 3.3. We thus get $x = xaxay = xay = y$. Hence, $cax = cay = c$.

(2) $\Rightarrow$ (1) Since a is strongly left (b,c)-invertible, there exists $x \in R$ such that

$$x = xax,\ l(x) = l(b),\ Rx \subseteq Rc,\ xR \subseteq bR, \text{ and } x \in a_l^{s\|(b,c)}.$$

It follows that $cax = c$. Write $x = dc = bt$. We have

$$x = xax = btax \in bRx, \text{ and } x = xax = xadc \in xRc.$$

Namely, $b = xab$ because $1 - xa \in l(x) = l(b)$. Thus, a is (b,c)-invertible and $a^{\|(b,c)} = x$. It is obvious that $a^{\|(b,c)} = x \in a_l^{s\|(b,c)}$. □

Proposition 3.7 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $Rc \cap l(ab) = 0$.

Proof (1) $\Rightarrow$ (2) It follows from Proposition 3.6 that a is strongly left (b,c)-invertible. Now let $a^{\|(b,c)}=y$. Then $y=yay$, $yab=b$, and $cay=c$. Assume that $z\in Rc\cap l(ab)$. Then we have $z=dc$ and $zab=0$, where $d\in R$. Thus, $dcab=0$. Set $y=bs$. Then $z=dc=dcay=zay=zabs=0$.

(2) $\Rightarrow$ (1) Let $x\in a_l^{s\|(b,c)}$. Then by Proposition 3.2, we get $xax=x$, $x=bvc$, $l(x)=l(b)$, and $cax=caxaxax=caxa(bvc)ax$. Hence, $ca-caxabvca\in l(x)=l(b)$. This gives $cab=caxabvcab$. We thus get $c-caxabvc\in l(ab)\cap Rc=0$. This yields that $c=caxabvc=caxax=cax$. By Proposition 3.6, we have that a is (b,c)-invertible. □

Corollary 3.8 *Let R be a ring and $a,b,c\in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $l(c)=l(cab)$.

Proof (1) $\Rightarrow$ (2) Take any $x\in l(cab)$. We have $xcab=0$. Thus, $xc\in Rc\cap l(ab)=0$ by Proposition 3.7. That is, $x\in l(c)$.

(2) $\Rightarrow$ (1) For any $y\in Rc\cap l(ab)$, we know that $y=dc$ and $yab=0$, where $d\in R$. Thus, $dcab=0$. This means that $d\in l(cab)=l(c)$. Therefore, $y=dc=0$. It follows from Proposition 3.7 that a is (b,c)-invertible. □

Corollary 3.9 *Let R be a ring and $a,b,c\in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $R=Rc\oplus l(ab)$.

Proof Assume that a is (b,c)-invertible. By Proposition 3.7, we know that $Rc\cap l(ab)=0$. Write $a^{\|(b,c)}=y$. Then we have $y\in yRc$ and $b=yab$. Hence, $ab=ayab$. It follows that $1-ay\in l(ab)$. We thus get $1\in Ry+l(ab)\subseteq Rc+l(ab)$. Then $R=Rc+l(ab)$. That $R=Rc\oplus l(ab)$ follows from Proposition 3.7. The converse is obvious. □

Corollary 3.10 *Let R be a ring and $a,b,c\in R$. If a is (b,c)-invertible, then $R=Ra_l^{s\|(b,c)}\oplus l(ab)$.*

Proof Since a is (b,c)-invertible, $c=caa_l^{s\|(b,c)}$ by Proposition 3.6 and $R=Rc\oplus l(ab)$ by Corollary 3.9. Hence, $R=Ra_l^{s\|(b,c)}+l(ab)$. For any $z\in Ra_l^{s\|(b,c)}\cap l(ab)$, we have $z=ya_l^{s\|(b,c)}$ and $zab=0$, where $y\in R$. This gives $ya_l^{s\|(b,c)}ab=0$. Write $a_l^{s\|(b,c)}=btc$ for $t\in R$. Since $b=a_l^{s\|(b,c)}ab$, we have $z=ya_l^{s\|(b,c)}=ybtc=ya_l^{s\|(b,c)}abtc=0$. The result is $Ra_l^{s\|(b,c)}\cap l(ab)=0$. Therefore, $R=Ra_l^{s\|(b,c)}\oplus l(ab)$. □

Naturally, is the converse of the Corollary 3.10 true? The problem has not yet been solved.

Question 3.11 *If a is strongly left invertible and $Ra_l^{s\|(b,c)}\oplus l(ab)=R$, then is a (b,c)-invertible?*

4. Left min-Abel ring and (b,c)-inverse of a

This section is devoted to the study of left (resp. strongly left) min-Abel ring.

Let R be a ring and $e^2 = e \in R$. We denote by $E(R)$ the set of all idempotents of R. If Re is a left minimal ideal of R, then e is called a left minimal idempotent of R. Denote by $ME_l(R)$ the set of all left minimal idempotents of R. If either $ME_l(R)$ is an empty set or every element of $ME_l(R)$ is left (resp. right) semicentral in R, then R is called a left (resp. strongly left) min-Abel ring.

We first give some conditions to ensure that a ring R is a left min-Abel ring, by means of left semicentral elements and left (b,c)-invertible elements in R.

Lemma 4.1 *Let R be a ring and $e \in ME_l(R)$ a left semicentral idempotent. If $e = abe$ for $a, b \in R$, then $e = bae$.*

Proof Since e is left semicentral and $e = abe$, we have $e = aebe$. Thus, $ae \neq 0$. This gives $Re = Rae$. Writing $e = cae$ for $c \in R$, we can assert that $ce = c(aebe) = (cae)be = ebe = be$. It is obvious that $bae = beae = ceae = cae = e$. □

Proposition 4.2 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a left min-Abel ring;
(2) $e_a^- \subseteq {}_ae^-$ for any $e \in ME_l(R)$ and $a \in R$.

Proof (1) $\Rightarrow$ (2) Assume that R is a left min-Abel ring, $e \in ME_l(R)$, and $a \in R$. Fix $x \in e_a^-$. Then we have $e = (ex)ae$. Since R is a left min-Abel ring, we deduce that e is left semicentral. That $e = aexe = axe$ follows from Lemma 4.1. Thus, $e = eaxe$. That is, $x \in {}_ae^-$.

(2) $\Rightarrow$ (1) For any $e \in ME_l(R)$ and $a \in R$, writing $h = (1-e)ae$, we can assert that $he = h$, $eh = 0$, and $h^2 = 0$. If $h \neq 0$, then $Rh = Re$. Taking $e = ch$ for $c \in R$, we get $e = eche$. That is, $c \in e_h^-$. From the hypothesis, we obtain that $e_h^- \subseteq {}_he^-$. It follows that $c \in {}_he^-$. We thus get $e = ehce = 0$. This contradicts our assumption. From this, we see that $h = 0$. It follows that $(1-e)ae = h = 0$ for any $a \in R$. This gives $(1-e)Re = 0$. Consequently, R is a left min-Abel ring.

□

Proposition 4.3 *Let R be a ring and $k \in E(R)$. Then the following conditions are equivalent:*

(1) k is a left minimal idempotent of R;
(2) if $ak \neq 0$ for $a \in R$, then a is left $(k,1)$-invertible.

Proof (1) $\Rightarrow$ (2) Suppose that k is a left minimal idempotent of R and $ak \neq 0$. Then we get $Rk = Rak$. It follows that a is left $(k,1)$-invertible.

(2) $\Rightarrow$ (1) Let $0 \neq L$ be any left ideal of R contained in Rk. Then we get $0 \neq y \in L \subseteq Rk$. Write $y = ak$. It follows that $ak \neq 0$. From the assumption, we know that a is left $(k,1)$-invertible and $k \neq 0$. Then it is easy to see that $0 \neq Rk \subseteq R1ak = Ry \subseteq L$. That is, $Rk = L$. Hence, Rk is a left minimal ideal of R. □

Proposition 4.4 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a left min-Abel ring;
(2) if $ae \neq 0$ for $e \in ME_l(R)$ and $a \in R$, then there exists $c \in Re$ such that $e = cae$.

Proof (1) $\Rightarrow$ (2) Suppose that $ae \neq 0$. It follows from Proposition 4.3 that a is left $(e,1)$-invertible. For each $x \in a_l^{\|(e,1)}$, we get $e = xae$. Since R is a left min-Abel ring, we know that e is a left semicentral idempotent, i.e. $e = xeae$. Taking $c = xe \in Re$, the result holds.

(2) $\Rightarrow$ (1) For any $e \in ME_l(R)$, if $(1-e)Re \neq 0$, then there exists $a \in R$ such that $h = (1-e)ae \neq 0$. By assumption, there exists $c \in Re$ such that $e = che$ for $he = h \neq 0$. Write $c = te$. It is easy to show that $e = tehe = te(1-e)ae = 0$. It is a contradiction, so we have $(1-e)Re = 0$. Hence, R is a left min-Abel ring. □

Motivated by Propositions 4.2–4.4, in the following, we give the main result for this section.

Theorem 4.5 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a strongly left min-Abel ring;
(2) if $ea \neq 0$ for $e \in ME_l(R)$ and $a \in R$, then a is right (e,e)-invertible.

Proof (1) $\Rightarrow$ (2) We first show that eR is a minimal right ideal of R. Assume that $0 \neq K$ is an arbitrary right ideal of R contained in eR. For every $0 \neq x \in K$, we know $x = ex$. Since R is a strongly left min-Abel ring, e is a right semicentral idempotent. It follows that $x = xe$ and $0 \neq Rx = Rxe = Re$. Write $e = yx$ and $g = xy$, where $y \in R$. It is clear that

$$g^2 = xyxy = xey = xy = g,\ g = xy = exy = eg \text{ and } e = (yx)(yx) = ygx.$$

Moreover, $ge = ege = eg = g$. It follows that $0 \neq Rg = Rge \subseteq Re$. That is, $Rg = Re$. Thus, $g \in ME_l(R)$. This means that g is also a right semicentral idempotent. Furthermore, we get

$$e = ygx = ygxg = eg = g, \text{ and } eR = gR = xyR \subseteq xR \subseteq K \subseteq eR.$$

Thus, eR is a minimal right ideal of R.

Now we assume that $ea \neq 0$. Then we get $eaR = eR$ and write $e = eac$ for some $c \in R$. Since e is central, we have $e = eaec$, which means that a is right (e,e)-invertible.

(2) $\Rightarrow$ (1) Suppose that $e \in ME_l(R)$. If $eR(1-e) \neq 0$, then there exists some $a \in R$ such that $h = ea(1-e) \neq 0$. Since $eh = h$, we have that h is right (e,e)-invertible by (2). This clearly forces $e \in eheR$, so $e = 0$, which is a contradiction. It follows that $eR(1-e) = 0$. Hence, R is a strongly left min-Abel ring. □

Corollary 4.6 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a strongly left min-Abel ring;
(2) for each $e \in ME_l(R)$ and $x, y \in R$, $e = xy$ implies that $e = yx$.

Proof (1) $\Rightarrow$ (2) The proof is straightforward from Theorem 4.5.

(2) $\Rightarrow$ (1) For any $a \in R$, we denote $g = e + ea(1-e)$. It follows that $eg = g$ and $ge = e$. By assumption, we get $e = ge = eg = g$. It is obvious that $eR(1-e) = 0$. □

Acknowledgments

This work was supported by the National Science Foundation of China under Grant Nos. 11471282, 11661014, 11701499, and 11871063, and the Research Involution Project of Academic Degree Graduate Students in Jiangsu Province of China under Grant No. XKYCX17_029.

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