

1-1-2018

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Recommended Citation

YENER, ABDULLAH (2018) "Several Hardy-type inequalities with weights related to Baouendi–Grushin operators," *Turkish Journal of Mathematics*: Vol. 42: No. 6, Article 18. <https://doi.org/10.3906/mat-1802-18>

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Several Hardy-type inequalities with weights related to Baouendi–Grushin operators

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Received: 07.02.2018

Accepted/Published Online: 03.10.2018

Final Version: 27.11.2018

Abstract: In this paper we shall prove several weighted L^p Hardy-type inequalities associated to the Baouendi–Grushin-type operators $\Delta_\gamma = \Delta_x + |x|^{2\gamma} \Delta_y$, where Δ_x and Δ_y are the classical Laplace operators in the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, respectively, and γ is a positive real number.

Key words: Baouendi–Grushin operator, weighted Hardy inequality

1. Introduction

The well-known Hardy inequality in \mathbb{R}^n , $n \geq 3$, asserts that for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx. \quad (1.1)$$

Though the constant $\left(\frac{n-2}{2}\right)^2$ is sharp, equality in (1.1) is never achieved by any function $u \in H_0^1(\mathbb{R}^n)$. Hardy [13] originally discovered this inequality in 1920 for the one-dimensional case. Since then it has attracted the attention of many mathematicians and has been comprehensively analyzed in several directions; see, for example, [2, 3, 5, 10, 11, 14, 15, 21] and the references therein.

The sharp Hardy inequality (1.1) arises very naturally in the study of degenerate elliptic differential operators and it was first extended in [9] by Garofalo to the Baouendi–Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ with $n, k \geq 1$, $\gamma > 0$. To be explicit, the author in [9] proved the following Hardy inequality

$$\int_{\mathbb{R}^{n+k}} |\nabla_\gamma u|^2 dx dy \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^{n+k}} \frac{|x|^{2\gamma} u^2}{\rho^{2\gamma} \rho^2} dx dy \quad (1.2)$$

for every $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^k \setminus \{(0,0)\})$. Here, $Q = n + (1 + \gamma)k$ is the homogeneous dimension, $\nabla_\gamma = (X_1, \dots, X_n, Y_1, \dots, Y_k)$ is the subelliptic gradient, and $\rho = (|x|^{2(1+\gamma)} + (1 + \gamma)^2 |y|^2)^{\frac{1}{2(1+\gamma)}}$ is the gauge function

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2010 AMS Mathematics Subject Classification: 26D10, 35H10, 46E35

associated to the fundamental solution for the subelliptic operator $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y$, where Δ_x and Δ_y are the Laplacians on \mathbb{R}^n and \mathbb{R}^k , respectively.

After the seminal work of Garofalo [9], much effort has been devoted to Hardy-type inequalities for the Baouendi–Grushin vector fields; see [6–8, 16–20]. For instance, in [6], D’Ambrosio obtained a weighted L^p analogue of (1.2). Later, Niu et al. [19] considered various types of Hardy inequalities on some special domains in $\mathbb{R}^n \times \mathbb{R}^k$ via a Picone-type identity for the Baouendi–Grushin vector fields $\{X_i, Y_j\}$. In [16], Kombe proved weighted Hardy-type inequalities with remainder terms. On the other hand, Laptev et al. [18] recently established weighted Hardy inequalities for the quadratic form of the magnetic Baouendi–Grushin operator with Aharonov–Bohm-type magnetic field. More recently, Kombe and Yener [17] introduced a unified approach to the weighted Hardy inequalities related to Baouendi–Grushin operators.

In the literature, Hardy-type inequalities mostly involve the weights of the form $\rho^\alpha |x|^\beta$ for some real numbers α and β . Our goal in this paper is to prove several L^p Hardy-type inequalities with more general and nonstandard weights associated to the Baouendi–Grushin-type operators Δ_γ . For this goal we shall mainly use a technique developed in [17].

2. Preliminaries and notations

In this section we shall introduce some notations and preliminary facts. Let \mathbb{R}^N be split in $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^k$ with $n, k \geq 1$ and $N = n + k$. We denote by $|x|$ and $|y|$ the usual Euclidean norms in \mathbb{R}^n and \mathbb{R}^k , respectively. The corresponding vector field is

$$\nabla_\gamma = (X_1, \dots, X_n, Y_1, \dots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y),$$

where

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k \tag{2.1}$$

and γ is a positive real number. The Baouendi–Grushin operator is of the form

$$\Delta_\gamma = \sum_{i=1}^n X_i^2 + \sum_{j=1}^k Y_j^2 = \nabla_\gamma \cdot \nabla_\gamma = \Delta_x + |x|^{2\gamma} \Delta_y.$$

Here, Δ_x and Δ_y stand for the usual Laplace operators in the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, respectively. The operator Δ_γ was first studied by Baouendi [1] and Grushin [12] when γ is a positive integer. We note that if γ is not an even positive integer, then Δ_γ is not a sum of squares of C^∞ vector fields satisfying Hörmander’s condition:

$$\text{rank Lie}[X_1, \dots, X_n, Y_1, \dots, Y_k] = N.$$

The nonlinear p -degenerate subelliptic operator associated with the vector fields (2.1) is defined by

$$\Delta_{\gamma,p}u = \nabla_\gamma \cdot \left(|\nabla_\gamma u|^{p-2} \nabla_\gamma u \right), \quad p > 1.$$

The operator Δ_γ possesses a natural family of anisotropic dilations, namely

$$\delta_\lambda(z) = (\lambda x, \lambda^{\gamma+1}y), \quad \lambda > 0, \quad z = (x, y) \in \mathbb{R}^N.$$

It is easy to verify that

$$d\delta_\lambda(x, y) = \lambda^Q dx dy = \lambda^Q dz,$$

where

$$Q = n + (1 + \gamma)k$$

is the homogeneous dimension with respect to the dilation δ_λ and $dz = dx dy$ denotes the Lebesgue measure on \mathbb{R}^N . For $z = (x, y) \in \mathbb{R}^N$, define the distance between z and the origin of \mathbb{R}^N as follows:

$$\rho = \rho(z) = \left(|x|^{2(1+\gamma)} + (1 + \gamma)^2 |y|^2 \right)^{\frac{1}{2(1+\gamma)}}.$$

We remark that ρ is positive, smooth away from the origin, and symmetric.

A simple calculation shows that

$$\nabla_\gamma \rho = \frac{|x|^\gamma}{\rho^{2\gamma+1}} (|x|^\gamma x_1, \dots, |x|^\gamma x_n, (1 + \gamma)y_1, \dots, (1 + \gamma)y_k)$$

and hence

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}.$$

A function u on \mathbb{R}^N is said to be radial when u has the form $u = u(\rho)$. If u is radial, then by a straightforward computation we have

$$|\nabla_\gamma u(\rho)| = |\nabla_\gamma \rho| |u'(\rho)|$$

and

$$\Delta_\gamma u(\rho) = |\nabla_\gamma \rho|^2 \left[u''(\rho) + (Q - 1) \frac{u'(\rho)}{\rho} \right].$$

In particular, when $u(\rho) = \rho^\alpha$ one gets

$$|\nabla_\gamma \rho^\alpha| = |\alpha| |\nabla_\gamma \rho| \rho^{\alpha-1} \tag{2.2}$$

and

$$\Delta_\gamma \rho^\alpha = \alpha(Q + \alpha - 2) |\nabla_\gamma \rho|^2 \rho^{\alpha-2} \tag{2.3}$$

with $\alpha \in \mathbb{R}$. Moreover, the gauge ρ is infinite harmonic in $\mathbb{R}^N - \{0\}$ (see [4]); that is, ρ is the solution of the following equation:

$$\nabla_\gamma (|\nabla_\gamma \rho|^2) \cdot \nabla_\gamma \rho = 0. \tag{2.4}$$

3. Weighted Hardy-type inequalities

For the convenience of the reader, we begin by quoting a known result from [17].

Theorem 3.1 *Let $\vartheta \in C^1(\mathbb{R}^N)$ and $w \in L^1_{loc}(\mathbb{R}^N)$ be nonnegative functions and $f \in C^\infty(\mathbb{R}^N)$ be a positive function such that*

$$-\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f \right) \geq w f^{p-1} \tag{3.1}$$

almost everywhere in \mathbb{R}^N . Then for all $u \in C_0^\infty(\mathbb{R}^N)$ and $p > 1$, one has

$$\int_{\mathbb{R}^N} \vartheta |\nabla_\gamma u|^p dz \geq \int_{\mathbb{R}^N} w |u|^p dz. \tag{3.2}$$

Remark 3.2 Let $\epsilon > 0$ be given. Define

$$\rho_\epsilon := \left(|x|_\epsilon^{2(1+\gamma)} + (1+\gamma)^2 |y|^2 \right)^{\frac{1}{2(1+\gamma)}},$$

where $|x|_\epsilon := (\epsilon^2 + \sum_{i=1}^n x_i^2)^{1/2}$. In the proof of the following theorems, if the functions ϑ and f are not smooth enough, by standard argument one can consider ρ_ϵ a regularization of ρ and after the computation takes the limit as $\epsilon \rightarrow 0$. However, for the sake of simplicity we will proceed formally.

Recall that the following weighted Hardy-type inequalities in the Euclidean setting were proved by Ghoussoub and Moradifam [10]: Let $a, b > 0$ and α, β, m be real numbers.

- If $\alpha\beta > 0$ and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 dx \geq \left(\frac{n-2m-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx. \tag{3.3}$$

- If $\alpha\beta < 0$ and $2m - \alpha\beta \leq n - 2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 dx \geq \left(\frac{n+\alpha\beta-2m-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx. \tag{3.4}$$

We now generalize and improve the above inequalities (3.3) and (3.4) to the L^p case for the Baouendi–Grushin operator Δ_γ by specializing the choice of functions ϑ and f in the differential inequality (3.1). Here is the first result in this direction.

Theorem 3.3 Let $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta > 0$ and $1 < p \leq Q - pm$, then for all $u \in C_0^\infty(\mathbb{R}^N)$ one has

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}} |\nabla_\gamma u|^p dz &\geq C_{Q,p,m}^p \int_{\mathbb{R}^N} \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm+p}} |\nabla_\gamma \rho|^p |u|^p dz \\ &\quad + C_{Q,p,m}^{p-1} \alpha\beta b \int_{\mathbb{R}^N} \frac{(a+b\rho^\alpha)^{\beta-1}}{\rho^{pm+p-\alpha}} |\nabla_\gamma \rho|^p |u|^p dz, \end{aligned}$$

where $C_{Q,p,m} = \left(\frac{Q-pm-p}{p} \right)$.

Proof Set $\vartheta = \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}}$ and $f = \rho^{-\left(\frac{Q-pm-p}{p}\right)}$ in Theorem 3.1. Clearly,

$$\nabla_\gamma f = - \left(\frac{Q-pm-p}{p} \right) \rho^{-\left(\frac{Q-pm-p}{p}\right)} \nabla_\gamma \rho.$$

Noting that $1 < p \leq Q - pm$, one gets

$$|\nabla_\gamma f|^{p-2} = \left(\frac{Q - pm - p}{p}\right)^{p-2} \rho^{-\left(\frac{Q-pm}{p}\right)(p-2)} |\nabla_\gamma \rho|^{p-2}.$$

Hence,

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f &:= -\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f\right) \\ &= C_{Q,p,m}^{p-1} \nabla_\gamma \cdot \left(\frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}} \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right), \end{aligned} \tag{3.5}$$

where $C_{Q,p,m} := \left(\frac{Q-pm-p}{p}\right)$. Using the identities (2.3) and (2.4), a direct calculation yields

$$\begin{aligned} \nabla_\gamma \left(\frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}}\right) &= \frac{\alpha\beta b\rho^{\alpha+pm-1} (a + b\rho^\alpha)^{\beta-1} \nabla_\gamma \rho - pm\rho^{pm-1} (a + b\rho^\alpha)^\beta \nabla_\gamma \rho}{\rho^{2pm}} \\ &= \alpha\beta b (a + b\rho^\alpha)^{\beta-1} \rho^{\alpha-pm-1} \nabla_\gamma \rho - pm (a + b\rho^\alpha)^\beta \rho^{-pm-1} \nabla_\gamma \rho \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \nabla_\gamma \cdot \left(\rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right) &= -\left(\frac{Q - pm}{p}\right) (p - 1) \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)-1} |\nabla_\gamma \rho|^p \\ &\quad + \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} \left(\nabla_\gamma \left(|\nabla_\gamma \rho|^{p-2}\right) \cdot \nabla_\gamma \rho + |\nabla_\gamma \rho|^{p-2} \Delta_\gamma \rho\right) \\ &= \left(\frac{Q - pm - p}{p} + pm\right) \rho^{\frac{Q-pm-p}{p} - Q + pm} |\nabla_\gamma \rho|^p. \end{aligned} \tag{3.7}$$

It therefore follows from (3.5), (3.6), and (3.7) that

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f &= C_{Q,p,m}^{p-1} \nabla_\gamma \left(\frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}}\right) \cdot \left(\rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right) \\ &\quad + C_{Q,p,m}^{p-1} \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}} \nabla_\gamma \cdot \left(\rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right) \\ &= C_{Q,p,m}^{p-1} \alpha\beta b (a + b\rho^\alpha)^{\beta-1} \rho^{\frac{Q-pm-p}{p} - Q + \alpha} |\nabla_\gamma \rho|^p \\ &\quad - C_{Q,p,m}^{p-1} pm (a + b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ &\quad + C_{Q,p,m}^{p-1} \left(\frac{Q - pm - p}{p} + pm\right) (a + b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ &= C_{Q,p,m}^p (a + b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ &\quad + C_{Q,p,m}^{p-1} \alpha\beta b (a + b\rho^\alpha)^{\beta-1} \rho^{\frac{Q-pm-p}{p} - Q + \alpha} |\nabla_\gamma \rho|^p. \end{aligned}$$

Since

$$f^{p-1} = \rho^{\frac{Q-pm-p}{p} - Q + pm + p},$$

one can put

$$w = C_{Q,p,m}^p \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm-p}} |\nabla_\gamma \rho|^p + C_{Q,p,m}^{p-1} \alpha \beta b \frac{(a + b\rho^\alpha)^{\beta-1}}{\rho^{pm+p-\alpha}} |\nabla_\gamma \rho|^p.$$

Hence, by Theorem 3.1 we complete the proof. □

Theorem 3.4 *Let $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta < 0$ and $1 < p \leq Q + \alpha\beta - pm$, then for all $u \in C_0^\infty(\mathbb{R}^N)$ one has*

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}} |\nabla_\gamma u|^p dz &\geq C_{Q,p,m,\alpha,\beta}^p \int_{\mathbb{R}^N} \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm+p}} |\nabla_\gamma \rho|^p |u|^p dz \\ &\quad - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha \beta a \int_{\mathbb{R}^N} \frac{(a + b\rho^\alpha)^{\beta-1}}{\rho^{pm+p}} |\nabla_\gamma \rho|^p |u|^p dz, \end{aligned}$$

where $C_{Q,p,m,\alpha,\beta} = \left(\frac{Q + \alpha\beta - pm - p}{p}\right)$.

Proof Choose $\vartheta = \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}}$ and $f = \rho^{-\left(\frac{Q + \alpha\beta - pm - p}{p}\right)}$, where $a, b > 0$, $\alpha\beta < 0$ and $1 < p \leq Q + \alpha\beta - pm$.

One easily sees

$$\nabla_\gamma f = -\left(\frac{Q + \alpha\beta - pm - p}{p}\right) \rho^{-\left(\frac{Q + \alpha\beta - pm}{p}\right)} \nabla_\gamma \rho$$

and

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f &= -\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f\right) \\ &= C_{Q,p,m,\alpha,\beta}^{p-1} \nabla_\gamma \cdot \left(\frac{(a + b\rho^\alpha)^\beta}{\rho^{pm}} \rho^{-\left(\frac{Q + \alpha\beta - pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right), \end{aligned} \tag{3.8}$$

where $C_{Q,p,m,\alpha,\beta} = \left(\frac{Q + \alpha\beta - pm - p}{p}\right)$. By a similar computation as in the proof of Theorem 3.3, we derive

$$\nabla_\gamma \cdot \left(\rho^{-\left(\frac{Q + \alpha\beta - pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho\right) = \left(\frac{Q + \alpha\beta - pm - p}{p} + pm - \alpha\beta\right) \rho^{\frac{Q + \alpha\beta - pm - p}{p} - Q - \alpha\beta + pm} |\nabla_\gamma \rho|^p. \tag{3.9}$$

Combining (3.6) and (3.9) with (3.8), we deduce that

$$\begin{aligned}
 -\Delta_{\gamma,p,\vartheta} f &= C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta b (a + b\rho^\alpha)^{\beta-1} \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta+\alpha} |\nabla_\gamma \rho|^p \\
 &\quad - C_{Q,p,m,\alpha,\beta}^{p-1} pm (a + b\rho^\alpha)^\beta \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta} |\nabla_\gamma \rho|^p \\
 &\quad + C_{Q,p,m,\alpha,\beta}^{p-1} \left(\frac{Q+\alpha\beta-pm-p}{p} + pm - \alpha\beta \right) \frac{(a + b\rho^\alpha)^\beta}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} |\nabla_\gamma \rho|^p \\
 &= C_{Q,p,m,\alpha,\beta}^p (a + b\rho^\alpha)^\beta \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta} |\nabla_\gamma \rho|^p \\
 &\quad + C_{Q,p,m,\alpha,\beta}^{p-1} (\alpha\beta b\rho^\alpha - \alpha\beta (a + b\rho^\alpha)) \frac{(a + b\rho^\alpha)^{\beta-1}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} |\nabla_\gamma \rho|^p \\
 &= C_{Q,p,m,\alpha,\beta}^p \frac{(a + b\rho^\alpha)^\beta}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} |\nabla_\gamma \rho|^p \\
 &\quad - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta a \frac{(a + b\rho^\alpha)^{\beta-1}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} |\nabla_\gamma \rho|^p.
 \end{aligned}$$

This shows that one can put

$$w = C_{Q,p,m,\alpha,\beta}^p \frac{(a + b\rho^\alpha)^\beta}{\rho^{pm+p}} |\nabla_\gamma \rho|^p - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta a \frac{(a + b\rho^\alpha)^{\beta-1}}{\rho^{pm+p}} |\nabla_\gamma \rho|^p$$

and hence finishes the proof. □

Remark 3.5 Note that if $\alpha = 0$ or $\beta = 0$ in the above two inequalities, then they reduce to Hardy-type inequalities with classical weights. Therefore, we are interested in the case where $\alpha\beta \neq 0$.

Even though the literature has mostly focused on power radial weights $\rho^\alpha |x|^\beta$ for some $\alpha, \beta \in \mathbb{R}$, we now establish L^p Hardy-type inequalities with nonradial weights related to Baouendi–Grushin operator Δ_γ . Here is the first result in this direction.

Theorem 3.6 For any $u \in C_0^\infty(\Omega)$ and $p > 1$, one has

$$\int_\Omega \left(\frac{y_1}{|x|^\gamma} \right)^{p-2} \log x_1 |\nabla_\gamma u|^p dz \geq \int_\Omega \frac{|x|^{2\gamma} \log x_1}{y_1^2 \log^{p-1} y_1} |u|^p dz, \tag{3.10}$$

where $\Omega = \{z = (x, y) \in \mathbb{R}^N : x_1 > 1, y_1 > 1\}$.

Proof Let us first set $\vartheta = \left(\frac{y_1}{|x|^\gamma} \right)^{p-2} \log x_1$ and $f = \log y_1$ in Theorem 3.1 with $x_1, y_1 > 1$. Observing that

$$X_i(\log y_1) = \frac{\partial}{\partial x_i} \log y_1 = 0 \quad \forall i = 1, \dots, n$$

and

$$Y_j(\log y_1) = \left(|x|^\gamma \frac{\partial}{\partial y_j} \right) (\log y_1) = \begin{cases} \frac{|x|^\gamma}{y_1}, & \text{if } j = 1, \\ 0, & \text{if } j \neq 1, \end{cases}$$

we immediately have

$$\nabla_\gamma \log y_1 = \left(0, \dots, 0, \frac{|x|^\gamma}{y_1}, 0, \dots, 0 \right).$$

It follows that

$$|\nabla_\gamma \log y_1|^{p-2} = \left(\frac{|x|^\gamma}{y_1} \right)^{p-2}$$

and therefore

$$\begin{aligned} -\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f \right) &= -\nabla_\gamma \cdot \left(0, \dots, 0, \frac{|x|^\gamma \log x_1}{y_1}, 0, \dots, 0 \right) \\ &= -Y_1 \left(\frac{|x|^\gamma \log x_1}{y_1} \right) \\ &= -|x|^{2\gamma} \log x_1 \frac{\partial}{\partial y_1} \left(\frac{1}{y_1} \right) \\ &= \frac{|x|^{2\gamma} \log x_1}{y_1^2}. \end{aligned}$$

This means that one can put

$$w = \frac{|x|^{2\gamma} \log x_1}{y_1^2 \log^{p-1} y_1}.$$

We have thus proved the inequality (3.10). □

Theorem 3.7 For any $u \in C_0^\infty(\Omega)$ and $p > 1$, one has

$$\int_\Omega \cosh^\alpha y_1 |\nabla_\gamma u|^p dz \geq (p-1) \int_\Omega \frac{\cosh^\alpha y_1}{x_1^p \log^{p-1} x_1} |u|^p dz, \tag{3.11}$$

where $\Omega = \{z = (x, y) \in \mathbb{R}^N : x_1 > 1\}$ and $\alpha \in \mathbb{R}$.

Proof Let us now choose $\vartheta = \cosh^\alpha y_1$ and $f = \log x_1$ with $x_1 > 1$ and $\alpha \in \mathbb{R}$. Then we directly compute

$$X_i(\log x_1) = \frac{\partial}{\partial x_i} \log x_1 = \begin{cases} \frac{1}{x_1}, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1 \end{cases} \tag{3.12}$$

and

$$Y_j(\log x_1) = \left(|x|^\gamma \frac{\partial}{\partial y_j} \right) (\log x_1) = 0 \quad \forall j = 1, \dots, k. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\nabla_\gamma \log x_1 = \left(\frac{1}{x_1}, 0, \dots, 0 \right), \quad |\nabla_\gamma \log x_1|^{p-2} = x_1^{2-p}.$$

Therefore,

$$\begin{aligned} -\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f \right) &= -\nabla_\gamma \cdot \left(\frac{\cosh^\alpha y_1}{x_1^{p-1}}, 0, \dots, 0 \right) \\ &= -X_1 \left(\frac{\cosh^\alpha y_1}{x_1^{p-1}} \right) \\ &= -\cosh^\alpha y_1 \frac{\partial}{\partial x_1} \left(x_1^{1-p} \right) \\ &= (p-1) \frac{\cosh^\alpha y_1}{x_1^p \log^{p-1} x_1} f^{p-1}. \end{aligned}$$

By Theorem 3.1, the inequality (3.11) is deduced. □

Finally, we now prove the following L^p Hardy-type inequality including the power of the hyperbolic sine function of ρ as a weight.

Theorem 3.8 *Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $Q + \alpha + \beta > p > 1$. Then the following inequality holds:*

$$\int_{\mathbb{R}^N} \rho^\alpha \sinh^\beta \rho |\nabla_\gamma u|^p dz \geq \left(\frac{Q + \alpha + \beta - p}{p} \right)^p \int_{\mathbb{R}^N} \rho^\alpha \sinh^\beta \rho |\nabla_\gamma \rho|^p \frac{|u|^p}{\rho^p} dz$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Proof Letting $\vartheta = \rho^\alpha \sinh^\beta \rho$ and $f = \rho^{-\left(\frac{Q+\alpha+\beta-p}{p}\right)}$ in Theorem 3.1 with $\beta \geq 0$ and $Q + \alpha + \beta > p > 1$, we have

$$\nabla_\gamma f = -\left(\frac{Q + \alpha + \beta - p}{p} \right) \rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)} \nabla_\gamma \rho$$

and

$$|\nabla_\gamma f|^{p-2} = \left(\frac{Q + \alpha + \beta - p}{p} \right)^{p-2} \rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-2)} |\nabla_\gamma \rho|^{p-2}.$$

It therefore follows that

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f &= -\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f \right) \\ &= \left(\frac{Q + \alpha + \beta - p}{p} \right)^{p-1} \nabla_\gamma \cdot \left(\rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-1)+\alpha} \sinh^\beta \rho |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right). \end{aligned} \tag{3.14}$$

From the identities (2.3) and (2.4), it is not hard to see that

$$\nabla_\gamma \cdot \left(|\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right) = (Q-1) \frac{|\nabla_\gamma \rho|^p}{\rho}. \tag{3.15}$$

On the other hand, using the formula (2.2), one can readily obtain

$$\begin{aligned} \nabla_\gamma \left(\rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-1)+\alpha} \sinh^\beta \rho \right) &= \left(\frac{Q + \alpha + \beta}{p} - Q - \beta \right) \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho \nabla_\gamma \rho \\ &\quad + \beta \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta} \sinh^{\beta-1} \rho \cosh \rho \nabla_\gamma \rho. \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) with (3.14) leads to

$$\begin{aligned}
 -\Delta_{\gamma,p,\vartheta} f &= \left(\frac{Q + \alpha + \beta - p}{p}\right)^{p-1} \left(\frac{Q + \alpha + \beta}{p} - Q - \beta\right) \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho |\nabla_\gamma \rho|^p \\
 &\quad + \left(\frac{Q + \alpha + \beta - p}{p}\right)^{p-1} \beta \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta} \sinh^{\beta-1} \rho \cosh \rho |\nabla_\gamma \rho|^p \\
 &\quad + \left(\frac{Q + \alpha + \beta - p}{p}\right)^{p-1} (Q - 1) \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho |\nabla_\gamma \rho|^p \\
 &= \left(\frac{Q + \alpha + \beta - p}{p}\right)^p \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho |\nabla_\gamma \rho|^p \\
 &\quad + \left(\frac{Q + \alpha + \beta - p}{p}\right)^{p-1} \left(\rho \frac{\cosh \rho}{\sinh \rho} - 1\right) \beta \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho |\nabla_\gamma \rho|^p \\
 &\geq \left(\frac{Q + \alpha + \beta - p}{p}\right)^p \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho |\nabla_\gamma \rho|^p,
 \end{aligned}$$

where we have used the fact that $\rho \coth \rho \geq 1$. Hence,

$$-\nabla_\gamma \cdot \left(\vartheta |\nabla_\gamma f|^{p-2} \nabla_\gamma f\right) \geq \left(\frac{Q + \alpha + \beta - p}{p}\right)^p \rho^{\alpha-p} \sinh^\beta \rho |\nabla_\gamma \rho|^p f^{p-1};$$

that is, according to the assumption in Theorem 3.1, we can put

$$w = \left(\frac{Q + \alpha + \beta - p}{p}\right)^p \rho^{\alpha-p} \sinh^\beta \rho |\nabla_\gamma \rho|^p.$$

This ends the proof. □

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