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Jackson and Stechkin type inequalities of trigonometric approximation in $A_{w,\vartheta}^{p,q(\cdot)}$

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Abstract: In this paper, we study Jackson and Stechkin type theorems of trigonometric polynomial approximation in the space $A_{w,\vartheta}^{p,q(\cdot)}$ by considering a modulus of smoothness defined by virtue of the Steklov operator.

Key words: Modulus of smoothness, trigonometric polynomials, Muckenhoupt weight, direct and inverse theorems

1. Introduction and main results

Let $\mathbb{T} := [-\pi, \pi]$. The space $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$ is the intersection of the well-known weighted Lebesgue space $L_w^p(\mathbb{T})$ and the variable exponent weighted Lebesgue space $L_{\vartheta}^{q(\cdot)}(\mathbb{T})$, i.e. $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T}) = L_w^p(\mathbb{T}) \cap L_{\vartheta}^{q(\cdot)}(\mathbb{T})$. In the literature, the variable exponent Lebesgue spaces were first investigated by Orlicz [23]. In the space $L_w^p(\mathbb{T})$ the direct and inverse theorems of trigonometric approximation were proved in the papers [1, 2, 5, 6, 19, 20, 29]. In the space $L_{\vartheta}^{q(\cdot)}(\mathbb{T})$, similar theorems were obtained in the papers [3, 7, 10, 18, 23, 25]. Variable exponent weighted Lebesgue spaces have important applications in the theory of elasticity, fluid dynamics, and differential equations [15, 24]. Also, in weighted Lorentz spaces $L_w^{p,q}(\mathbb{T})$, which is a generalization of $L_w^p(\mathbb{T})$, direct and inverse theorems of approximation theory were proved in [8, 9, 21, 28]. In this paper, we prove the direct and inverse theorems of trigonometric approximation in $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$. First, we give some definitions and notations.

Let $\varphi(\mathbb{T})$ be the class of all measurable functions $p(\cdot) : \mathbb{T} \rightarrow [1, \infty]$ called the variable exponent on \mathbb{T} . In this paper, the notation $p(\cdot)$ is used always for a variable exponent. Let $p(\cdot) \in \varphi(\mathbb{T})$ and we set

$$p^- = \operatorname{ess\,inf}_{x \in \mathbb{T}} p(x), \quad p^+ = \operatorname{ess\,sup}_{x \in \mathbb{T}} p(x).$$

A measurable function $w : \mathbb{T} \rightarrow [0, \infty]$ is called a weight function if the preimage set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero.

Let $1 \leq p < \infty$ and f be a measurable function on \mathbb{T} . The weighted Lebesgue space L_w^p is defined as the set of all measurable functions f such that

$$\|f\|_{p,w} := \left(\int_{\mathbb{T}} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

It is known that L_w^p is a Banach space with respect to the norm $\|f\|_{p,w}$.

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Weighted variable exponent Lebesgue space $L_{\vartheta}^{p(\cdot)}$ is defined as the set of all measurable functions f on \mathbb{T} such that

$$\|f\|_{p(\cdot),\vartheta} = \left\| f \vartheta^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} < \infty$$

where

$$\|f\|_{p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{\mathbb{T}} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

It is a Banach space with respect to the norm $\|f\|_{p(\cdot),\vartheta}$.

Let $1 < p < \infty$, $q(\cdot) \in \wp(\mathbb{T})$. We set

$$A_{w,\vartheta}^{p,q(\cdot)} := \left\{ f : f \in L_w^p \cap L_{\vartheta}^{q(\cdot)} \right\}$$

and equip this Banach space with the norm

$$\|f\|_{w,\vartheta}^{p,q(\cdot)} = \|f\|_{p,w} + \|f\|_{q(\cdot),\vartheta}$$

for any $f \in A_{w,\vartheta}^{p,q(\cdot)}$.

We denote by $C_c(\mathbb{T})$ the space of all continuous, complex valued functions on \mathbb{T} . $C^\infty(\mathbb{T})$ consists of functions that are continuously differentiable arbitrarily many times. The set $C_0^\infty(\mathbb{T})$ is the subset of $C^\infty(\mathbb{T})$ of functions that have compact support. $C(\mathbb{T})$ is the set of all continuous functions on \mathbb{T} .

Some properties of $A_{w,\vartheta}^{p,q(\cdot)}$ were given in [26]:

I) $(A_{w,\vartheta}^{p,q(\cdot)}, \|\cdot\|_{w,\vartheta}^{p,q(\cdot)})$ is a Banach space with $q^+ < \infty$,

II) The space $A_{w,\vartheta}^{p,q(\cdot)}$ is a Banach function space on \mathbb{T} .

The set $C(\mathbb{T})$ is dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$.

Indeed: The set $C_0^\infty(\mathbb{T})$ is dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$ [26, Corollary 2], so for every $f \in A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$ there exists $g \in C_0^\infty(\mathbb{T})$ such that $\|f - g\|_{A_{w,\vartheta}^{p,q(\cdot)}} < \varepsilon$. On the other hand, as the embedding $C_0^\infty(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ is valid [13, p. 15], we have $g \in C(\mathbb{T})$. Thus, the set $C(\mathbb{T})$ is a dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$.

The weight functions used in this paper satisfy Muckenhoupt condition A_p [22] defined as

$$\sup \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w^{1-p'}(x) dx \right)^{p-1} = C_{A_p} < \infty, \quad p' = \frac{p}{p-1}, \quad 1 < p < \infty,$$

where the supremum is taken with respect to all intervals I , which are any subintervals of \mathbb{T} . The constant C_{A_p} is called the Muckenhoupt constant of w .

For weights w , the class $A_{q(\cdot)}$ is defined as

$$\|w\|_{q(\cdot)} := \sup_{I \in \varsigma} |I|^{-q_I} \|w\|_{L^1(I)} \left\| \frac{1}{w} \right\|_{L^{\frac{q'(\cdot)}{q(\cdot)}}(I)} < \infty,$$

where ς denotes the set of all intervals of \mathbb{T} , $q_I = \left(\frac{1}{|I|} \int_I \frac{1}{q(x)} dx \right)$ and $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ [11, 14].

We define the modulus of smoothness of a function $f \in A_{w,\vartheta}^{p,q(\cdot)}$ as

$$\Omega_r(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} := \sup_{0 \leq h \leq \delta} \|(I - \sigma_h)^r f\|_{A_{w,\vartheta}^{p,q(\cdot)}}, \quad r \in \mathbb{N} \tag{1.1}$$

where

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(u) du, \quad x \in \mathbb{T}.$$

If the condition

$$|q(x) - q(y)| \leq \frac{C}{-\ln|x - y|}, \quad x, y \in \Omega, \quad |x - y| \leq \frac{1}{2}$$

holds, then we say that the exponent $q(x)$ satisfies the Log-Hölder continuous function where $\Omega \subset \mathbb{T}$. We denote the family of all Log-Hölder continuous functions by the symbol $P^{\log}(\mathbb{T})$.

Denote by Mf the Hardy-Littlewood maximal operator, defined for $f \in L^1(\mathbb{T})$ by

$$Mf(x) := \sup_I \frac{1}{|I|} \int_I |f(y)| dy, \quad x \in \mathbb{T}$$

where the supremum is taken with respect to all the intervals I , which are any subintervals of \mathbb{T} .

Let $1 < p < \infty$, $q \in P^{\log}$ and $1 < q^- \leq q^+ < \infty$. The maximal operator $M : A_{w,\vartheta}^{p,q(\cdot)} \rightarrow A_{w,\vartheta}^{p,q(\cdot)}$ is bounded iff $w \in A_p$ and $\vartheta \in A_{q(\cdot)}$ [26, Theorem 10]. Therefore, the shift operator $\sigma_h(f)$ belongs to $A_{w,\vartheta}^{p,q(\cdot)}$ and thus $\Omega_r(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}}$ makes sense under these conditions.

The best approximation $E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}}$ of $f \in A_{w,\vartheta}^{p,q(\cdot)}$ is given by

$$E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} = \inf_{T_n \in \mathbf{T}_n} \|f - T_n\|_{A_{w,\vartheta}^{p,q(\cdot)}},$$

where \mathbf{T}_n is the set of trigonometric polynomials T_n of degree not more than n .

For an $f \in A_{w,\vartheta}^{p,q(\cdot)}$ the K -functional is defined as

$$K\left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^r\right) := \inf_{g \in W_{p,q(\cdot),w,\vartheta}^r} \left\{ \|f - g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + t^r \left\| \frac{d^r}{dx^r} g \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right\}, \quad t > 0$$

where

$$W_{p,q(\cdot),w,\vartheta}^r := \left\{ g \in A_{w,\vartheta}^{p,q(\cdot)} : \frac{d^r}{dx^r} g \in A_{w,\vartheta}^{p,q(\cdot)} \right\}.$$

The notation \lesssim indicates that “ $A \lesssim B$ iff there exists a positive constant C , independent of essential parameters, such that $A \leq CB$ ”.

If $A \lesssim B$ and $B \lesssim A$, simultaneously, we will write $A \approx B$.

The aim of this paper is to prove the Jackson and Stechkin type theorems of polynomial approximation in $A_{w,\vartheta}^{p,q(\cdot)}$ by considering modulus of smoothness (1.1).

Theorem 1.1 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, and $n, r \in \mathbb{N}$. Then

$$E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \tag{1.2}$$

holds for some constant depending only on $r, p, q(\cdot)$ and C_{A_p} .

Theorem 1.2 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, and $n, r \in \mathbb{N}$. Then

$$\Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \frac{1}{n^{2r}} \sum_{k=0}^n (k+1)^{2r-1} E_k(f)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

holds with some constant depending only on $r, p, q(\cdot)$ and C_{A_p} and $C_{A_{q(\cdot)}}$ are constants corresponding to L_p and $L_{q(\cdot)}$, respectively.

From Theorem 1.1 and Theorem 1.2, we obtain the following Marchaud type inequality.

Corollary 1.3 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$. Then we have

$$\Omega_r(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \delta^{2r} \int_{\delta}^1 u^{-2r-1} \Omega_k(f, u)_{A_{w,\vartheta}^{p,q(\cdot)}} du, \quad 0 < \delta < 1,$$

for $r, k \in \mathbb{N}$ with $r < k$.

Corollary 1.4 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$. If

$$E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$, then, for a given $r \in \mathbb{N}$, we have the estimations

$$\Omega_r(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \begin{cases} \delta^\alpha & , \quad r > \alpha/2; \\ \delta^{2r} \log \frac{1}{\delta} & , \quad r = \alpha/2; \\ \delta^{2r} & , \quad r < \alpha/2. \end{cases}$$

If we define the generalized Lipschitz class $Lip(\alpha, A_{w,\vartheta}^{p,q(\cdot)})$ for $\alpha > 0$ as

$$Lip(\alpha, A_{w,\vartheta}^{p,q(\cdot)}) := \left\{ f \in A_{w,\vartheta}^{p,q(\cdot)} : \Omega_k(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \delta^\alpha, \delta > 0 \right\},$$

then by virtue of Theorem 1.1 and Corollary 1.3 we obtain the following result, which gives a constructive characterization of the Lipschitz classes $Lip(\alpha, A_{w,\vartheta}^{p,q(\cdot)})$ in the case of $k := [\alpha/2]+1$, $[x] := \max \{n \in \mathbb{Z} : n \leq x\}$, $\alpha > 0$.

Corollary 1.5 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 \leq \alpha < \infty$, $1 < p < \infty$, $q \in P^{\log}$, and $1 < q^- \leq q^+ < \infty$. The following conditions are equivalent:

$$(i) f \in Lip\left(\alpha, A_{w,\vartheta}^{p,q(\cdot)}\right); \quad (ii) E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}.$$

2. Auxiliary results

To prove the above theorems we need the following auxiliary results.

Lemma 2.1 Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, and $t, k > 0$. Then

$$\Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \approx K\left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2\right)$$

and

$$\Omega_1(f, kt)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim (1 + [k])^2 \Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

hold for some constants depending only on $p, q(\cdot)$, and C_{A_p} and $C_{A_{q(\cdot)}}$ are constants corresponding to L_p and $L_{q(\cdot)}$, respectively.

Proof Let $t > 0$. Then there exists an $n \in \mathbb{N}$ such that $(1/n) < t \leq (2/n)$.

For this $n \in \mathbb{N}$, we shall consider an operator L_n on $A_{w,\vartheta}^{p,q(\cdot)}$ as follows:

$$(L_n f)(x) := 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s) ds du dt, \quad x \in \mathbb{T}, f \in A_{w,\vartheta}^{p,q(\cdot)}.$$

Then from [16] and [3, p. 14], with some constant $c \in \mathbb{R}$,

$$\frac{d^2}{dx^2} (L_n f)(x) = cn^2 ((I - \sigma_{1/n})) f(x) \tag{2.1}$$

is valid for a.e. $x \in \mathbb{T}$.

The operator L_n is bounded in $A_{w,\vartheta}^{p,q(\cdot)}$. In fact, using the uniform boundedness of the operator $f \rightarrow \sigma_{1/n} f$ in $A_{w,\vartheta}^{p,q(\cdot)}$ we obtain that

$$\frac{d^2}{dx^2} L_n f(x) \in A_{w,\vartheta}^{p,q(\cdot)}$$

and so $L_n f \in W_{p,q(\cdot),w,\vartheta}^2$. From the generalized Minkowski inequality and the boundedness of $\sigma_{1/n}$ we obtain

$$\begin{aligned} \|L_n f\|_{A_{w,\vartheta}^{p,q(\cdot)}} &= \left\| 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s) ds du dt \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim 3n^3 \int_0^{1/n} \int_0^t 2u \|\sigma_u f\|_{A_{w,\vartheta}^{p,q(\cdot)}} du dt \\ &\lesssim 3n^3 \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \int_0^{1/n} \int_0^t 2u du dt = \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{2.2}$$

By (2.2) we have $f - L_n f \in A_{w,\vartheta}^{p,q(\cdot)}$ for $f \in A_{w,\vartheta}^{p,q(\cdot)}$ and

$$\begin{aligned} K\left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2\right) &\leq 4K\left(f, 1/n; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2\right) \\ &\lesssim \|f - L_n f\|_{A_{w,\vartheta}^{p,q(\cdot)}} + n^{-2} \left\| \frac{d^2}{dx^2} L_n f(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ &=: I_1 + I_2. \end{aligned} \tag{2.1}$$

Using the generalized Minkowski inequality, we have

$$\begin{aligned} I_1 &= \|f - L_n f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim n^3 \int_0^{1/n} \int_0^t 2u \|(I - \sigma_u) f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \, dudt \\ &\lesssim \sup_{0 \leq u \leq 1/n} \|(I - \sigma_u) f\|_{A_{w,\vartheta}^{p,q(\cdot)}} 3n^3 \int_0^{1/n} \int_0^t 2u \, dudt \\ &\lesssim \sup_{0 \leq u \leq 1/n} \|(I - \sigma_u) f\|_{A_{w,\vartheta}^{p,q(\cdot)}} = \Omega_1\left(f, \frac{1}{n}\right)_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{2.2}$$

Using (2.1), we have

$$\begin{aligned} I_2 &= n^{-2} \left\| \frac{d^2}{dx^2} L_n f(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} = \left\| n^{-2} \frac{d^2}{dx^2} L_n f(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ &= \|C(I - \sigma_{1/n}) f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \sup_{0 \leq u \leq 1/n} \|(I - \sigma_u) f\|_{A_{w,\vartheta}^{p,q(\cdot)}} = \Omega_1\left(f, \frac{1}{n}\right)_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{2.5}$$

From (2.1)–(2.5),

$$K\left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2\right) \lesssim \Omega_1\left(f, \frac{1}{n}\right)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

Now we obtain the converse estimate of the last inequality. For $g \in W_{p,q(\cdot),w,\vartheta}^2$,

$$(I - \sigma_h)g(x) = \frac{1}{2h} \int_{-h}^h (g(x) - g(x+t)) \, dt = -\frac{1}{8h} \int_0^h \int_0^t \int_{-u}^u \left(\frac{d^2}{dx^2} g\right)(x+s) \, dsdudt.$$

Then

$$\begin{aligned} & \| (I - \sigma_h) g \|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \frac{1}{8h} \int_0^h \int_0^t 2u \left\| \frac{1}{2u} \int_{-u}^u \left(\frac{d^2}{dx^2} g \right) (x+s) ds \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} dudt \\ & \lesssim \frac{1}{8h} \int_0^h \int_0^t 2u \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} dudt = h^2 \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \end{aligned} \tag{2.3}$$

and we get

$$\Omega_1(g, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim t^2 \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}}$$

for $g \in W_{p,q(\cdot),w,\vartheta}^2$.

Then for $g \in W_{p,q(\cdot),w,\vartheta}^2$,

$$\Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \|f - g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + t^2 \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

If we take the infimum on $g \in W_{p,q(\cdot),w,\vartheta}^2$ in the last inequality,

$$\Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim K \left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2 \right).$$

Consequently,

$$\Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \approx K \left(f, t; A_{w,\vartheta}^{p,q(\cdot)}, W_{p,q(\cdot),w,\vartheta}^2 \right).$$

Using the last equivalence we have

$$\begin{aligned} \Omega_1(f, kt)_{A_{w,\vartheta}^{p,q(\cdot)}} & \lesssim \inf_{g \in W_{p,q(\cdot),w,\vartheta}^2} \left\{ \|f - g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + (kt)^2 \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right\} \\ & \lesssim (1 + [k])^2 \inf_{g \in W_{p,q(\cdot),w,\vartheta}^2} \left\{ \|f - g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + t^2 \left\| \frac{d^2}{dx^2} g(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right\} \\ & \lesssim (1 + [k])^2 \Omega_1(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \end{aligned}$$

and the lemma is proved. □

Lemma 2.2 (a) Let $F \in C(\mathbb{T})$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, and $n, m, r \in \mathbb{N}$. Then there exists a number $\delta \in (0, 1)$, depending only on $p, q(\cdot)$ and C_{A_p} , such that

$$\Omega_r(F, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1}(F, t)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

holds for any $t \in (0, 1/n)$, where the constant $C_1 > 0$ depends only on $r, p, q(\cdot)$ and C_{A_p} , and the constant $C_2 > 0$ depends only on $r, m, p, q(\cdot)$ and C_{A_p} .

(b) Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$. Then there exists $F \in C(\mathbb{T})$ so that

$$\Omega_r(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1}(f, t)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

where the constant $C_1 > 0$ depends only on $r, p, q(\cdot)$ and C_{A_p} , and the constant $C_2 > 0$ depends only on $r, m, p, q(\cdot)$ and C_{A_p} .

Proof (a) We will follow as in Lemma 3.2 of [4]. There exists a constant $\mathbf{C} > 1$, such that boundedness of $\sigma_h F$ in $A_{w,\vartheta}^{p,q(\cdot)}$ implies

$$\|\sigma_h F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \mathbf{C} \|F\|_{A_{w,\vartheta}^{p,q(\cdot)}}$$

for any $h > 0$. We define $\delta := \mathbf{C}/(\mathbf{C} + 1)$.

We can write the equalities

$$I - \sigma_h = \frac{1}{2} (I - \sigma_h)(I + \sigma_h) + \frac{1}{2} (I - \sigma_h)^2$$

and

$$\sigma_h(I - \sigma_h) = \frac{1}{2} (I - \sigma_h)(I + \sigma_h) - \frac{1}{2} (I - \sigma_h)^2, \quad h > 0.$$

Hence,

$$\begin{aligned} & \|(I - \sigma_h)g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|\sigma_h(I - \sigma_h)g\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \|(I - \sigma_h)(I + \sigma_h)g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^2g\|_{A_{w,\vartheta}^{p,q(\cdot)}}, \end{aligned} \tag{2.4}$$

for $g \in C(\mathbb{T})$ and $h > 0$. On the other hand,

$$\begin{aligned} & \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & = \delta \left((1/\mathbf{C}) \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & \leq \delta \left(\|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & = \delta \left(\|(I - \sigma_h)(I - \sigma_h)^{r-1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & = \delta \left(\left\| \left(\sigma_h(I - \sigma_h) + (I - \sigma_h)^2 \right) (I - \sigma_h)^{r-1} F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & \leq \delta \left(\left\| \sigma_h(I - \sigma_h)(I - \sigma_h)^{r-1} F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \left\| (I - \sigma_h)^2 (I - \sigma_h)^{r-1} F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & \quad + \delta \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \delta \left(\|\sigma_h(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \left\| (I - \sigma_h)^{r+1} F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right). \end{aligned} \tag{2.5}$$

If we take $g := (I - \sigma_h)^{r-1} F$ in (2.5), then

$$\begin{aligned} & \|\sigma_h (I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \|(I - \sigma_h)^r (I + \sigma_h) F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned}$$

Applying this in (2.4),

$$\begin{aligned} & \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \delta \left(\|\sigma_h (I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & \leq \delta \left(\|(I - \sigma_h)^r (I + \sigma_h) F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right) \\ & \quad + \delta \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \delta \|(I - \sigma_h)^r (I + \sigma_h) F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + 2\delta \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{2.6}$$

If we repeat r times the last inequality, then

$$\begin{aligned} \|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} & \leq \delta \|(I - \sigma_h)^r (I + \sigma_h) F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + 2\delta \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \delta^2 \|(I - \sigma_h)^r (I + \sigma_h)^2 F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \quad + 2\delta^2 \|(I - \sigma_h)^{r+1} (I + \sigma_h) F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + 2\delta \|(I - \sigma_h)^{r+1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \leq \dots \leq \delta^r \|(I - \sigma_h)^r (I + \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & \quad + 2 \sum_{k=1}^r \delta^k \|(I - \sigma_h)^{r+1} (I + \sigma_h)^{k-1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ & = \delta^r \|(I - \sigma_h^2)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + 2 \sum_{k=1}^r \delta^k \|(I - \sigma_h)^{r+1} (I + \sigma_h)^{k-1} F\|_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned}$$

Hence, the last inequality gives

$$\|(I - \sigma_h)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \delta^r \|(I - \sigma_h^2)^r F\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C \left(r, A_{w,\vartheta}^{p,q(\cdot)} \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} \tag{2.10}$$

for $0 < h \leq 1/n$. From (2.10) and applying

$$\|F\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \|F\|_{C(\mathbb{T})},$$

we get

$$\begin{aligned}
 & \| (I - \sigma_h)^r F \|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \\
 & \lesssim \delta^r \left\| (I - \sigma_h^2)^r F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C \left(r, A_{w,\vartheta}^{p,q(\cdot)} \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} \\
 & \lesssim \delta^{2r} \left\| (I - \sigma_h^4)^r F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + (\delta^r + 1) C \left(r, A_{w,\vartheta}^{p,q(\cdot)} \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} \\
 & \lesssim \dots \lesssim \delta^{mr} \left\| (I - \sigma_h^{2^m})^r F \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C \left(r, A_{w,\vartheta}^{p,q(\cdot)}, m \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} \\
 & \lesssim \delta^{mr} \left\| (I - \sigma_h^{2^m})^r F \right\|_{C(\mathbb{T})} + C \left(r, A_{w,\vartheta}^{p,q(\cdot)}, m \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} \\
 & \lesssim \delta^{mr} \|F\|_{C(\mathbb{T})} + C \left(r, A_{w,\vartheta}^{p,q(\cdot)}, m \right) \Omega_{r+1} (F, h)_{A_{w,\vartheta}^{p,q(\cdot)}} .
 \end{aligned} \tag{2.11}$$

Taking the supremum, from the last inequality we obtain

$$\Omega_r (F, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \delta^{mr} \|F\|_{C(\mathbb{T})} + \Omega_{r+1} (F, t)_{A_{w,\vartheta}^{p,q(\cdot)}} ,$$

since $\left\| (I - \sigma_h^{2^m})^r F \right\|_{C(\mathbb{T})} \leq 2^r \|F\|_{C(\mathbb{T})}$.

(b) Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$. For any $\varepsilon > 0$, there is $F \in C(\mathbb{T})$ such that $\|f - F\|_{C(\mathbb{T})} < \varepsilon$. From (a) and density of $C(\mathbb{T})$ in $A_{w,\vartheta}^{p,q(\cdot)}$ we get

$$\Omega_r (f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1} (f, t)_{A_{w,\vartheta}^{p,q(\cdot)}} .$$

□

Lemma 2.3 ([27, p. 344]) *Let σ^* be the Fejer operator, $M(x)$ be a maximal function, and $f \in L^1$. Then there exists an absolute constant c such that*

$$\sigma^*(x, f) \leq cM(x).$$

Lemma 2.4 *Let $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$. If $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, then for any trigonometric polynomial T_n of degree n the following inequality holds:*

$$\|T'_n\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq cn \|T_n\|_{A_{w,\vartheta}^{p,q(\cdot)}} ,$$

with a constant c independent of n .

Proof It is known that $|T'_n(x)| \leq 2n\sigma^*(|T'_n(x)|)$.

From the last inequality, Lemma 2.3, and [26, Theorem 10] we can finish the proof of lemma. □

3. Proofs of main results

Proof of Theorem 1.1. The case $r = 1$. Let $n \in \mathbb{N}$ and $f \in A_{w,\vartheta}^{p,q(\cdot)}$ be fixed. We will use the operator $L_n f$. We know that the relations $L_w^p \subset L^1$ and $L_\vartheta^{q(\cdot)} \subset L^1$ hold, so $A_{w,\vartheta}^{p,q(\cdot)} \subset L^1$. Let $S_n(f)$ be the n th partial sum of the Fourier series of f . Using (2.2), (2.5), and the uniformly boundedness (in n) of the operator $f \rightarrow S_n$ on $A_{w,\vartheta}^{p,q(\cdot)}$ ([3, 17]), we get

$$\begin{aligned} E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} &= E_n(f - L_n f + L_n f)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq E_n(f - L_n f)_{A_{w,\vartheta}^{p,q(\cdot)}} + E_n(L_n f)_{A_{w,\vartheta}^{p,q(\cdot)}} \\ &\lesssim \|f - L_n f\|_{A_{w,\vartheta}^{p,q(\cdot)}} + n^{-2} \left\| \frac{d^2}{dx^2} L_n f(x) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_1 \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{3.1}$$

The case $r \geq 2$. We will use induction on r as in [12]. We obtained above that the estimate (1.2) holds for $r = 1$. We suppose that the inequality (1.2) holds for some $r = 2, 3, 4, \dots$. We will show that the inequality (1.2) holds for $r + 1$.

We set $g(\cdot) := f(\cdot) - S_n f(\cdot)$. Then

$$S_n(g) = S_n(f - S_n(f)) = S_n(f) - S_n(S_n(f)) = S_n(f) - S_n(f) = 0$$

and

$$\|S_n f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \|S_n f\|_{p,w} + \|S_n f\|_{q(\cdot),\vartheta} \lesssim \|f\|_{p,w} + \|f\|_{q(\cdot),\vartheta} \lesssim \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

Using the induction hypothesis, [17], and [3],

$$\|g\|_{A_{w,\vartheta}^{p,q(\cdot)}} = \|g - S_n(g)\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C E_n(g)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C \Omega_r \left(g, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

From Lemma 2.2 we have that

$$\Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \tag{2.13}$$

for any $f \in A_{w,\vartheta}^{p,q(\cdot)}$.

Indeed: If $\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} = 1$, then

$$\begin{aligned} \Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} &\leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \\ &= C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C_2 \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}. \end{aligned} \tag{3.2}$$

If $\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} = 0$, then

$$\Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} = C_1 \delta^{mr} \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

In the case of $\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} > 0$ and $\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} \neq 1$, let $f^* = \frac{f}{\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}}$. Then $\|f^*\|_{A_{w,\vartheta}^{p,q(\cdot)}} = 1$. Thus, from (3.2) we get

$$\Omega_r \left(f^*, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} \|f^*\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C_2 \Omega_{r+1} \left(f^*, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

Hence,

$$\frac{1}{\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}} \Omega_r \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} \frac{1}{\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}} \|f\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \frac{C_2}{\|f\|_{A_{w,\vartheta}^{p,q(\cdot)}}} \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

and (2.13) follows.

If we choose m so big that $\mathbf{C}C_1\delta^{mr} \|F\|_{C(\mathbb{T})} < 1/2$, we obtain

$$\|g\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \mathbf{C}\Omega_r \left(g, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \mathbf{C}C_1\delta^{mr} \|F\|_{C(\mathbb{T})} \|g\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \mathbf{C}C_2\Omega_{r+1} \left(g, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

Then

$$\|g\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_{r+1} \left(g, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

From [17] and [3] and using

$$\Omega_r \left(S_n f; \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C\Omega_r \left(f; \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

we have

$$\Omega_{r+1} \left(f - S_n f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

Consequently,

$$E_n(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \|f - S_n(f)\|_{A_{w,\vartheta}^{p,q(\cdot)}} = \|g\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \Omega_{r+1} \left(f, \frac{1}{n} \right)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

This completes the proof. □

Proof of Theorem 1.2. Using (2.3) and the equalities [3]

$$\begin{aligned} \frac{d^2}{dx^2} ((I - \sigma_h)g)(x) &= (I - \sigma_h) \frac{d^2}{dx^2} g(x) \\ (I - \sigma_h)^{m+n} f(x) &= (I - \sigma_h)^m (I - \sigma_h)^n f(x), \quad m, n \in \mathbb{N} \end{aligned}$$

we obtain

$$\Omega_r(g, t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \delta^{2r} \left\| g^{(2r)} \right\|_{A_{w,\vartheta}^{p,q(\cdot)}}, \quad r \in \mathbb{N}, \tag{2.15}$$

for $g^{(2r)} \in A_{w,\vartheta}^{p,q(\cdot)}$ and $\delta > 0$. On the other hand, for any $m \in \mathbb{N}$

$$\Omega_r(f, \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq \Omega_r(f - S_{2^{m+1}}(f), \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} + \Omega_r(S_{2^{m+1}}(f), \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \tag{2.16}$$

and

$$\Omega_r (f - S_{2^{m+1}} (f), \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \|f - S_{2^{m+1}} (f)\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim E_{2^{m+1}}(f)_{A_{w,\vartheta}^{p,q(\cdot)}}. \tag{2.17}$$

Then by (2.15) and the weighted version of Bernstein’s inequality in $A_{w,\vartheta}^{p,q(\cdot)}$ [28, Lemma 2.2],

$$\begin{aligned} \Omega_r (S_{2^{m+1}} (f), \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} &\lesssim \delta^{2r} \left\| S_{2^{m+1}}^{(2r)} (f) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \\ &\lesssim \delta^{2r} \left\{ \left\| S_1^{(2r)} (f) - S_0^{(2r)} (f) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \sum_{i=1}^m \left\| S_{2^{i+1}}^{(2r)} (f) - S_{2^i}^{(2r)} (f) \right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \right\} \\ &\lesssim \delta^{2r} \left\{ E_0(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + \sum_{i=1}^m 2^{(i+1)2r} E_{2^i}(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \right\} \\ &\lesssim \delta^{2r} \left\{ E_0(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + 2^{2r} E_1(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + \sum_{i=1}^m 2^{(i+1)2r} E_{2^i}(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \right\}. \end{aligned}$$

Applying here the inequality and choosing m as $2^m \leq n < 2^{m+1}$, from (2.16) – (2.19), we obtain the following result:

$$2^{(i+1)2r} E_{2^i}(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \sum_{k=2^{i-1}+1}^{2^m} k^{2r-1} E_k(f)_{A_{w,\vartheta}^{p,q(\cdot)}}, \quad i \geq 1, \tag{2.18}$$

and we obtain

$$\begin{aligned} \Omega_r (S_{2^{m+1}} (f), \delta)_{A_{w,\vartheta}^{p,q(\cdot)}} &\lesssim \delta^{2r} \left\{ E_0(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + 2^{2r} E_1(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + \sum_{k=2}^{2^m} k^{2r-1} E_k(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \right\} \\ &\lesssim \delta^{2r} \left\{ E_0(f)_{A_{w,\vartheta}^{p,q(\cdot)}} + \sum_{k=1}^{2^m} k^{2r-1} E_k(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \right\}, \end{aligned} \tag{2.19}$$

using the estimate

$$E_{2^{m+1}}(f)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \frac{1}{n^{2r}} \sum_{k=2^{m-1}+1}^{2^m} k^{2r-1} E_k(f)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

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