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## Free modules and crossed modules of $R$ -algebroids

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**Abstract:** In this paper, first, we construct the free modules and precrossed modules of  $R$ -algebroids. Then we introduce the Peiffer ideal of a precrossed module and use it to construct the free crossed module.

**Key words:** R-category, R-algebroid, crossed modules, free modules

### 1. Introduction

Crossed modules, algebraic models of two types, were first invented by Whitehead [23, 24] in his study on homotopy groups and have been studied by many mathematicians. Various studies on crossed modules over groups and groupoids can be found in papers and books such as [7, 8, 21], and those over algebras in [4, 5, 19, 20, 22] and in [11, 13, 14] in different names. Kassel and Loday [12] studied crossed modules of Lie algebras and higher dimensional analogues were proposed by Ellis [10] for use in homotopical and homological algebras. Mosa [18] studied crossed modules of  $R$ -algebroids and double algebroids. Pullback and pushout crossed modules of algebroids can be found in [1] and [2], respectively. Provided that  $P$  is a group and  $K$  is a set, the construction of the free  $P$ -group on  $K$  and the constructions of the free precrossed and crossed modules on a function  $\omega : K \rightarrow P$  were handled in [7]. Shammu constructed the free crossed module on a function  $f : K \rightarrow A$  where, with our notations,  $K$  is a set and  $A$  is an  $R$ -algebra for a commutative ring  $R$  in [22].

The basic goal of this paper is to construct the free  $R$ -algebroid crossed module. For this goal, after giving some basic data in the second section, we define the category  $\text{Sets}_0/\text{Alg}(R)$  whose objects are all functions  $\omega : K \rightarrow A_0 \times A_0$ , where  $K$  is a set and  $A$  is an  $R$ -algebroid, and its subcategory  $\text{Sets}_0/(\text{Alg}(R)/A)$  formed by a fixed  $R$ -algebroid  $A$  in the third section. Then we construct the free  $R$ -algebroid  $A$ -module determined by an object  $\omega : K \rightarrow A_0 \times A_0$  of  $\text{Sets}_0/(\text{Alg}(R)/A)$  in the same section. In the fourth section we define the category  $\text{Sets}/\text{Alg}(R)$ , whose objects are formed by all functions of the form  $\omega : K \rightarrow A$  where  $K$  is a set and  $A$  is an  $R$ -algebroid and its subcategory  $\text{Sets}/(\text{Alg}(R)/A)$ , for a fixed  $R$ -algebroid  $A$ . Then, in the same section, we construct the free  $R$ -algebroid precrossed  $A$ -module determined by an object of  $\text{Sets}/(\text{Alg}(R)/A)$ .

In Section 5, we introduce the Peiffer ideal for an  $R$ -algebroid precrossed module to construct a crossed module and this procedure gives us the functor  $(-)^{\text{ct}}$  from the category of precrossed to the category of crossed modules of  $R$ -algebroids.

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In the last section, we construct the free  $R$ -algebroid crossed  $A$ -module determined by an object  $\omega : K \rightarrow A$  of  $\text{Sets}/(\text{Alg}(R)/A)$ , from the corresponding precrossed module, using the functor  $(-)^{\text{cr}}$ .

**2. Preliminaries**

$R$ -algebroids were especially studied by Mitchell [15–17] and by Amgott [3]. Mitchell gave a categorical definition of  $R$ -algebroids and obtained some interesting results. Mosa defined crossed modules of  $R$ -algebroids and proved the equivalence of crossed modules of algebroids and special double algebroids with connections in [18]. Alp constructed the pullback and pushout crossed modules of algebroids in [1] and [2], respectively. In this section, we give some basic definitions concerning crossed modules of  $R$ -algebroids.

**Definition 1** [15–17]. *Let  $R$  be a commutative ring. A category of which each homset has an  $R$ -module structure and of which composition is  $R$ -bilinear is called an ‘ $R$ -category’. A small  $R$ -category is called an ‘ $R$ -algebroid’. Moreover, if we omit the axiom of the existence of identities from an  $R$ -algebroid structure then the remaining structure is called a ‘pre- $R$ -algebroid’.*

A pre- $R$ -algebroid  $A$  comes with an object set  $Ob(A) = A_0$ , a morphism set  $Mor(A)$ , and two functions  $s, t : Mor(A) \rightarrow Ob(A)$ , the source and target functions respectively, such that if  $sa = x$  and  $ta = y$  then we say that ‘ $a$  is from  $x$  to  $y$ ’ and write  $a \in A(x, y)$  where  $A(x, y)$  is a homset, the set of all morphisms of  $A$  from  $x$  to  $y$ . From the definition,  $A(x, y)$  is an  $R$ -module for all  $x, y \in A_0$ . Moreover, we say that  $A$  is over  $A_0$ .

**Definition 2** [15–17]. *An  $R$ -linear functor between two  $R$ -categories is called an ‘ $R$ -functor’ and an  $R$ -functor between two  $R$ -algebroids is called an ‘ $R$ -algebroid morphism’. Moreover, an assignment between two pre- $R$ -algebroids satisfying all axioms of an  $R$ -functor except for the identity preservation axiom is called a ‘pre- $R$ -algebroid morphism’.*

All  $R$ -algebroids and their morphisms form the category  $\text{Alg}(R)$ .

**Remark 3** *Throughout this paper, for a (pre-)  $R$ -algebroid  $A$ ,  $a \in A$  will mean that  $a$  is a morphism of  $A$ . Moreover, if  $a, a' \in A$  with  $ta = sa'$  then their composition will be denoted by  $aa'$ .*

**Definition 4** [18]. *Let  $A$  be a pre- $R$ -algebroid and*

$$I = \{I(x, y) \subseteq A(x, y) : x, y \in A_0\}$$

*be a family of  $R$ -submodules of  $A$ . For all  $w, x, y, z \in A_0$ ,  $a' \in A(w, x)$ ,  $a'' \in A(y, z)$  and  $a \in I(x, y)$  if  $a'a \in I(w, y)$ , and  $aa'' \in I(x, z)$  then  $I$  is said to be a ‘two-sided ideal’ of  $A$ .*

**Definition 5** [18]. *Let  $A$  be an  $R$ -algebroid and  $M$  be a pre- $R$ -algebroid with the same object set  $A_0$ . A family of maps defined for all  $x, y, z \in A_0$  as*

$$\begin{array}{ccc} M(x, y) \times A(y, z) & \longrightarrow & M(x, z) \\ (m, a) & \longmapsto & m^a \end{array}$$

*is called a ‘right action’ of  $A$  on  $M$ , if the conditions*

1.  $(m^a)^{a'} = m^{aa'}$
2.  $m^{a_1+a_2} = m^{a_1} + m^{a_2}$
3.  $(m'm)^a = m'm^a$
4.  $(m_1 + m_2)^a = m_1^a + m_2^a$
5.  $(r \cdot m)^a = r \cdot m^a = m^{r \cdot a}$
6.  $m^{1 \cdot m} = m$

*are satisfied for all  $r \in R$ ,  $a, a', a_1, a_2 \in A$ ,  $m, m', m_1, m_2 \in M$  with compatible sources and targets.*

A ‘left action’ of  $A$  on  $M$  can be defined in a similar way.

If  $A$  has a right and a left action on  $M$  and if the condition

$$({}^a m)^{a'} = {}^a (m^{a'})$$

is satisfied for all  $m \in M$  and  $a, a' \in A$  with  $ta = sm, tm = sa'$  then  $A$  is said to have an ‘associative action’ on  $M$ .

**Definition 6** Let  $A$  be an  $R$ -algebroid and  $M$  be a pre- $R$ -algebroid with the same object set  $A_0$ . If  $A$  has an associative action on  $M$  then  $M$  is called an ‘ $A$ -module’. If  $M$  is an  $A$ -module we usually write  $(M, A)$  and call it an ‘ $R$ -algebroid module’ or an ‘ $R$ -algebroid  $A$ -module’. Moreover, for any two  $R$ -algebroid modules  $(M, A)$  and  $(N, B)$  a pair  $(f, g) : (M, A) \rightarrow (N, B)$  is called an  $R$ -algebroid module morphism if  $f : M \rightarrow N$  is a pre- $R$ -algebroid morphism,  $g : A \rightarrow B$  is an  $R$ -algebroid morphism and the conditions

1.  $fm \in N(g(sm), g(tm)),$
2.  $f({}^a m) = {}^{ga}(fm)$  and  $f(m^{a'}) = (fm)^{ga'}$

are satisfied for all  $m \in M, a, a' \in A$  with  $ta = sm, tm = sa'$ .

Thus, we get a category, denoted by  $\text{ModAlg}(R)$ , whose objects are all  $R$ -algebroid modules and morphisms are all  $R$ -algebroid module morphisms. Furthermore, all  $R$ -algebroid  $A$ -modules with the identity morphism  $I_A$  on  $A$  form a subcategory  $\text{ModAlg}(R)/A$  of  $\text{ModAlg}(R)$ .

**Definition 7** [18]. Let  $A$  be an  $R$ -algebroid and  $M$  be a pre- $R$ -algebroid with the same set of objects  $A_0$  and let  $A$  have an associative action on  $M$ . A pre- $R$ -algebroid morphism  $\mu : M \rightarrow A$  is called an ‘ $R$ -algebroid precrossed module’ or an ‘ $R$ -algebroid precrossed  $A$ -module’ if the condition

$$\text{CM1) } \mu({}^a m) = a(\mu m) \text{ and } \mu(m^{a'}) = (\mu m)a'$$

is satisfied, and  $\mu : M \rightarrow A$  is called an ‘ $R$ -algebroid crossed module’ or an ‘ $R$ -algebroid crossed  $A$ -module’ if a second condition,

$$\text{CM2) } m^{\mu m'} = mm' = \mu^m m',$$

is satisfied, for all  $a, a' \in A$  and  $m, m' \in M$  with  $ta = sm, tm = sa' = sm'$ . Thus, a crossed module is a precrossed module satisfying CM2.

Let  $\mathcal{M} = (\mu : M \rightarrow A)$  and  $\mathcal{N} = (\eta : N \rightarrow B)$  be two (pre)crossed modules of  $R$ -algebroids and let  $f : M \rightarrow N$  be a pre- $R$ -algebroid morphism and  $g : A \rightarrow B$  be an  $R$ -algebroid morphism. The pair  $(f, g) : \mathcal{M} \rightarrow \mathcal{N}$  is called a (pre)crossed module morphism if the conditions

1.  $f({}^a m) = {}^{ga}(fm)$  and  $f(m^{a'}) = (fm)^{ga'}$
2.  $(\eta f)(m) = (g\mu)(m)$

are satisfied, for all  $a, a' \in A$  and  $m \in M$  with  $ta = sm, tm = sa'$ . The meaning of the second condition is that the diagram in Figure 1 is commutative.

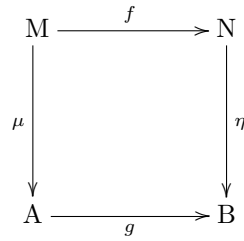


Figure 1

Note, also, that if  $\mu : M \rightarrow A$  is a (pre)crossed module then  $M$  is an  $A$ -module and a (pre)crossed module morphism is a module morphism satisfying the second condition.

Thus, all  $R$ -algebroid precrossed modules and their morphisms form a category denoted by  $PXAlg(R)$ . Moreover, all  $R$ -algebroid precrossed  $A$ -modules with the identity morphism on  $A$  form a subcategory  $PXAlg(R)/A$  of  $PXAlg(R)$ . Similarly, all  $R$ -algebroid crossed modules form the category  $XAlg(R)$  and all  $R$ -algebroid crossed  $A$ -modules form the category  $XAlg(R)/A$ , which is a subcategory of  $XAlg(R)$ . Obviously,  $XAlg(R)$  is a full subcategory of  $PXAlg(R)$  and  $XAlg(R)/A$  is a full subcategory of  $PXAlg(R)/A$ .

**Example 8** [18]. If  $A$  is an  $R$ -algebroid and  $I$  is a two-sided ideal of  $A$ , then the inclusion morphism

$$i : I \rightarrow A$$

is a crossed module with the action of  $A$  on  $I$  defined by

$${}^a b = ab \quad \text{and} \quad b^{a'} = ba'$$

for all  $a, a' \in A, b \in I$  with  $ta = sb, tb = sa'$ .

### 3. Free $R$ -algebroid modules

Clearly an  $R$ -algebroid module  $(M, A)$  comes with a function  $\xi_M : Mor(M) \rightarrow A_0 \times A_0$  defined as  $\xi_M m = (sm, tm)$  for all  $m \in M$ . This motivates us to form a category,  $Sets_0/Alg(R)$ , whose objects are all functions  $\omega : K \rightarrow A_0 \times A_0$  defined as  $\omega k = (\omega_1 k, \omega_2 k)$ , where  $K$  is a set and  $A$  is an  $R$ -algebroid, and whose morphisms are all pairs  $(f, g_0 \times g_0) : \omega \rightarrow \omega'$  where if  $\omega' : K' \rightarrow B_0 \times B_0$  then  $f : K \rightarrow K'$  is a function,  $g : A \rightarrow B$  is an  $R$ -algebroid morphism,  $g_0$  is the restriction of  $g$  on  $A_0$ , and  $g_0 \times g_0 : A_0 \times A_0 \rightarrow B_0 \times B_0$  is defined as  $(g_0 \times g_0)(x, y) = (g_0 x, g_0 y)$  for all  $x, y \in A_0$ , making the diagram in Figure 2 commutative.

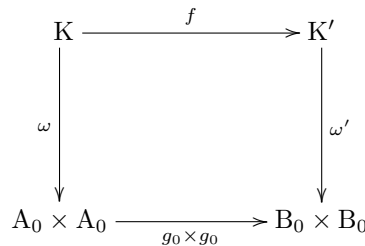


Figure 2

By fixing the  $R$ -algebroid  $A$  and taking  $g_0 \times g_0$  as  $I_{A_0 \times A_0}$ , the identity function on  $A_0 \times A_0$ , we obtain a subcategory  $Sets_0/(Alg(R)/A)$  of  $Sets_0/Alg(R)$ .

Note that, for each  $R$ -algebroid module  $(M, A)$ , the function  $\xi_M$  is an object of  $Sets_0/(Alg(R)/A)$ .

**Proposition 9** For any object  $\omega : K \rightarrow A_0 \times A_0$  of  $\text{Sets}_0 / (\text{Alg}(R)/A)$  there exists an  $R$ -algebroid  $A$ -module  $(F(\omega), A)$  and a morphism  $(i_m, I_{A_0 \times A_0}) : \omega \rightarrow \xi_{F(\omega)}$  such that for all  $R$ -algebroid  $A$ -modules  $(N, A)$  and for all morphisms  $(f, I_{A_0 \times A_0}) : \omega \rightarrow \xi_N$  there exists a unique  $A$ -module morphism  $(\alpha, I_A) : (F(\omega), A) \rightarrow (N, A)$  satisfying  $f = \alpha i_m$ , which means that the diagram in Figure 3 is commutative.

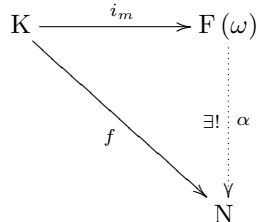


Figure 3

$(F(\omega), A)$ , with the morphism  $(i_m, I_{A_0 \times A_0})$ , is called the free  $R$ -algebroid  $A$ -module determined by  $\omega$ . The free module is unique up to isomorphism.

**Proof** Provided that  $n \in \mathbb{N}^+$ ,  $k, k_1, \dots, k_n \in K$  and  $a, a_1, \dots, a_n, a', a'_1, \dots, a'_n \in A$ , consider all elements of the form  $aka'$  under the conditions  $ta = \omega_1 k$  and  $sa' = \omega_2 k$ , and tying such elements construct all words of the form  $a_1 k_1 a'_1 a_2 k_2 a'_2 \dots a_n k_n a'_n$  under the conditions  $ta'_1 = sa_2, \dots, ta'_{n-1} = sa_n$ . For any word  $p_i = a_{i_1} k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} a'_{i_n}$  define its source as  $sp_i = sa_{i_1}$  and its target as  $tp_i = ta'_{i_n}$ , and for all  $x, y \in A_0$  denote the free additive abelian group generated by all words with source  $x$  and target  $y$  by  $G(\omega)(x, y)$ . Obviously, each element of  $G(\omega)(x, y)$  is of the form  $\sum_i p_i$  where  $p_i$ s are words with source  $x$  and target  $y$ .

Now we consider the normal subgroup  $N(x, y)$  of  $G(\omega)(x, y)$  generated by all elements of forms

$$\begin{aligned}
 & a_1 k_1 a'_1 \dots (a_i + a'_i) k_i a'_i \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots a_i k_i a'_i \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots a'_i k_i a'_i \dots a_n k_n a'_n \\
 & a_1 k_1 a'_1 \dots a_i k_i (a'_i + a''_i) \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots a_i k_i a'_i \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots a_i k_i a''_i \dots a_n k_n a'_n \\
 & (r \cdot a_1) k_1 a'_1 \dots a_i k_i a'_i \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots (r \cdot a_i) k_i a'_i \dots a_n k_n a'_n \\
 & (r \cdot a_1) k_1 a'_1 \dots a_i k_i a'_i \dots a_n k_n a'_n - a_1 k_1 a'_1 \dots a_i k_i (r \cdot a'_i) \dots a_n k_n a'_n
 \end{aligned}$$

for all  $r \in R$ . If we divide  $G(\omega)(x, y)$  by  $N(x, y)$  then we get an abelian quotient group  $[G(\omega)(x, y)]$  of which elements are cosets of  $N(x, y)$ . We denote  $[G(\omega)(x, y)]$  with  $F(\omega)(x, y)$ , and the cosets  $p_i + N(x, y)$

and  $\sum_i p_i + N(x, y)$  with  $[p_i]$  and  $\left[ \sum_i p_i \right]$ , respectively, for all  $p_i, \sum_i p_i \in G(\omega)(x, y)$ . It is obvious that  $\left[ \sum_i p_i \right] = \sum_i [p_i]$ .

Now we can define an  $R$ -action on  $F(\omega)(x, y)$  as  $r \cdot [p_i] = [(r \cdot a_{i_1}) k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} a'_{i_n}]$  and  $r \cdot \left( \sum_i [p_i] \right) = \sum_i [r \cdot p_i]$  for all  $r \in R$ , and with this action the quotient group  $F(\omega)(x, y)$  is clearly an  $R$ -module.

Hence, the family  $F(\omega) = \{F(\omega)(x, y) : x, y \in A_0\}$  becomes a pre- $R$ -algebroid by the composition defined for all  $x, y, z \in A_0$  as

$$\begin{aligned}
 F(\omega)(x, y) \times F(\omega)(y, z) & \rightarrow F(\omega)(x, z) \\
 \left( \sum_i [p_i], \sum_j [p_j] \right) & \mapsto \left( \sum_i [p_i] \right) \left( \sum_j [p_j] \right) = \sum_{i,j} [p_i p_j] = \sum_i \sum_j [p_i p_j]
 \end{aligned}$$

where if  $p_i = a_{i_1} k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} a'_{i_n}$  and  $p_j = a_{j_1} k_{j_1} a'_{j_1} \dots a_{j_n} k_{j_n} a'_{j_n}$ , then  $p_i p_j = a_{i_1} k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} a'_{i_n} a_{j_1} k_{j_1} a'_{j_1} \dots a_{j_n} k_{j_n} a'_{j_n}$ .

Moreover, an associative  $A$ -action on  $F(\omega)$  can be defined as  ${}^a \left( \sum_i [p_i] \right) = \sum_i [{}^a p_i]$  and  $\left( \sum_i [p_i] \right)^{a'} = \sum_i [p_i^{a'}]$  where  ${}^a p_i = (a a_{i_1}) k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} a'_{i_n}$  and  $p_i^{a'} = a_{i_1} k_{i_1} a'_{i_1} \dots a_{i_n} k_{i_n} (a'_{i_n} a')$  under the condition  $ta = sp_i$ ,  $tp_i = sa'$ , and this action makes  $F(\omega)$  an  $A$ -module.

Define  $i_m : K \rightarrow F(\omega)$  as  $i_m(k) = [1k1]$  ( $= [1_{\omega_1} k 1_{\omega_2} k]$ ) and  $\alpha : F(\omega) \rightarrow N$  as  $\alpha[aka'] = {}^a (fk)^{a'}$ ,  $\alpha[p_i] = \alpha[a_{i_1} k_{i_1} a'_{i_1}] \dots \alpha[a_{i_n} k_{i_n} a'_{i_n}]$  and  $\alpha \left( \sum_i [p_i] \right) = \sum_i \alpha[p_i]$  for all  $(f, I_{A_0 \times A_0}) : \omega \rightarrow \xi_N$ . It can easily be shown that  $(i_m, I_{A_0 \times A_0})$  is a morphism from  $\omega$  to  $\xi_{F(\omega)}$  and  $(\alpha, I_A)$  is an  $A$ -module morphism from  $(F(\omega), A)$  to  $(N, A)$  satisfying  $f = \alpha i_m$ . Obviously,  $\alpha$  is unique from its definition. Moreover, it can be shown that  $(F(\omega), A)$  with the morphism  $(i_m, I_{A_0 \times A_0})$  is unique up to isomorphism.  $\square$

The construction of the free module gives a functor  $F$  from  $\text{Sets}_0 / (\text{Alg}(R)/A)$  to  $\text{ModAlg}(R)/A$  defined as  $F(\omega) = (F(\omega), A)$  on objects and as  $F(f, I_{A_0 \times A_0}) = (Ff, I_A)$  on morphisms such that  $Ff([aka']) = [a(fk)a']$  on generators.

**Proposition 10** *The functor  $F$  is the left adjoint of the forgetful functor  $U : \text{ModAlg}(R)/A \rightarrow \text{Sets}_0 / (\text{Alg}(R)/A)$ , which is defined as  $U(N, A) = \xi_N$  for each  $R$ -algebroid module  $(N, A)$  and is defined as  $U(g, I_A) = (Ug, I_{A_0 \times A_0})$  on morphisms such that  $(Ug)(n) = gn$  for all  $n \in N$ .*

**Proof** We must find a natural equivalence

$$\Phi : (\text{ModAlg}(R)/A)(F(-), (-)) \cong (\text{Sets}_0 / (\text{Alg}(R)/A))(-, U(-)),$$

which is required to give a map

$$\begin{aligned} \Phi : \text{Ob}(\text{Sets}_0 / (\text{Alg}(R)/A)) \times \text{Ob}(\text{ModAlg}(R)/A) &\rightarrow \text{Sets} \\ (\omega : K \rightarrow A_0 \times A_0, (N, A)) &\mapsto \Phi(\omega, (N, A)) \end{aligned}$$

such that  $\Phi(\omega, (N, A))$  is a bijection from  $(\text{ModAlg}(R)/A)(F(\omega), (N, A))$  to  $(\text{Sets}_0 / (\text{Alg}(R)/A))(\omega, U(N, A) = \xi_N)$  and is natural in both  $\omega$  and  $(N, A)$  for all  $\omega \in \text{Ob}(\text{Sets}_0 / (\text{Alg}(R)/A))$  and  $(N, A) \in \text{Ob}(\text{ModAlg}(R)/A)$ .

We abbreviate  $\Phi(\omega, (N, A))$  as  $\Phi(\omega, A)$  and define  $\Phi(\omega, N)$  as  $\Phi(\omega, N)(f, I_A) = (\Phi(\omega, N)(f), I_{A_0 \times A_0})$  such that

$$\begin{aligned} \Phi(\omega, N)(f) : K &\rightarrow N \\ k &\mapsto \Phi(\omega, N)(f)(k) = f[1k1] \end{aligned}$$

for all  $(f, I_A) \in (\text{ModAlg}(R)/A)((F(\omega), A), (N, A))$  where  $\omega : K \rightarrow A_0 \times A_0$ . Clearly,  $\Phi(\omega, N)$  is well defined and 1-1. It is also onto since each morphism

$$(h, I_{A_0 \times A_0}) : (\omega : K \rightarrow A_0 \times A_0) \rightarrow (\xi_N : N \rightarrow A_0 \times A_0)$$

is the image of the morphism  $(f, I_A)$  under  $\Phi(\omega, N)$ , where  $f : F(\omega) \rightarrow N$  is defined as  $f[aka'] = {}^a (hk)^{a'}$  on generators.

Moreover, provided that  $(-)^{\bullet}$  is a composition with  $(-)$  from right, for all  $(g, I_{A_0 \times A_0}) : \omega \rightarrow \omega'$ ,  $(f, I_A) : ((F(\omega'), A) \rightarrow (N, A))$  and  $k \in K$

$$\begin{aligned} (\Phi(\omega, N) (Fg)^{\bullet}) (f) (k) &= (\Phi(\omega, N) (Fg)^{\bullet} (f)) (k) = (\Phi(\omega, N) (f (Fg))) (k) \\ &= (f (Fg)) [1k1] = f [1 (gk) 1] \\ &= (\Phi(\omega', N) (f)) (gk) = ((\Phi(\omega', N) (f)) g) (k) \\ &= (g^{\bullet} (\Phi(\omega', N) (f))) (k) = (g^{\bullet} \Phi(\omega', N)) (f) (k), \end{aligned}$$

i.e. the diagram in Figure 4 is commutative and  $\Phi(\omega, N)$  is natural in  $\omega$ .

$$\begin{array}{ccc} (\text{ModAlg}(R)/A)((F(\omega), A), (N, A)) & \xrightarrow{\Phi(\omega, N)} & (\text{Sets}_0/(\text{Alg}(R)/A))(\omega, \xi_N) \\ \uparrow (Fg)^{\bullet} & & \uparrow g^{\bullet} \\ (\text{ModAlg}(R)/A)((F(\omega'), A), (N, A)) & \xrightarrow{\Phi(\omega', N)} & (\text{Sets}_0/(\text{Alg}(R)/A))(\omega', \xi_N) \end{array}$$

Figure 4

A similar calculation shows that the diagram in Figure 5 is commutative for each  $(g, I_A) \in (\text{ModAlg}(R)/A)((N, A), (N', A))$ , where  $(-)^{\bullet}$  is composition with  $(-)$  from left, and  $\Phi(\omega, N)$  is natural in  $(N, A)$ .

$$\begin{array}{ccc} (\text{ModAlg}(R)/A)((F(\omega), A), (N, A)) & \xrightarrow{\Phi(\omega, N)} & (\text{Sets}_0/(\text{Alg}(R)/A))(\omega, \xi_N) \\ \downarrow g^{\bullet} & & \downarrow (Ug)^{\bullet} \\ (\text{ModAlg}(R)/A)((F(\omega), A), (N', A)) & \xrightarrow{\Phi(\omega, N')} & (\text{Sets}_0/(\text{Alg}(R)/A))(\omega, \xi_{N'}) \end{array}$$

Figure 5

□

#### 4. Free $R$ -algebroid precrossed modules

The fact that if  $\eta : N \rightarrow A$  is a (pre)crossed module then there is a restricted function  $\eta_m : \text{Mor}(N) \rightarrow A$  as  $\eta_m(n) = \eta n$  motivates us to form a category  $\text{Sets}/\text{Alg}(R)$  whose objects are all functions  $\omega : K \rightarrow A$  where  $K$  is a set and  $A$  is an  $R$ -algebroid such that  $\omega k$  is a morphism of  $A$  for all  $k \in K$  and whose morphisms are all pairs  $(f, g) : \omega \rightarrow \omega'$  where if  $\omega' : K' \rightarrow B$  then  $f : K \rightarrow K'$  is a function and  $g : A \rightarrow B$  is an  $R$ -algebroid morphism making the diagram in Figure 6 commutative.

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ \omega \downarrow & & \downarrow \omega' \\ A & \xrightarrow{g} & B \end{array}$$

Figure 6

By fixing the  $R$ -algebroid  $A$  and taking  $g$  as  $I_A$ , we obtain a subcategory  $\text{Sets}/(\text{Alg}(R)/A)$  of  $\text{Sets}/\text{Alg}(R)$ .



Note that, for each precrossed or crossed  $A$ -module  $\mathcal{N} = (\eta : N \rightarrow A)$ , the function  $\eta_m : Mor(N) \rightarrow A$  is an object of  $Sets/Alg(R)$ .

**Proposition 11** *For any object  $\omega : K \rightarrow A$  of  $Sets/(Alg(R)/A)$  there exists an  $R$ -algebroid precrossed  $A$ -module  $F_P(\omega) = (\omega_P : F_P(\omega) \rightarrow A)$  and a morphism  $(i_p, I_A) : \omega \rightarrow \omega_{F_P}$  such that for all  $R$ -algebroid precrossed  $A$ -modules  $\mathcal{N} = (\eta : N \rightarrow A)$  and for all morphisms  $(f, I_A) : \omega \rightarrow \eta_m$  there exists a unique precrossed  $A$ -module morphism  $(\alpha, I_A) : F_P(\omega) \rightarrow \mathcal{N}$  satisfying  $f = \alpha i_p$ , which means the diagram in Figure 7 is commutative.*

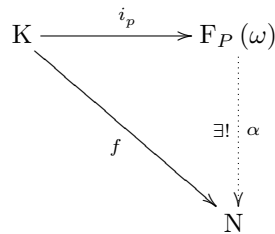


Figure 7

$F_P(\omega)$ , with the morphism  $(i_p, I_A)$ , is called the free  $R$ -algebroid precrossed  $A$ -module determined by  $\omega$ . The free precrossed module is unique up to isomorphism.

**Proof**  $\omega$  determines a function  $\omega_{A_0} : K \rightarrow A_0 \times A_0$  as  $\omega_{A_0}(k) = (s(\omega k), t(\omega k))$  and from the previous section there exists a free  $R$ -algebroid  $A$ -module  $F(\omega_{A_0})$  determined by  $\omega_{A_0}$ , with an  $A$ -action defined as  $a'' [aka'] = [(a''a)ka']$  and  $[aka']^{a'''} = [ak(a'a''')]$  on generators with  $ta'' = sa$  and  $ta' = sa'''$ . Now, taking  $F_P(\omega) = F(\omega_{A_0})$ , define  $\omega_P : F_P(\omega) \rightarrow A$  as  $\omega_P [aka'] = a(\omega k) a'$  on generators and  $i_p : K \rightarrow F_P(\omega)$  as  $i_p k = [1k1]$  for all  $k \in K$ . It can easily be checked that, by these definitions,  $F_P(\omega) = (\omega_P : F_P(\omega) \rightarrow A)$  is a precrossed module and  $(i_p, I_A)$  is a morphism from  $\omega$  to  $\omega_{F_P}$ .

Defining  $\alpha : F_P(\omega) \rightarrow N$  as  $\alpha [aka'] = {}^a(fk)a'$  on generators, since the rest are detail, completes the proof. □

As in the case of free modules, the construction of free precrossed module gives a functor  $F_P : Sets/(Alg(R)/A) \rightarrow P X Alg(R)/A$  defined as  $F_P(\omega) = (\omega_P : F_P(\omega) \rightarrow A)$  on objects and as  $F_P(f, I_A) = (F_P f, I_A)$  on morphisms such that  $F_P f [aka'] = [a(fk)a']$  on generators.

**Proposition 12** *The functor  $F_P$  is the left adjoint of the forgetful functor  $U : P X Alg(R)/A \rightarrow Sets/(Alg(R)/A)$ , which for a precrossed module  $\mathcal{N} = (\eta : N \rightarrow A)$  gives the function  $\eta_m$  and for a precrossed  $A$ -module morphism  $(f, I_A) : \mathcal{N} \rightarrow \mathcal{N}'$  gives the morphism  $U(f, I_A) = (Uf, I_A) : \eta_m \rightarrow \eta'_m$  such that  $Uf(n) = fn$  for all  $n \in N$ .*

**Proof** We omit the proof, since the constructions are almost the same as those in the proof of Proposition 10. □

**5. Peiffer ideal of a precrossed module**

Since our aim in the next section is to obtain the free  $R$ -algebroid crossed modules, in this section we construct the Peiffer ideal for a precrossed module of  $R$ -algebroids to get a crossed module. The term ‘Peiffer element’ was first used by Brown and Huebschmann [9], and Baus and Conduché [6] gave a substantial theory of Peiffer commutator calculus. Brown et al. used the Peiffer subgroup to obtain crossed modules of groups in [7] and Shammu used Peiffer commutators to get crossed modules of algebras in [22].

**Definition 13** Let  $\mathcal{M}=(\mu : M \longrightarrow A)$  be a precrossed module of  $R$ -algebroids and let  $m, m'$  be two morphisms of  $M$  satisfying the condition  $tm = sm'$ . The Peiffer commutators of  $m$  and  $m'$  are defined as  $[[m, m']_1 = m^{\mu m'} - mm'$  and  $[[m, m']_2 = {}^{\mu}m m' - mm'$ .

If  $\mathcal{M}$  is a crossed module then both of these commutators are zero. Conversely, a precrossed module in which all of these commutators are zero is a crossed module.

For all  $x, y \in A_0$ , we denote the subgroup of  $M(x, y)$  generated by  $[[M, M]]_g(x, y) = \{[[m, m']_1, [[m, m']_2 : m, m' \in M, x = sm, tm' = y\}$ , the set of all Peiffer commutators of  $M(x, y)$ , by  $[[M, M]](x, y)$ . Since  $M(x, y)$  is abelian,  $[[M, M]](x, y)$  is also abelian. By a direct calculation, it can be shown that  $r \cdot [[m, m']_1 = [[r \cdot m, m']_1 = [[m, r \cdot m']_1$  and  $r \cdot [[m, m']_2 = [[r \cdot m, m']_2 = [[m, r \cdot m']_2$  for all  $[[m, m']_1, [[m, m']_2 \in [[M, M]](x, y)$  and for all  $r \in R$ , which means  $[[M, M]](x, y)$  is closed under the action of  $R$ , and this results in that  $[[M, M]](x, y)$  is an  $R$ -module, an  $R$ -submodule of  $M(x, y)$ .

**Proposition 14** (i) The family  $[[M, M]] = \{[[M, M]](x, y) : x, y \in A_0\}$  is a two sided ideal of  $M$ .

(ii)  $[[M, M]]$  is closed under the action of  $A$ .

**Proof** For all  $w, x, y, z \in A_0, [[m, m']_1, [[m, m']_2 \in [[M, M]](x, y), m'' \in M(w, x), m''' \in M(y, z), a \in A(w, x),$  and  $a' \in A(y, z)$ , a direct calculation gives that

$$\begin{aligned}
 (i) \quad & m''[[m, m']_1 = [[m''m, m']_1 \in [[M, M]](w, y) \\
 & [[m, m']_1 m''' = [[m, m' m''']_1 - [[m^{\mu m'}, m''']_1 \in [[M, M]](x, z) \\
 & m''[[m, m']_2 = [[m''m, m']_2 - [[m'', {}^{\mu}m m']_2 \in [[M, M]](w, y) \\
 & [[m, m']_2 m''' = [[m, m' m''']_2 \in [[M, M]](x, z) \\
 (ii) \quad & {}^a [[m, m']_1 = [[{}^a m, m']_1 \in [[M, M]](w, y) \\
 & [[m, m']_1^{a'} = [[m, (m')^{a'}]_1 \in [[M, M]](x, z) \\
 & {}^a [[m, m']_2 = [[{}^a m, m']_2 \in [[M, M]](w, y) \\
 & [[m, m']_2^{a'} = [[m, (m')^{a'}]_2 \in [[M, M]](x, z).
 \end{aligned}$$

□

The ideal  $[[M, M]]$  is called the ‘Peiffer’ ideal of  $M$ .

Now construct the family

$$\frac{M}{[[M, M]]} = \left\{ \frac{M}{[[M, M]]}(x, y) = \frac{M(x, y)}{[[M, M]](x, y)} : x, y \in A_0 \right\}$$

of quotient  $R$ -modules. Clearly,  $\frac{M}{[[M, M]]}$  is a pre- $R$ -algebroid which is an  $A$ -module thanks to the addition, multiplication,  $R$ -action and associative  $A$ -action induced by those defined on  $M$ .

We write  $M^{cr}$  instead of  $\frac{M}{[[M, M]]}$  and  $\bar{m}$  instead of  $m + [[M, M]](x, y)$  for all  $m \in M(x, y)$ , to abbreviate.  $\mu$  induces a map

$$\begin{aligned} \mu^{cr} : M^{cr} &\longrightarrow A \\ \bar{m} &\longmapsto \mu^{cr}\bar{m} = \mu m \end{aligned}$$

since  $\mu$  maps  $[[M, M]]$  to  $0_A = \{0_{A(x,y)} : x, y \in A_0\}$ , where  $0_{A(x,y)}$  is the additive identity of  $A(x, y)$ .

**Proposition 15** (i) *If  $\mathcal{M} = (\mu : M \rightarrow A)$  is a precrossed module of  $R$ -algebroids, then  $\mathcal{M}^{cr} = (\mu^{cr} : M^{cr} \rightarrow A)$  is a crossed module.*

(ii) *Provided that  $\phi : M \rightarrow M^{cr}$  is the quotient morphism, for all crossed  $A$ -modules  $\mathcal{N} = (\eta : N \rightarrow A)$  and for all precrossed  $A$ -module morphisms  $(\alpha, I_A) : \mathcal{M} \rightarrow \mathcal{N}$ , there exists a unique crossed  $A$ -module morphism  $(\alpha', I_A) : \mathcal{M}^{cr} \rightarrow \mathcal{N}$  satisfying  $\alpha = \alpha' \phi$ .*

**Proof** (i) It can easily be shown that  $\mu^{cr}$  is a pre- $R$ -algebroid morphism. We show that it satisfies the crossed module conditions: For all  $m, m' \in M$  and for all  $a, a' \in A$  with  $ta = sm, tm = sm' = sa'$

$$\text{CM1) } \mu^{cr}(a\bar{m}) = \mu^{cr}(\overline{am}) = \mu(am) = a(\mu m) = a(\mu^{cr}\bar{m})$$

and similarly  $\mu^{cr}(\overline{m}a') = (\mu^{cr}\bar{m})a'$ ,

$$\begin{aligned} \text{CM2) } \overline{m}^{\mu^{cr}\bar{m}'} &= \overline{m}^{\mu m'} = \overline{m^{\mu m'}} = m^{\mu m'} + [[M, M]](sm, tm') \\ &= m^{\mu m'} + (-[[m, m']_1 + [[M, M]](sm, tm')) \\ &= m^{\mu m'} + (- (m^{\mu m'} - mm')) + [[M, M]](sm, tm') \\ &= mm' + [[M, M]](sm, tm') = \overline{mm'} = \overline{m}m' \end{aligned}$$

and similarly  $\mu^{cr}\overline{m}m' = \overline{m}m'$ .

(ii) Define  $\alpha' : M^{cr} \rightarrow N$  as  $\alpha'\bar{m} = \alpha m$ . Obviously,  $(\alpha', I_A)$  is a crossed  $A$ -module morphism and for all  $m \in M$

$$(\alpha' \phi)(m) = \alpha'(\phi m) = \alpha'\bar{m} = \alpha m.$$

The uniqueness of  $\alpha'$  comes from its definition. □

Thus, we get a functor  $(-)^{cr} : \text{PXAlg}(R) \rightarrow \text{XAlg}(R)$ , which gives a crossed module  $\mathcal{M}^{cr}$  for any precrossed module  $\mathcal{M}$  and is defined as  $(f, g)^{cr} = (f^{cr}, g)$  on morphisms where if  $(f, g) : \mathcal{M} \rightarrow \mathcal{M}'$  then  $(f^{cr}, g) : \mathcal{M}^{cr} \rightarrow \mathcal{M}'^{cr}$  such that  $f^{cr}\bar{m} = \overline{fm}$  for all  $m \in M$ .

**Proposition 16** *The functor  $(-)^{cr} : \text{PXAlg}(R) \rightarrow \text{XAlg}(R)$  is the left adjoint of the inclusion functor  $I_n : \text{XAlg}(R) \rightarrow \text{PXAlg}(R)$ .*

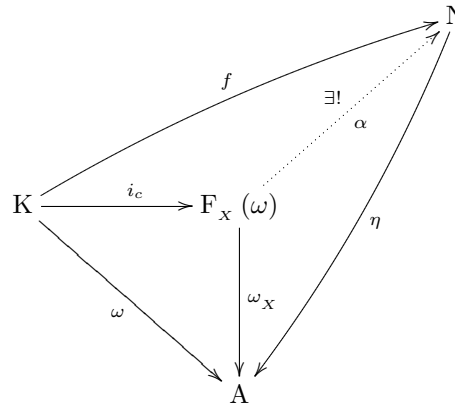
**Proof** For all  $\mathcal{M} \in \text{Ob}(\text{PXAlg}(R))$ ,  $\mathcal{N} \in \text{Ob}(\text{XAlg}(R))$  and crossed module morphisms  $g = (g_1, g_2) : \mathcal{M}^{cr} \rightarrow \mathcal{N}$  the pair  $h = (h_1, g_2) : \mathcal{M} \rightarrow \mathcal{N}$  with  $h_1 m = g_1 \bar{m}$  for all  $m \in M$  is clearly a precrossed module morphism. Then the map  $\Phi(\mathcal{M}, \mathcal{N})$  defined as

$$\begin{aligned} \Phi(\mathcal{M}, \mathcal{N}) : \text{XAlg}(R)(\mathcal{M}^{cr}, \mathcal{N}) &\longrightarrow \text{PXAlg}(R)(\mathcal{M}, \mathcal{N}) \\ g = (g_1, g_2) &\longmapsto \Phi(\mathcal{M}, \mathcal{N})(g) = h = (h_1, g_2) \end{aligned}$$

can be shown to be a bijection, which is natural in both  $\mathcal{M}$  and  $\mathcal{N}$ , and this completes the proof. □

**6. Free  $R$ -algebroid crossed modules**

**Proposition 17** For any object  $\omega : K \rightarrow A$  of  $\text{Sets}/(\text{Alg}(R)/A)$  there exists an  $R$ -algebroid crossed  $A$ -module  $F_X(\omega) = (\omega_X : F_X(\omega) \rightarrow A)$  and a morphism  $(i_c, I_A) : \omega \rightarrow \omega_{X_m}$  such that for all  $R$ -algebroid crossed  $A$ -modules  $\mathcal{N} = (\eta : N \rightarrow A)$  and for all morphisms  $(f, I_A) : \omega \rightarrow \eta_m$  there exists a unique crossed  $A$ -module morphism  $(\alpha, I_A) : F_X(\omega) \rightarrow \mathcal{N}$  such that  $f = \alpha i_c$ , i.e. the diagram in Figure 8 is commutative.



**Figure 8**

$F_X(\omega)$ , with the morphism  $(i_c, I_A)$ , is called the free  $R$ -algebroid crossed  $A$ -module determined by  $\omega$ . The free crossed module is unique up to isomorphism.

**Proof** In the fourth section we got the free  $R$ -algebroid precrossed  $A$ -module  $F_P(\omega) = (\omega_P : F_P(\omega) \rightarrow A)$  determined by  $\omega$ , with the morphism  $(i_p, I_A) : \omega \rightarrow \omega_{P_m}$ .

Then, taking  $F_X(\omega) = (F_P(\omega))^{cr}$ , where  $\omega_X = \omega_P^{cr}$ , and then defining  $i_c : K \rightarrow F_X(\omega)$  as  $i_c k = \overline{[1k1]}$  for all  $k \in K$  and  $\alpha : F_X(\omega) \rightarrow N$  as  $\alpha \overline{[aka']} = {}^a(fk)^{a'}$  on generators completes the proof.  $\square$

Composing the free precrossed module functor  $F_P$  and the functor  $(-)^{cr}$  we get a functor  $F_X : \text{Sets}/(\text{Alg}(R)/A) \rightarrow \text{XAlg}(R)/A$  defined as  $F_X(\omega) = (\omega_X : F_X(\omega) \rightarrow A)$  on objects and as  $F_X(f, I_A) = (F_X f, I_A)$  on morphisms where  $(F_X f) \overline{[aka']} = \overline{[a(fk)a']}$  on generators.

**Proposition 18** If  $\omega : K \rightarrow A$  and  $(g, I_A) : \omega \rightarrow \omega'$  in  $\text{Sets}/(\text{Alg}(R)/A)$  then  $\omega'_{X_g} : (F_X g)(F_X(\omega)) \rightarrow A$  where  $\omega'_{X_g}$  is the restriction of  $\omega'_X$  on  $(F_X g)(F_X(\omega))$ , with  $i_g : g(K) \rightarrow (F_X g)(F_X(\omega))$  defined as  $i_g(gk) = \overline{[1gk1]}$ , is the free  $R$ -algebroid crossed  $A$ -module determined by  $\omega'_g : g(K) \rightarrow A$  where  $\omega'_g$  is the restriction of  $\omega'$  on  $g(K)$ .

**Proof** For any  $R$ -algebroid crossed  $A$ -module  $\mathcal{N} = (\eta : N \rightarrow A)$  and for any morphism  $(f, I_A) : \omega'_g \rightarrow \mathcal{N}$  the map  $\alpha_g : (F_X g)(F_X(\omega)) \rightarrow N$  defined as  $\alpha_g \overline{[a(gk)a']}$  clearly forms a unique crossed module morphism with  $I_A$  and makes the universal diagram commutative, completing the proof.  $\square$

**Proposition 19** As in the case of free precrossed modules, the functor  $F_X$  is the left adjoint of the forgetful functor  $U : \text{XAlg}(R)/A \rightarrow \text{Sets}/(\text{Alg}(R)/A)$ , which for a crossed module  $\mathcal{N} = (\eta : N \rightarrow A)$  gives the

function  $\eta_m$  and for a crossed  $A$ -module morphism  $(f, I_A) : \mathcal{N} \rightarrow \mathcal{N}'$  gives the morphism  $U(f, I_A) = (Uf, I_A) : \eta_m \rightarrow \eta'_m$  such that  $(Uf)(n) = fn$  for all  $n \in \mathcal{N}$ .

**Proof** For all  $\mathcal{N} \in Ob(\text{XAlg}(R)/A)$  and  $\omega \in Ob(\text{Sets}/(\text{Alg}(R)/A))$  we have bijections  $(\text{XAlg}(R)/A)(F_x(\omega), \mathcal{N}) \cong (\text{PXAlg}(R)/A)(F_P(\omega), \mathcal{N})$  from Proposition 16 and  $(\text{PXAlg}(R)/A)(F_P(\omega), \mathcal{N}) \cong (\text{Sets}/(\text{Alg}(R)/A))(\omega, \eta_m)$  from Proposition 12, and their composition gives the needed isomorphism which is natural in  $\mathcal{N}$  and  $\omega$ .  $\square$

**Proposition 20** *i) There exists a natural transformation*

$$\delta = \{(\delta_\omega, I_A) : \omega \in \text{Sets}/(\text{Alg}(R)/A)\} : I_{\text{Sets}/(\text{Alg}(R)/A)} \implies UF_x$$

where  $(\delta_\omega, I_A) : \omega \rightarrow (UF_x)(\omega)$  is a morphism for all  $\omega \in \text{Sets}/(\text{Alg}(R)/A)$  and  $I_{\text{Sets}/(\text{Alg}(R)/A)}$  is the identity functor on  $\text{Sets}/(\text{Alg}(R)/A)$ .

*ii) For each  $\omega \in \text{Sets}/(\text{Alg}(R)/A)$ ,  $\mathcal{N} \in \text{XAlg}(R)/A$  and morphism  $(g, I_A) : \omega \rightarrow U(\mathcal{N}) = \eta_m$  there exists a unique crossed  $A$ -module morphism  $(f, I_A) : F_x(\omega) \rightarrow \mathcal{N}$  such that  $g = (Uf)\delta_\omega$ .*

**Proof** *i)* If  $\omega : K \rightarrow A$ , defining  $\delta_\omega k = \overline{[1k1]}$  for all  $k \in K$  completes the proof since the rest are clear.

*ii)* Define  $f[\overline{aka'}] = {}^a(gk)^{a'}$  on generators. Then obviously  $(f, I_A)$  is a crossed  $A$ -module morphism and  $gk = f[\overline{1k1}] = (Uf)[\overline{1k1}] = (Uf)\delta_\omega k$  for all  $k \in K$ . Moreover,  $(Uf)\delta_\omega = (Uf')\delta_\omega$  implies  $gk = f[\overline{1k1}] = f'[\overline{1k1}]$  and  $f[\overline{aka'}] = f'[\overline{aka'}]$  for all  $k \in K$  and for all generators  $\overline{aka'} \in F_x(\omega)$  and this ensures the uniqueness of  $f$  for fixed  $g$ .  $\square$

**Proposition 21** *i) There exists a natural transformation*

$$\theta = \{(\theta_{\mathcal{N}}, I_A) : \mathcal{N} \in \text{XAlg}(R)/A\} : F_x U \implies I_{\text{XAlg}(R)/A}$$

where  $(\theta_{\mathcal{N}}, I_A) : (F_x U)(\mathcal{N}) \rightarrow \mathcal{N}$  is a crossed  $A$ -module morphism for all  $\mathcal{N} \in \text{XAlg}(R)/A$  and  $I_{\text{XAlg}(R)/A}$  is the identity functor on  $\text{XAlg}(R)/A$ .

*ii) For all  $\omega \in \text{Sets}/(\text{Alg}(R)/A)$ ,  $\mathcal{N} \in \text{XAlg}(R)/A$  and crossed  $A$ -module morphism  $(f, I_A) : F_x(\omega) \rightarrow \mathcal{N}$  there exists a unique morphism  $(g, I_A) : \omega \rightarrow U(\mathcal{N}) = \eta_m$  such that  $f = \theta_{\mathcal{N}}(F_x g)$ .*

**Proof** *i)* For each  $\mathcal{N} = (\eta : N \rightarrow A)$ , defining  $\theta_{\mathcal{N}}(\overline{[ana']}) = {}^a n^{a'}$  on generators completes the proof since the rest are clear.

*ii)* Define  $gk = f[\overline{1k1}]$ . Then

$$f[\overline{aka'}] = {}^a(f[\overline{1k1}])^{a'} = {}^a(gk)^{a'} = \theta_{\mathcal{N}}[\overline{a(gk)a'}] = \theta_{\mathcal{N}}(F_x g)[\overline{aka'}]$$

for all generators  $\overline{aka'} \in F_x(\omega)$ . Moreover,  $g$  is unique since if  $(g', I_A) : \omega \rightarrow U(\mathcal{N})$  is another morphism with  $f = \theta_{\mathcal{N}}(F_x g')$  then

$$gk = f[\overline{1k1}] = (\theta_{\mathcal{N}}(F_x g'))(\overline{[1k1]}) = \theta_{\mathcal{N}}(\overline{[1g'k1]}) = {}^1(g'k)^1 = g'k$$

for all  $k \in K$ .  $\square$

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