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Weighted composition operators between vector-valued Bloch-type spaces

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Abstract: Let X and Y be complex Banach spaces and $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. Let φ be an analytic self-map of $\mathbb{D}$ and ψ be an analytic operator-valued function from $\mathbb{D}$ into the space of all bounded linear operators from X to Y. The weighted composition operator $W_{\psi,\varphi} : \mathcal{H}(\mathbb{D}, X) \to \mathcal{H}(\mathbb{D}, Y)$ is defined by

$$W_{\psi,\varphi}(f)(z) = \psi(z)(f(\varphi(z))), \qquad (z \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D}, X)),$$

where $\mathcal{H}(\mathbb{D}, X)$ is the space of all analytic X-valued functions on $\mathbb{D}$. In this paper we provide necessary and sufficient conditions for the boundedness and compactness of weighted composition operators $W_{\psi,\varphi}$ between vector-valued Bloch-type spaces $\mathcal{B}_\alpha(X)$ and $\mathcal{B}_\beta(Y)$ for $\alpha, \beta > 0$ in terms of ψ, φ, their derivatives, and the nth power φ^n of φ.

Key words: Vector-valued Bloch-type spaces, weighted Banach spaces of analytic functions, weighted composition operators, compact operators

1. Introduction and preliminaries

The study of composition and weighted composition operators between Banach spaces of vector-valued functions has received recently very much attention. For instance, the (weakly) compact composition operators on analytic vector-valued function spaces such as weighted Bergman spaces, Bloch spaces, and BMOA were studied in [1, 7, 8, 10, 11]. The compact weighted composition operators between vector-valued Lipschitz function spaces were investigated in [3]. Laitila and Tylli in [9] characterized boundedness and (weak) compactness of weighted composition operators $W_{\psi,\varphi} : \mathcal{H}^\infty_\nu(X) \to \mathcal{H}^\infty_\omega(Y)$, whenever X and Y are complex Banach spaces. Bonet et al. in [2] studied weighted composition operators on unweighted $\mathcal{H}(\mathbb{D}, X)$ and weighted $\mathcal{H}^\infty_\nu(X)$ spaces, where X is a complete barrelled locally convex space. Boundedness and compactness of weighted composition operators between Bloch-type spaces in the scalar valued case were discussed in [12]. The aim of the present paper is to find some necessary and sufficient conditions for boundedness and compactness of weighted composition operators $W_{\psi,\varphi}$ between vector-valued Bloch-type spaces in terms of ψ, φ, their derivatives, and the nth power φ^n of φ.

Let $(X, \|\cdot\|_X)$ be a complex Banach space and $\mathcal{H}(\mathbb{D}, X)$ be the space of all analytic X-valued functions on the open unit disc $\mathbb{D}$. We consider the weighted Banach spaces of X-valued analytic functions

$$\mathcal{H}^\infty_\nu(X) = \{f \in \mathcal{H}(\mathbb{D}, X) : \|f\|_{\nu,X} = \sup_{z\in\mathbb{D}} \nu(z)\|f(z)\|_X < \infty\},$$

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and

$$\mathcal{H}_\nu^0(X) = \{f \in \mathcal{H}_\nu^\infty(X) : \lim_{|z|\to 1} \nu(z)\|f(z)\|_X = 0\},$$

endowed with the norm $\|\cdot\|_{\nu,X}$, where $\nu : \mathbb{D} \to (0,\infty)$ is a bounded continuous weight function. The weight ν is called radial if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. The associated weight $\tilde{\nu}$ of a weight function ν is defined as

$$\tilde{\nu}(z) := \left(\sup\{|f(z)| : f \in \mathcal{H}_\nu^\infty, \|f\|_\nu \le 1\}\right)^{-1}, \qquad z \in \mathbb{D},$$

where $\mathcal{H}_\nu^\infty = \mathcal{H}_\nu^\infty(\mathbb{C})$ and $\|\cdot\|_\nu = \|\cdot\|_{\nu,\mathbb{C}}$. We consider the standard weights $\nu_\alpha(z) = (1-|z|^2)^\alpha$ with $0 < \alpha < \infty$ and the logarithmic weight $\nu_{\log}(z) = \left(\log \frac{2}{1-|z|^2}\right)^{-1}$. It is well known that $\tilde{\nu}_\alpha = \nu_\alpha$ and $\tilde{\nu}_{\log} = \nu_{\log}$. In this paper, we denote the spaces $(\mathcal{H}_{\nu_\alpha}^\infty(X), \|\cdot\|_{\nu_\alpha,X})$ and $(\mathcal{H}_{\nu_{\log}}^\infty(X), \|\cdot\|_{\nu_{\log},X})$ by $(\mathcal{H}_\alpha^\infty(X), \|\cdot\|_{\alpha,X})$ and $(\mathcal{H}_{\log}^\infty(X), \|\cdot\|_{\log,X})$, respectively.

For $0 < \alpha < \infty$, the vector-valued Bloch-type spaces $\mathcal{B}_\alpha(X)$ and $\mathcal{B}_\alpha^0(X)$ are the Banach spaces of all functions $f \in \mathcal{H}(\mathbb{D}, X)$ whose derivatives f' are in $\mathcal{H}_\alpha^\infty(X)$ and $\mathcal{H}_\alpha^0(X)$, respectively, endowed with the norm $\|f\|_{\mathcal{B}_\alpha(X)} := \|f(0)\|_X + \|f'\|_{\alpha,X}$. We denote $\mathcal{B}_\alpha(\mathbb{C})$ by $\mathcal{B}_\alpha$. We will also abbreviate $\mathcal{B}_1(X) = \mathcal{B}(X)$ and $\mathcal{B}_1 = \mathcal{B}$. For $f \in \mathcal{B}_\alpha$ and $x \in X$, the function $fx : \mathbb{D} \to X$ given by $(fx)(z) = f(z)x$ for all $z \in \mathbb{D}$ belongs to $\mathcal{B}_\alpha(X)$ and $\|fx\|_{\mathcal{B}_\alpha(X)} = \|f\|_{\mathcal{B}_\alpha}\|x\|_X$. In particular, for any $x \in X$, the constant function $(1x)(z) = x$ for all $z \in \mathbb{D}$ is in $\mathcal{B}_\alpha(X)$ and $\|1x\|_{\mathcal{B}_\alpha(X)} = \|x\|_X$.

It is easy to check that for every $f \in \mathcal{B}_\alpha(X)$ and $z \in \mathbb{D}$,

$$\|f(z)\|_X \lesssim \|f\|_{\mathcal{B}_\alpha(X)} \begin{cases} 1 & 0 < \alpha < 1 \\ \log \frac{2}{1-|z|^2} & \alpha = 1 \\ \frac{1}{(1-|z|^2)^{\alpha-1}} & \alpha > 1 \end{cases}, \tag{1.1}$$

see [12]. The notations $A \lesssim B$ and $A \approx B$ mean that $A \le cB$ and $cB \le A \le CB$, respectively, for some positive constants c and C.

We will assume throughout this paper that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces and ν and ω are radial nonincreasing weights tending to zero at the boundary of $\mathbb{D}$. Let $\mathcal{L}(X,Y)$ ($\mathcal{K}(X,Y)$) be the Banach space of all bounded (compact) linear operators from X to Y. Let φ be a nonconstant analytic self-map of $\mathbb{D}$ and $\psi : \mathbb{D} \to \mathcal{L}(X,Y)$ be an analytic operator-valued function. The weighted composition operator $W_{\psi,\varphi}$ from $\mathcal{H}(\mathbb{D}, X)$ to $\mathcal{H}(\mathbb{D}, Y)$ is defined to be the linear operator of the form $W_{\psi,\varphi}(f)(z) = \psi(z)(f(\varphi(z))$ for all $f \in \mathcal{H}(\mathbb{D}, X)$ and $z \in \mathbb{D}$.

For simplicity of notation, we write ψ_z instead of $\psi(z)$. Note that if ψ_z is the identity map on X for every $z \in \mathbb{D}$, then $W_{\psi,\varphi}$ is the composition operator on $\mathcal{H}(\mathbb{D}, X)$. We also use $||T||_{X\to Y}$ for the norm of a linear operator $T : X \to Y$. The essential norm of a bounded linear operator $T : X \to Y$ is defined as the distance from T to $\mathcal{K}(X,Y)$, and it is denoted by $\|T\|_{e;X\to Y}$.

In [9] Laitila and Tylli characterized the boundedness of weighted composition operators $W_{\psi,\varphi} : \mathcal{H}_\nu^\infty(X) \to \mathcal{H}_\omega^\infty(Y)$. They proved the following:

Theorem 1.1 *[9, Theorem 2.1]*

$$\|W_{\psi,\varphi}\|_{\mathcal{H}^\infty_\nu(X)\to\mathcal{H}^\infty_\omega(Y)} = \sup_{z\in\mathbb{D}} \frac{\omega(z)}{\tilde{\nu}(\varphi(z))}\|\psi_z\|_{X\to Y}. \tag{1.2}$$

In particular, $\psi \in \mathcal{H}^\infty_\omega(\mathcal{L}(X,Y))$ *if* $W_{\psi,\varphi} : \mathcal{H}^\infty_\nu(X) \to \mathcal{H}^\infty_\omega(Y)$ *is bounded.*

The following theorem estimates the norm of $W_{\psi,\varphi} : \mathcal{H}^\infty_\nu(X) \to \mathcal{H}^\infty_\omega(Y)$ in terms of ψ, φ, their derivatives, and the nth power φ^n of φ. The proof is exactly the same as in [4, Theorem 2.4 (a)] and we just examine their proof for the vector-valued case. For this, let

$$\overline{\nu}(\xi) = \left(\sup_{n\geq 0} \frac{|\xi|^n}{\|z^n\|_\nu}\right)^{-1},$$

where z^n is a monomial on $\mathbb{D}$. Then $\overline{\nu}$ is a radial weight and it is equivalent to $\tilde{\nu}$, see [4, Corollary 2.3].

Theorem 1.2 *Let* $\psi : \mathbb{D} \to \mathcal{L}(X,Y)$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$ *be analytic maps. The weighted composition operator* $W_{\psi,\varphi}$ *maps* $\mathcal{H}^\infty_\nu(X)$ *into* $\mathcal{H}^\infty_\omega(Y)$ *boundedly if and only if*

$$\|W_{\psi,\varphi}\|_{\mathcal{H}^\infty_\nu(X)\to\mathcal{H}^\infty_\omega(Y)} = \sup_{z\in\mathbb{D}} \frac{\omega(z)}{\tilde{\nu}(\varphi(z))}\|\psi_z\|_{X\to Y} \approx \sup_{n\geq 0} \frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} < \infty.$$

Proof By [4, Corollary 2.3] $\tilde{\nu}$ is equivalent to $\overline{\nu}$. Hence,

$$c_1 \sup_{z\in\mathbb{D}} \frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} \leq \sup_{z\in\mathbb{D}}\sup_{n\geq 0} \frac{\omega(z)|\varphi(z)|^n\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu} \leq c_2 \sup_{z\in\mathbb{D}} \frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))}, \tag{1.3}$$

for some positive constants c_1 and c_2 and for each $z \in \mathbb{D}$. It follows from Theorem 1.1 that $W_{\psi,\varphi} : \mathcal{H}^\infty_\nu(X) \to \mathcal{H}^\infty_\omega(Y)$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)}{\tilde{\nu}(\varphi(z))}\|\psi_z\|_{X\to Y} < \infty.$$

By (1.3), this inequality is valid if and only if

$$\sup_{n\geq 0} \frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} = \sup_{n\geq 0}\sup_{z\in\mathbb{D}} \frac{\omega(z)|\varphi(z)|^n\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu} < \infty.$$

Furthermore,

$$\sup_{n\geq 0} \frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} \approx \sup_{z\in\mathbb{D}} \frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))}.$$

□

We will use the following lemma to get our main results.

Lemma 1.3 *[5, Lemma 2.1] For every* $\alpha > 0$,

(i) $\lim_{n\to\infty} (n+1)^\alpha \|z^n\|_\alpha = (\frac{2\alpha}{e})^\alpha$,

(ii) $\lim_{n\to\infty} \log(n)\|z^n\|_{\log} = 1$.

Corresponding to an operator-valued analytic function $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$, we define two integral operators,

$$I_\psi f(z) = \int_0^z f'(\xi)\psi_\xi d\xi \qquad \text{and} \qquad J_\psi f(z) = \int_0^z f(\xi)\psi'_\xi d\xi,$$

for every $f \in \mathcal{H}(\mathbb{D}) := \mathcal{H}(\mathbb{D}, \mathbb{C})$ and $z \in \mathbb{D}$. It can be easily seen that $I_\psi, J_\psi : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ are well defined, and for every $z \in \mathbb{D}$,

$$(I_\psi f)'(z) = f'(z)\psi_z \qquad \text{and} \qquad (J_\psi f)'(z) = f(z)\psi'_z,$$

see [13, Theorem 3.27].

The following lemma is the vector-valued version of [5, Lemma 2.2].

Lemma 1.4 *Let $n \in \mathbb{N}$ be fixed and $\alpha > 0$. If $\varphi'\varphi^{n-1}\psi$ and $\varphi^n\psi'$ are in $\mathcal{H}^\infty_\alpha(\mathcal{L}(X,Y))$, then $I_\psi\varphi^n$ and $J_\psi\varphi^n$ belong to $\mathcal{B}_\alpha(\mathcal{L}(X,Y))$. Moreover,*

$$\|\varphi'\varphi^{n-1}\psi\|_{\alpha,\mathcal{L}(X,Y)} = \frac{1}{n}\|I_\psi\varphi^n\|_{\mathcal{B}_\alpha(\mathcal{L}(X,Y))} \ \textit{and} \ \|\varphi^n\psi'\|_{\alpha,\mathcal{L}(X,Y)} = \|J_\psi\varphi^n\|_{\mathcal{B}_\alpha(\mathcal{L}(X,Y))}.$$

Proof We prove the first equality. The other one is proved similarly.

Let $\varphi'\varphi^{n-1}\psi \in \mathcal{H}^\infty_\alpha(\mathcal{L}(X,Y))$. Then,

$$\begin{aligned}\|\varphi'\varphi^{n-1}\psi\|_{\alpha,\mathcal{L}(X,Y)} &= \sup_{z\in\mathbb{D}} \nu_\alpha(z)\|\varphi'(z)\varphi^{n-1}(z)\psi_z\|_{X\to Y} \\ &= \frac{1}{n}\sup_{z\in\mathbb{D}} \nu_\alpha(z)\|(\varphi^n)'(z)\psi_z\|_{X\to Y} \\ &= \frac{1}{n}\sup_{z\in\mathbb{D}} \nu_\alpha(z)\|(I_\psi\varphi^n)'(z)\|_{X\to Y} \\ &= \frac{1}{n}\|(I_\psi\varphi^n)'\|_{\alpha,\mathcal{L}(X,Y)}.\end{aligned}$$

Since $I_\psi\varphi^n(0) = 0$, we get

$$\|\varphi'\varphi^{n-1}\psi\|_{\alpha,\mathcal{L}(X,Y)} = \frac{1}{n}\|I_\psi\varphi^n\|_{\mathcal{B}_\alpha(\mathcal{L}(X,Y))}.$$

□

We will frequently use the following test functions to prove our main results. For $a, b \geq 0$ and $w \in \mathbb{D}$ we define

$$K_w^{a,b}(z) = \frac{(1-|\varphi(w)|^2)^a}{(1-\overline{\varphi(w)}z)^b} \quad \text{and} \quad \lambda_w(z) = \log\frac{2}{1-\overline{\varphi(w)}z}.$$

It can be easily seen that

$$(K_w^{a,b})' = b\overline{\varphi(w)}K_w^{a,b+1}, \qquad K_w^{a,b}(\varphi(w)) = (1-|\varphi(w)|^2)^{a-b},$$

and

$$\lambda'_w = \overline{\varphi(w)}K_w^{0,1}, \qquad \lambda_w(\varphi(w)) = \log\frac{2}{1-|\varphi(w)|^2}.$$

Specifically, for $a \geq 0$ and $0 < \alpha < \infty$ with $a + \alpha > 1$, we see that $\|K_w^{a,a+\alpha-1}\|_{\mathcal{B}_\alpha} \leq 2^a(1 + 2^\alpha)$ and $\|\lambda_w\|_{\mathcal{B}} \leq 2 + \log 2$. Thus, $\{K_w^{a,a+\alpha-1} : w \in \mathbb{D}\}$ and $\{\lambda_w : w \in \mathbb{D}\}$ are bounded subsets of $\mathcal{B}_\alpha^0$ and $\mathcal{B}^0$, respectively.

2. Bounded weighted composition operators from $\mathcal{B}_\alpha(X)$ into $\mathcal{B}_\beta(Y)$

In this section we investigate the boundedness of the weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$. Our approach is inspired by the techniques given in [6]. We consider the derivative operator $D : \mathcal{B}_\alpha(X) \to \mathcal{H}_\alpha^\infty(X)$, $f \mapsto f'$. It is easy to see that D is a bounded linear operator with $\|D\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\alpha^\infty(X)} \leq 1$. For each $f \in \mathcal{H}_\alpha^\infty(X)$, we define $g : \mathbb{D} \to X$ by $g(z) = \int_0^z f(\xi)d\xi$. Applying [13, Theorem 3.31] one can show that $g \in \mathcal{H}(\mathbb{D}, X)$ and $g' = f$. Hence, $g \in \mathcal{B}_\alpha(X)$ and $Dg = f$, which ensures that D is onto.

Considering the operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$, for each $f \in \mathcal{B}_\alpha(X)$ and each $z \in \mathbb{D}$ we have

$$(W_{\psi,\varphi}f)'(z) = \varphi'(z)\psi_z f'(\varphi(z)) + \psi_z' f(\varphi(z)),$$

from which we reach

$$DW_{\psi,\varphi}f = W_{\varphi'\psi,\varphi}Df + W_{\psi',\varphi}f.$$

Thus,

$$DW_{\psi,\varphi} = W_{\varphi'\psi,\varphi} \circ D + W_{\psi',\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\beta^\infty(Y).$$

For each $f \in \mathcal{B}_\alpha(X)$, by (1.1) we observe that

$$\|W_{\psi,\varphi}f(0)\|_Y = \|\psi_0 f(\varphi(0))\|_Y \leq \|\psi_0\|_{X\to Y}\|f(\varphi(0))\|_X \leq c\|f\|_{\mathcal{B}_\alpha(X)},$$

where c is a positive constant depending on α, $\varphi(0)$ and ψ_0. For all $f \in \mathcal{B}_\alpha(X)$ with $\|f\|_{\mathcal{B}_\alpha(X)} \leq 1$, we have

$$\begin{aligned}
\|W_{\psi,\varphi}f\|_{\mathcal{B}_\beta(Y)} &= \|W_{\psi,\varphi}f(0)\|_Y + \|DW_{\psi,\varphi}f\|_{\beta,Y} \\
&\leq c + \|(W_{\varphi'\psi,\varphi} \circ D)f\|_{\beta,Y} + \|W_{\psi',\varphi}f\|_{\beta,Y} \\
&\leq c + \|W_{\varphi'\psi,\varphi}D\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)} + \|W_{\psi',\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)}.
\end{aligned}$$

Therefore,

$$\|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)} \leq c + \|W_{\varphi'\psi,\varphi}\|_{\mathcal{H}_\alpha^\infty(X)\to\mathcal{H}_\beta^\infty(Y)} + \|W_{\psi',\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)}. \tag{2.1}$$

Accordingly, a sufficient condition for the boundedness of $W_{\psi,\varphi}$ is the boundedness of both operators $W_{\varphi'\psi,\varphi} : \mathcal{H}_\alpha^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ and $W_{\psi',\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\beta^\infty(Y)$.

We break the problem of the boundedness of $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ into three different cases.

Theorem 2.1 *Let $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ and φ be an analytic self-map of $\mathbb{D}$. Then for $0 < \alpha < 1$, $\beta > 0$, the weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded if and only if*

(i) $\psi \in \mathcal{B}_\beta(\mathcal{L}(X,Y))$,

(ii) $\sup_{n\geq 1} n^{\alpha-1}\|I_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty$.

Proof Let $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ be bounded. Using the family of constant functions $\{1x : x\in X, \|x\|_X\le 1\}$, we see that $\psi\in\mathcal{B}_\beta(\mathcal{L}(X,Y))$. To show (ii), let $\alpha>0$, $w\in\mathbb{D}$ and define

$$f_w=\frac{1}{\overline{\varphi(w)}}(K_w^{2,1+\alpha}-K_w^{1,\alpha}).$$

Then $\{f_w : \ w\in\mathbb{D}\}$ is a bounded subset of $\mathcal{B}_\alpha$. Furthermore, $f_w(\varphi(w))=0$ and $f_w'(\varphi(w))=\frac{1}{(1-|\varphi(w)|^2)^\alpha}$. Thus,

$$\begin{aligned}\sup_{\substack{w\in\mathbb{D}\\ \|x\|_X\le 1}}\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha}|\varphi'(w)|\|\psi_w x\|_Y &= \sup_{\substack{w\in\mathbb{D}\\ \|x\|_X\le 1}}(1-|w|^2)^\beta|\varphi'(w)||f_w'(\varphi(w))|\|\psi_w x\|_Y\\ &\le \sup_{\substack{w\in\mathbb{D}\\ \|x\|_X\le 1}}\|DW_{\psi,\varphi}(f_w x)\|_{\beta,Y}\\ &\le \sup_{\substack{w\in\mathbb{D}\\ \|x\|_X\le 1}}\|W_{\psi,\varphi}(f_w x)\|_{\mathcal{B}_\beta(Y)}<\infty,\end{aligned}$$

which implies that

$$\sup_{w\in\mathbb{D}}\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha}|\varphi'(w)|\|\psi_w\|_{X\to Y}<\infty.$$

Thus, by Theorem 1.1, the operator $W_{\varphi'\psi,\varphi}:\mathcal{H}^\infty_\alpha(X)\to\mathcal{H}^\infty_\beta(Y)$ is bounded. It follows from Theorem 1.2 and Lemma 1.4 that

$$\sup_{n\ge 0}\frac{\frac{1}{n+1}\|I_\psi\varphi^{n+1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}}{\|z^n\|_\alpha}=\sup_{n\ge 0}\frac{\|\varphi'\varphi^n\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_\alpha}<\infty.$$

Applying Lemma 1.3, we obtain

$$\sup_{n\ge 0}(n+1)^{\alpha-1}\|I_\psi\varphi^{n+1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}\lesssim\sup_{n\ge 0}\frac{\frac{1}{n+1}\|I_\psi\varphi^{n+1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}}{\|z^n\|_\alpha}<\infty,$$

showing that (ii) is necessary.

Now let (i) and (ii) hold and $f\in\mathcal{B}_\alpha(X)$ with $\|f\|_{\mathcal{B}_\alpha(X)}\le 1$. Then by (i) and (1.1), for $\alpha<1$ we have

$$\begin{aligned}\|W_{\psi',\varphi}(f)\|_{\beta,Y} &=\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta\|\psi_w' f(\varphi(w))\|_Y\\ &\le\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta\|\psi_w'\|_{X\to Y}\|f(\varphi(w))\|_X\\ &\lesssim\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta\|\psi_w'\|_{X\to Y}\|f\|_{\mathcal{B}_\alpha(X)}\\ &\le\|\psi\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}.\end{aligned}$$

Hence, $W_{\psi',\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{H}^\infty_\beta(Y)$ is bounded. By hypothesis (ii), it follows from Lemmas 1.3 and 1.4 that

$$\sup_{n\ge 0}\frac{\|\varphi'\varphi^n\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_\alpha}\approx\sup_{n\ge 1}n^{\alpha-1}\|I_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}<\infty.$$

Thus, Theorem 1.2 implies that $W_{\varphi'\psi,\varphi}:\mathcal{H}^\infty_\alpha(X)\to\mathcal{H}^\infty_\beta(Y)$ is bounded. Considering (2.1), we conclude that $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ is bounded. □

Theorem 2.2 *Let $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ and φ be an analytic self-map of $\mathbb{D}$. Then for $\beta > 0$, the weighted composition operator $W_{\psi,\varphi} : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ is bounded if and only if*

(i) $\psi \in \mathcal{B}_\beta(\mathcal{L}(X,Y))$,

(ii) $\sup_{n\geq 1} \|I_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty$,

(iii) $\sup_{n\geq 1} \log n \|J_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty$.

Proof The necessity of (i) and (ii) can be shown in the same way as in the proof of Theorem 2.1. We show that (iii) is necessary. Let $W_{\psi,\varphi} : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ be bounded. Defining

$$g_w = 2\lambda_w - \frac{\lambda_w^2}{\lambda_w(\varphi(w))},$$

for each $w \in \mathbb{D}$, we see that $g_w(\varphi(w)) = \log \frac{2}{1-|\varphi(w)|^2}$ and $g_w'(\varphi(w)) = 0$. Since for each $w \in \mathbb{D}$, $(1+|\varphi(w)|)(1-|\varphi(w)|^2) < 2$, we have

$$\log \frac{2}{1-|\varphi(w)|} \leq 2 \log \frac{2}{1-|\varphi(w)|^2}.$$

Using this inequality, for each $w \in \mathbb{D}$ we get

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)|g_w'(z)| &\leq 2 \sup_{z\in\mathbb{D}}(1-|z|^2)|\lambda_w'(z)| \left(1 + \frac{|\lambda_w(z)|}{|\lambda_w(\varphi(w))|}\right) \\
&\leq 2 \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{\varphi(w)}z|} \left(1 + \frac{|\log \frac{2}{1-\overline{\varphi(w)}z}|}{\log \frac{2}{1-|\varphi(w)|^2}}\right) \\
&\leq 4 \left(1 + \frac{\log \frac{2}{1-|\varphi(w)|} + \pi}{\log \frac{2}{1-|\varphi(w)|^2}}\right) \\
&\leq 4 \left(1 + \frac{\log \frac{2}{1-|\varphi(w)|}}{\log \frac{2}{1-|\varphi(w)|^2}} + \frac{\pi}{\log \frac{2}{1-|\varphi(w)|^2}}\right) \\
&\leq 4(1 + 2 + \frac{\pi}{\log 2}) = 12 + \frac{4\pi}{\log 2}.
\end{aligned}$$

Moreover,

$$|g_w(0)| = \left| 2\log 2 - \frac{\log^2 2}{\log \frac{2}{1-|\varphi(w)|^2}} \right| \leq 3 \log 2,$$

for each $w \in \mathbb{D}$. Therefore, the family $\{g_w : \ w \in \mathbb{D}\}$ is a bounded subset of $\mathcal{B}$ with bound say M. Hence, for each $x \in X$ with $\|x\|_X \leq 1$ and each $w \in \mathbb{D}$,

$$\begin{aligned}(1-|w|^2)^\beta \log \frac{2}{1-|\varphi(w)|^2}\|\psi'_w x\|_Y &= (1-|w|^2)^\beta |g_w(\varphi(w))|\|\psi'_w x\|_Y \\ &\leq \|DW_{\psi,\varphi}(g_w x)\|_{\beta,Y} \\ &\leq \|W_{\psi,\varphi}(g_w x)\|_{\mathcal{B}_\beta(Y)} \\ &\leq M\|W_{\psi,\varphi}\|_{\mathcal{B}(X)\to\mathcal{B}_\beta(Y)} < \infty,\end{aligned}$$

from which we conclude that

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta}{\nu_{\log}(\varphi(w))}\|\psi'_w\|_{X\to Y} = \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta \log \frac{2}{1-|\varphi(w)|^2}\|\psi'_w\|_{X\to Y} < \infty.$$

Thus, by Theorem 1.1, the operator $W_{\psi',\varphi} : \mathcal{H}^\infty_{\log}(X) \to \mathcal{H}^\infty_\beta(Y)$ is bounded. Then by Theorem 1.2 and Lemma 1.4,

$$\sup_{n\geq 0} \frac{\|J_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}}{\|z^n\|_{\log}} = \sup_{n\geq 0} \frac{\|\varphi^n \psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\log}} < \infty,$$

and by Lemma 1.3 we observe that

$$\sup_{n\geq 1} \log(n)\|J_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty,$$

and (iii) holds.

Conversely, let (i)–(iii) hold. We show that $W_{\psi,\varphi} : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ is bounded. Condition (ii) along with Lemmas 1.3(i) and 1.4 imply that

$$\sup_{n\geq 0} \frac{\|\varphi'\varphi^n\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_1} \approx \sup_{n\geq 1} \|I_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty,$$

and using Theorem 1.2 we get that $W_{\varphi'\psi,\varphi} : \mathcal{H}^\infty_1(X) \to \mathcal{H}^\infty_\beta(Y)$ is bounded.

On the other hand, by condition (i), ψ' and hence $\varphi^n\psi'$ are in $\mathcal{H}^\infty_\beta(\mathcal{L}(X,Y))$. Then we can use Lemmas 1.4 and 1.3(ii) along with condition (iii) to conclude that

$$\sup_{n\geq 1} \frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\log}} = \sup_{n\geq 1} \frac{\|J_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}}{\|z^n\|_{\log}} \approx \sup_{n\geq 1} \log(n)\|J_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty.$$

Using (1.1) and Theorem 1.2, for each $f \in \mathcal{B}(X)$ with $\|f\|_{\mathcal{B}(X)} \leq 1$, we have

$$\begin{aligned}\|W_{\psi',\varphi}(f)\|_{\beta,Y} &= \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta \|\psi'_w(f(\varphi(w)))\|_Y \\ &\lesssim \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta \log\frac{2}{1-|\varphi(w)|^2}\|\psi'_w\|_{X\to Y}\|f\|_{\mathcal{B}(X)} \\ &\leq \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta}{\nu_{\log}(\varphi(w))}\|\psi'_w\|_{X\to Y} \\ &\approx \sup_{n\geq 1}\frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\log}} < \infty,\end{aligned}$$

which implies that $W_{\psi',\varphi} : \mathcal{B}(X) \to \mathcal{H}^\infty_\beta(Y)$ is bounded. Therefore, by (2.1) $W_{\psi,\varphi} : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ is bounded. □

For the case $\alpha > 1$, we need the following result, which is a modified version of [15, Proposition 7].

Proposition 2.3 *Let $\alpha > 1$. Then f is in $\mathcal{B}_\alpha(X)$ if and only if $f \in \mathcal{H}^\infty_{\alpha-1}(X)$. Moreover, $\|\cdot\|_{\alpha-1,X} \approx \|\cdot\|_{\mathcal{B}_\alpha(X)}$.*

Proof Let $f \in \mathcal{B}_\alpha(X)$. Then for each $x^* \in X^*$, $x^* \circ f \in \mathcal{B}_\alpha$, and by [15, Corollary 4] we have

$$x^* \circ f(z) = x^* \circ f(0) + \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha (x^* \circ f)'(w)}{\overline{w}(1-z\overline{w})^{\alpha+1}} dA(w), \quad (z \in \mathbb{D}),$$

where dA is the normalized area measure on $\mathbb{D}$. Since $(x^* \circ f)'(w) = x^*(f'(w))$, we get

$$x^* \circ f(z) - x^* \circ f(0) = \int_{\mathbb{D}} x^* \left(\frac{(1-|w|^2)^\alpha f'(w)}{\overline{w}(1-z\overline{w})^{\alpha+1}} \right) dA(w), \quad (z \in \mathbb{D}).$$

From [13, Theorem 3.27] we deduce that

$$f(z) - f(0) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f'(w)}{\overline{w}(1-z\overline{w})^{\alpha+1}} dA(w), \quad (z \in \mathbb{D}).$$

It follows that

$$\begin{aligned} \|f(z) - f(0)\|_X &\le \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha \|f'(w)\|_X}{|w||1-z\overline{w}|^{\alpha+1}} dA(w) \\ &\le \|f\|_{\mathcal{B}_\alpha(X)} \int_{\mathbb{D}} \frac{dA(w)}{|w||1-z\overline{w}|^{\alpha+1}}. \end{aligned}$$

The rest of the proof is similar to that of [15, Proposition 7] and we skip it. □

Theorem 2.4 *Let $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ and φ be an analytic self-map of $\mathbb{D}$. Then for $\alpha > 1$ and $\beta > 0$, the weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded if and only if*

(i) $\sup_{n \ge 1} n^{\alpha-1} \|I_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty,$

(ii) $\sup_{n \ge 1} n^{\alpha-1} \|J_\psi \varphi^{n-1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty.$

Proof Let $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ be bounded. The necessity of (i) can be shown in the same way as in the proof of Theorem 2.1(ii). To show that (ii) is necessary, we consider the functions h_w defined by

$$h_w = \alpha K_w^{0,\alpha-1} - (\alpha-1) K_w^{1,\alpha},$$

for $w \in \mathbb{D}$. We observe that $\{h_w : w \in \mathbb{D}\}$ is a bounded subset of $\mathcal{B}_\alpha$ such that $h_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}$

and $h'_w(\varphi(w)) = 0$. Hence,

$$\begin{aligned}\sup_{\substack{\|x\|_X\le 1\\ w\in\mathbb{D}}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}}\|\psi'_w x\|_Y &= \sup_{\substack{\|x\|_X\le 1\\ w\in\mathbb{D}}} (1-|w|^2)^\beta |h_w(\varphi(w))| \|\psi'_w x\|_Y \\ &\le \sup_{\substack{\|x\|_X\le 1\\ w\in\mathbb{D}}} \|DW_{\psi,\varphi}(h_w x)\|_{\beta,Y} \\ &\le \sup_{\substack{\|x\|_X\le 1\\ w\in\mathbb{D}}} \|W_{\psi,\varphi}(h_w x)\|_{\mathcal{B}_\beta(Y)},\end{aligned}$$

which implies that

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}}\|\psi'_w\|_{X\to Y} < \infty.$$

Therefore, by Theorem 1.2,

$$\sup_{n\ge 0} \frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\alpha-1}} < \infty,$$

and hence by Lemmas 1.3 and 1.4,

$$\begin{aligned}\sup_{n\ge 1} n^{\alpha-1}\|J_\psi\varphi^{n-1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} &= \sup_{n\ge 0}(n+1)^{\alpha-1}\|J_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} \\ &\approx \sup_{n\ge 0} \frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\alpha-1}} < \infty.\end{aligned}$$

Conversely, suppose (i) and (ii) hold. We show that $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded. By hypotheses (i) and (ii), Theorem 1.2, and Lemma 1.4, we have

$$\begin{aligned}\|W_{\varphi'\psi,\varphi}\|_{\mathcal{H}_\alpha^\infty(X)\to\mathcal{H}_\beta^\infty(Y)} &\approx \sup_{n\ge 0} \frac{\|\varphi^n\varphi'\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_\alpha} \\ &\approx \sup_{n\ge 0}(n+1)^{\alpha-1}\|I_\psi\varphi^{n+1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty\end{aligned}$$

and

$$\begin{aligned}\|W_{\psi',\varphi}\|_{\mathcal{H}_{\alpha-1}^\infty(X)\to\mathcal{H}_\beta^\infty(Y)} &\approx \sup_{n\ge 0} \frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\alpha-1}} \\ &\approx \sup_{n\ge 0}(n+1)^{\alpha-1}\|J_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} < \infty.\end{aligned}$$

By Proposition 2.3,

$$\|W_{\psi',\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)} \approx \|W_{\psi',\varphi}\|_{\mathcal{H}_{\alpha-1}^\infty(X)\to\mathcal{H}_\beta^\infty(Y)} < \infty.$$

Therefore, by (2.1), the operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded. □

Applying the arguments given in the proof of Theorems 2.1, 2.2, and 2.4, one can get the next theorem.

Theorem 2.5 *Let $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ and φ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\omega^\infty(Y)$ is bounded if and only if*

(i) $\psi \in \mathcal{H}_\omega^\infty(\mathcal{L}(X,Y))$, *whenever* $0 < \alpha < 1$;

(ii) $\sup\limits_{n\geq 0} \frac{\|\varphi^n \psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_{\log}} < \infty$, *whenever* $\alpha = 1$;

(iii) $\sup\limits_{n\geq 0} \frac{\|\varphi^n \psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_{\alpha-1}} < \infty$, *whenever* $\alpha > 1$.

We end this section with the following result.

Corollary 2.6 *Let $\psi \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X,Y))$ and φ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded if and only if both weighted composition operators*

$$W_{\psi',\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\beta^\infty(Y)$$

and

$$W_{\varphi'\psi,\varphi} : \mathcal{H}_\alpha^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$$

are bounded.

3. Compact weighted composition operators from $\mathcal{B}_\alpha(X)$ into $\mathcal{B}_\beta(Y)$

Laitila and Tylli in [9] provided some necessary and sufficient conditions under which every weighted composition operator $W_{\psi,\varphi} : \mathcal{H}_\alpha^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ is compact. We use their results to characterize the compact weighted composition operators between vector-valued Bloch-type spaces in terms of φ, ψ, their derivations, and the nth power φ^n of φ. We consider the linear operator $T_\psi : X \to \mathcal{B}_\beta(Y)$ defined by $T_\psi(x)(z) = \psi_z(x)$. Note that for each $x \in X$, considering the constant function $1x \in \mathcal{B}_\alpha(X)$ we have $T_\psi(x) = W_{\psi,\varphi}(1x)$. Hence,

$$\|T_\psi\|_{X\to\mathcal{B}_\beta(Y)} \leq \|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)}.$$

In particular, if $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded, then T_ψ is bounded.

We frequently use the following lemma to obtain our main results.

Lemma 3.1 *Let $W_{\psi,\varphi} : \mathcal{H}_\nu^\infty(X) \to \mathcal{H}_\omega^\infty(Y)$ be a bounded weighted composition operator. Then*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)}{\tilde{\nu}(\varphi(z))} \|\psi_z\|_{X\to Y} = 0, \tag{3.1}$$

if and only if

$$\lim_{n\to\infty} \frac{\|\varphi^n \psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} = 0. \tag{3.2}$$

Proof Let (3.1) hold and $\varepsilon > 0$. There exists $\delta > 0$ such that for every z with $1 - \delta < |\varphi(z)| < 1$,

$$\frac{\omega(z)}{\tilde{\nu}(\varphi(z))} \|\psi_z\|_{X\to Y} < \frac{\varepsilon}{2c},$$

where c is a positive constant for which $c^{-1}\tilde{\nu}^{-1} \le \overline{\nu}^{-1} \le c\tilde{\nu}^{-1}$. For every $z \in \mathbb{D}$ with $|\varphi(z)| > 1-\delta$,

$$\omega(z)\left(\sup_{n\ge 0}\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\right)\|\psi_z\|_{X\to Y} = \frac{\omega(z)\|\psi_z\|_{X\to Y}}{\overline{\nu}(\varphi(z))} \le c\frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} < \varepsilon/2.$$

Hence,

$$\sup_{|\varphi(z)|>1-\delta}\sup_{n\ge 0}\omega(z)\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\|\psi_z\|_{X\to Y} \le \varepsilon/2. \tag{3.3}$$

By Theorem 1.1, the boundedness of $W_{\psi,\varphi} : \mathcal{H}_\nu^\infty(X) \to \mathcal{H}_\omega^\infty(Y)$ implies that $\psi \in \mathcal{H}_\omega^\infty(\mathcal{L}(X,Y))$. Now if $|\varphi(z)| \le 1-\delta$, we may choose r such that $0 < 1-\delta < r < 1$. Then

$$\omega(z)\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\|\psi_z\|_{X\to Y} \le M\frac{(1-\delta)^n}{\|z^n\|_\nu} = M(\frac{1-\delta}{r})^n\frac{r^n}{\|z^n\|_\nu} \le \frac{M}{\overline{\nu}(r)}(\frac{1-\delta}{r})^n,$$

where $M = \|\psi\|_{\omega,\mathcal{L}(X,Y)}$. Hence,

$$\sup_{|\varphi(z)|\le 1-\delta}\omega(z)\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\|\psi_z\|_{X\to Y} \le \frac{M}{\overline{\nu}(r)}(\frac{1-\delta}{r})^n \to 0,$$

as $n \to \infty$. Choose a positive integer n_0 such that

$$\omega(z)\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\|\psi_z\|_{X\to Y} < \varepsilon/2, \tag{3.4}$$

for each $n \ge n_0$, and every z with $|\varphi(z)| \le 1-\delta$. Considering (3.3) and (3.4) for each $n \ge n_0$ we obtain

$$\frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} = \sup_{z\in\mathbb{D}}\omega(z)\frac{|\varphi(z)|^n}{\|z^n\|_\nu}\|\psi_z\|_{X\to Y} < \varepsilon,$$

which shows that (3.2) is valid.

Conversely, let (3.2) hold and $\varepsilon > 0$. Then there exists some positive integer n_0 such that $\frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} < \varepsilon$ for each $n > n_0$. For any $r \in (\frac{1}{2}, 1)$, $\frac{1}{\tilde{\nu}(r)} \le \frac{1}{r\overline{\nu}(r)}$ (see the proof of [4, Lemma 2.2]). Fixing $r \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned}
\sup_{|\varphi(z)|>r}\frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} &\le \sup_{|\varphi(z)|>r}\frac{1}{|\varphi(z)|}\sup_{n\ge 0}\frac{\omega(z)|\varphi(z)|^n\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu} \\
&\le \frac{1}{r}\sup_{|\varphi(z)|>r}\sup_{0\le n\le n_0}\frac{\omega(z)|\varphi(z)|^n\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu} \\
&\quad + \frac{1}{r}\sup_{|\varphi(z)|>r}\sup_{n>n_0}\frac{\omega(z)|\varphi(z)|^n\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu} \\
&\le \frac{1}{r}\sup_{0\le n\le n_0}\sup_{|\varphi(z)|>r}\frac{\omega(z)\tilde{\nu}(\varphi(z))\|\psi_z\|_{X\to Y}}{\|z^n\|_\nu\tilde{\nu}(\varphi(z))} \\
&\quad + \frac{1}{r}\sup_{n>n_0}\frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} \\
&\le \frac{1}{r}\|W_{\psi,\varphi}\|_{\mathcal{H}_\nu^\infty(X)\to\mathcal{H}_\omega^\infty(Y)}\sup_{|\varphi(z)|>r}\frac{\tilde{\nu}(\varphi(z))}{\|z^m\|_\nu} + \frac{\varepsilon}{r},
\end{aligned}$$

where $\|z^m\|_\nu = \min\{\|z^n\|_\nu : n = 0, 1, \cdots n_0\}$. Since $\lim_{r\to 1}\sup_{|\varphi(z)|>r}\tilde{\nu}(\varphi(z)) = 0$, taking the limit as $r \to 1$, we obtain

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} \le \varepsilon,$$

and since ε was an arbitrary positive number, we get

$$\limsup_{|\varphi(z)|\to 1}\frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} = \lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{\omega(z)\|\psi_z\|_{X\to Y}}{\tilde{\nu}(\varphi(z))} = 0.$$

□

Combining Lemma 3.1 and [9, Theorem 3.1] we conclude the following theorem.

Theorem 3.2 *Let $W_{\psi,\varphi} : \mathcal{H}_\nu^\infty(X) \to \mathcal{H}_\omega^\infty(Y)$ be a bounded weighted composition operator. Then $W_{\psi,\varphi}$ is compact if and only if $T_\psi : X \to \mathcal{H}_\omega^\infty(Y)$ is compact and*

$$\lim_{n\to\infty}\frac{\|\varphi^n\psi\|_{\omega,\mathcal{L}(X,Y)}}{\|z^n\|_\nu} = 0.$$

To investigate the compactness of operators $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$, we estimate the upper bound of their essential norms. To do this, we first find an upper bound for the essential norm of the weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\omega^\infty(Y)$.

Theorem 3.3 *Let $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\omega^\infty(Y)$ be bounded and $T_\psi : X \to \mathcal{H}_\omega^\infty(Y)$ be compact. Then*

$$\|W_{\psi,\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \lesssim \begin{cases} 0 & 0<\alpha<1 \\ \limsup\limits_{|\varphi(z)|\to 1}\omega(z)\|\psi_z\|_{X\to Y}\log\frac{2}{1-|\varphi(z)|^2} & \alpha = 1 \\ \limsup\limits_{|\varphi(z)|\to 1}\frac{\omega(z)\|\psi_z\|_{X\to Y}}{(1-|\varphi(z)|^2)^{\alpha-1}} & \alpha>1 \end{cases}.$$

Proof For each integer $n \ge 0$, consider the operators $q_n : \mathcal{B}_\alpha(X) \to X$ by $q_n(f) = \frac{f^{(n)}(0)}{n!}$ and $P_n : \mathcal{B}_\alpha(X) \to \mathcal{B}_\alpha(X)$ by $P_n(f) = \sum_{k=0}^n q_k(f)z^k$, for every $f \in \mathcal{B}_\alpha(X)$, where $\sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!}z^k$ is the Taylor expansion of f. It is easy to check that $\|q_0\|_{\mathcal{B}_\alpha(X)\to X} \le 1$. For every $n \ge 1$ and $f \in \mathcal{B}_\alpha(X)$ with $\|f\|_{\mathcal{B}_\alpha(X)} \le 1$, using the Cauchy integral formula we have

$$\begin{aligned}\|q_n f\|_X &= \frac{1}{2\pi n}\left\|\int_{|z|=\frac{1}{2}}\frac{f'(z)}{z^n}dz\right\|_X \\ &\le \frac{2^{\alpha+n-1}}{n}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\|f'(z)\|_X \\ &\le \frac{2^{\alpha+n-1}}{n}.\end{aligned}$$

Hence, $\|q_n\|_{\mathcal{B}_\alpha(X)\to X} \le \frac{2^{\alpha+n-1}}{n}$. Therefore, q_n and P_n are bounded linear operators.

For each $k \geq 0$, since φ is a self-map of $\mathbb{D}$, we deduce that the multiplier $M_{\varphi^k} : \mathcal{H}_\omega^\infty(Y) \to \mathcal{H}_\omega^\infty(Y)$ is bounded and hence $M_{\varphi^k} T_\psi : X \to \mathcal{H}_\omega^\infty(Y)$ is a compact operator. By the boundedness of $q_k : \mathcal{B}_\alpha(X) \to X$, for each m, the operator

$$W_{\psi,\varphi} P_m = \sum_{k=0}^{m} M_{\varphi^k} T_\psi q_k : \mathcal{B}_\alpha(X) \to \mathcal{H}_\omega^\infty(Y)$$

is compact.

For each positive integer n, let $r_n = \frac{n}{n+1}$ and consider the bounded operator $K_n : \mathcal{B}_\alpha(X) \to \mathcal{B}_\alpha^0(X)$ defined by $(K_n f)(z) = f(r_n z)$. It is straightforward to see that for every $f \in \mathcal{B}_\alpha(X)$, $K_n f \to f$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Furthermore, for every $f \in \mathcal{B}_\alpha^0(X)$,

$$\|K_n f - f\|_{\mathcal{B}_\alpha(X)} \to 0,$$

as $n \to \infty$. Fixing $n \geq 0$ and $f \in \mathcal{B}_\alpha(X)$ with $\|f\|_{\mathcal{B}_\alpha(X)} \leq 1$, for every m we have

$$\begin{aligned}
\|W_{\psi,\varphi}(K_n f - P_m K_n f)\|_{\omega,Y} &\leq \|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \|K_n f - P_m K_n f\|_{\mathcal{B}_\alpha(X)} \\
&\leq \|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \sum_{k=m+1}^{\infty} \frac{\|(K_n f)^{(k)}(0)\|_X}{k!} \|z^k\|_{\mathcal{B}_\alpha} \\
&\leq \|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \sum_{k=m+1}^{\infty} \frac{r_n^k \|f^{(k)}(0)\|_X}{(k-1)!}.
\end{aligned} \tag{3.5}$$

Let $r_n < \rho_n < 1$ for some ρ_n. Then by the Cauchy integral formula

$$\begin{aligned}
\frac{r_n^k \|f^{(k)}(0)\|_X}{(k-1)!} &\leq r_n \left(\frac{r_n}{\rho_n}\right)^{k-1} \sup_{|\xi|=\rho_n} \|f'(\xi)\|_X \\
&\leq \left(\frac{r_n}{\rho_n}\right)^k \frac{1}{(1-\rho_n^2)^\alpha} \sup_{|\xi|=\rho_n} (1-|\xi|^2)^\alpha \|f'(\xi)\|_X \\
&\leq \left(\frac{r_n}{\rho_n}\right)^k \frac{1}{(1-\rho_n^2)^\alpha}.
\end{aligned}$$

Thus, (3.5) implies that

$$\|(W_{\psi,\varphi} K_n - W_{\psi,\varphi} P_m K_n) f\|_{\omega,Y} \leq \|W_{\psi,\varphi}\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \frac{1}{(1-\rho_n^2)^\alpha} \sum_{k=m+1}^{\infty} \left(\frac{r_n}{\rho_n}\right)^k,$$

for each $f \in \mathcal{B}_\alpha(X)$ with $\|f\|_{\mathcal{B}_\alpha(X)} \leq 1$. Therefore,

$$\lim_{m\to\infty} \|W_{\psi,\varphi} K_n - W_{\psi,\varphi} P_m K_n\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} = 0,$$

which ensures the compactness of $W_{\psi,\varphi} K_n : \mathcal{B}_\alpha(X) \to \mathcal{H}_\omega^\infty(Y)$.

Fixing $0 < \delta < 1$, we have

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} &\leq \lim_{n\to\infty} \|W_{\psi,\varphi} - W_{\psi,\varphi}K_n\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \\ &= \lim_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} \|(W_{\psi,\varphi} - W_{\psi,\varphi}K_n)f\|_{\omega,Y} \\ &\leq \lim_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} \sup_{|\varphi(z)|\leq\delta} \omega(z)\|\psi_z\|_{X\to Y}\|f(\varphi(z)) - f(r_n\varphi(z))\|_X \\ &\quad + \lim_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} \sup_{|\varphi(z)|>\delta} \omega(z)\|\psi_z\|_{X\to Y}\|f(\varphi(z)) - f(r_n\varphi(z))\|_X \\ &= \lim_{n\to\infty}(I_{\delta,n} + J_{\delta,n}).\end{aligned}$$

Noticing that for every $\zeta \in \mathbb{D}$ and each $f \in \mathcal{B}_\alpha(X)$,

$$\begin{aligned}\|f(\zeta) - f(r_n\zeta)\|_X &= \| \int_{r_n}^1 \zeta f'(\zeta t)dt\|_X \\ &\leq \int_{r_n}^1 \frac{|\zeta|}{(1-t^2|\zeta|^2)^\alpha}\|f'(\zeta t)\|_X(1-t^2|\zeta|^2)^\alpha dt \\ &\leq \|f\|_{\mathcal{B}_\alpha(X)} \int_{r_n}^1 \frac{|\zeta|}{(1-t^2|\zeta|^2)^\alpha}dt, \end{aligned} \tag{3.6}$$

we observe

$$\sup_{|\zeta|\leq\delta} \|f(\zeta) - f(r_n\zeta)\|_X \leq \|f\|_{\mathcal{B}_\alpha(X)} \frac{\delta}{(1-\delta^2)^\alpha}(1-r_n).$$

The boundedness of $W_{\psi,\varphi}$ ensures that $\psi \in \mathcal{H}_\omega^\infty(\mathcal{L}(X,Y))$. Thus, for every $\alpha > 0$ we have

$$\begin{aligned}I_{\delta,n} &\leq \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} \|\psi\|_{\omega,\mathcal{L}(X,Y)} \sup_{|\zeta|\leq\delta} \|f(\zeta) - f(r_n\zeta)\|_X \\ &\leq \|\psi\|_{\omega,\mathcal{L}(X,Y)} \frac{\delta}{(1-\delta^2)^\alpha}(1-r_n) \to 0,\end{aligned}$$

as $n \to \infty$. For $J_{\delta,n}$, in the case $0 < \alpha < 1$, since

$$\int_{r_n}^1 \frac{|\zeta|}{(1-t|\zeta|)^\alpha}dt = \frac{(1-r_n|\zeta|)^{1-\alpha} - (1-|\zeta|)^{1-\alpha}}{1-\alpha} \leq \frac{(1-r_n)^{1-\alpha}}{1-\alpha},$$

by (3.6) we obtain

$$J_{\delta,n} \leq \|\psi\|_{\omega,\mathcal{L}(X,Y)} \frac{(1-r_n)^{1-\alpha}}{1-\alpha} \to 0, \quad \text{as} \quad n \to \infty.$$

By relation (1.1) when $\alpha = 1$,

$$\lim_{n\to\infty} J_{\delta,n} \lesssim \sup_{|\varphi(z)|>\delta} \omega(z)\|\psi_z\|_{X\to Y} \log \frac{2}{1-|\varphi(z)|^2},$$

and for $\alpha > 1$,

$$\lim_{n\to\infty} J_{\delta,n} \lesssim \sup_{|\varphi(z)|>\delta} \frac{\omega(z)\|\psi_z\|_{X\to Y}}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

Letting $\delta \to 1$ we observe that

$$\|W_{\psi,\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\omega^\infty(Y)} \lesssim \begin{cases} 0 & 0<\alpha<1 \\ \limsup\limits_{|\varphi(z)|\to 1} \omega(z)\log\frac{2}{1-|\varphi(z)|^2}\|\psi_z\|_{X\to Y} & \alpha=1 \\ \limsup\limits_{|\varphi(z)|\to 1} \frac{\omega(z)\|\psi_z\|_{X\to Y}}{(1-|\varphi(z)|^2)^{\alpha-1}} & \alpha>1 \end{cases}.$$

□

In the next lemma, we show that the compact open topology on $\mathcal{B}_\alpha^0(X)$ is stronger than the weak topology.

Lemma 3.4 *Let $\alpha > 0$ and $\{f_n\}$ be a bounded sequence in $\mathcal{B}_\alpha^0(X)$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Then $\{f_n\}$ converges weakly to zero.*

Proof For every $f \in \mathcal{B}_\alpha^0(X)$, define the function $\tilde{f}(z) = (1-|z|^2)^\alpha f'(z)$ on $\mathbb{D}$ and consider

$$\|\tilde{f}\|_\infty = \sup_{z\in\mathbb{D}} \|\tilde{f}(z)\|_X = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \|f'(z)\|_X = \|f-f(0)\|_{\mathcal{B}_\alpha(X)}.$$

Then $\tilde{\mathcal{B}_\alpha}(X) = \{\tilde{f} : f \in \mathcal{B}_\alpha^0(X)\}$ is a subspace of $C_0(\mathbb{D}, X)$. Let T be a bounded linear functional on $\mathcal{B}_\alpha^0(X)$. Define $\tilde{T}(\tilde{f}) = T(f-f(0))$. Clearly, $\tilde{T}$ is a well-defined bounded linear functional on $\tilde{\mathcal{B}_\alpha}(X)$. Using the Hahn–Banach theorem and the general form of a uniform continuous linear functional on $C_0(\mathbb{D}, X)$, we have $T(f-f(0)) = \int_{\mathbb{D}} \tilde{f} d\mu$, for some measure $\mu \in M(\mathbb{D}, X^*)$ and every $f \in \mathcal{B}_\alpha^0(X)$ (see [14, Lemma 4]). Fixing $\epsilon > 0$, let $\{r_m\}$ be an increasing sequence in $(0,1)$ converging to 1 and $D_m = \{z \in \mathbb{D} : |z| \le r_m\}$. Then $\mathbb{D} = \cup_{m=1}^\infty D_m$ and $|\mu|(\mathbb{D}\setminus D_m) < \frac{\epsilon}{2}$ for some m. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}_\alpha^0(X)$ with $\|f_n\|_{\mathcal{B}_\alpha(X)} \le 1$ and converging to zero uniformly on compact subsets of $\mathbb{D}$. Then $\lim_{n\to\infty}\sup_{z\in D_m}\|f_n'(z)\|_X = 0$ and $\sup_{z\in D_m}\|\tilde{f}_n(z)\|_X < \frac{\epsilon}{2\|\mu\|}$ for n sufficiently large. Therefore,

$$\begin{aligned} |T(f_n - f_n(0))| &\le \Big|\int_{\mathbb{D}\setminus D_m} \tilde{f}_n d\mu\Big| + \Big|\int_{D_m} \tilde{f}_n d\mu\Big| \\ &\le \int_{\mathbb{D}\setminus D_m} \|\tilde{f}_n(z)\|_X d|\mu|(z) + \int_{D_m} \|\tilde{f}_n(z)\|_X d|\mu|(z) \\ &\le |\mu|(\mathbb{D}\setminus D_m) + |\mu|(D_m)\frac{\epsilon}{2\|\mu\|} < \epsilon, \end{aligned}$$

which implies that $\lim_{n\to\infty} T(f_n - f_n(0)) = 0$. Since $\lim_{n\to\infty} f_n(0) = 0$, we obtain $\lim_{n\to\infty} T(f_n) = 0$ and hence $\{f_n\}$ converges weakly to zero as desired. □

We are now ready to characterize compact weighted composition operators between vector-valued Bloch-type spaces.

Theorem 3.5 *For $0<\alpha<1$ the bounded weighted composition operator $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ is compact if and only if*

(i) $T_{\psi'},T_{\varphi'\psi}:X\to\mathcal{H}^\infty_\beta(Y)$ *and* $\psi_0:X\to Y$ *are compact,*

(ii) $\limsup_{n\to\infty} n^{\alpha-1}\|I_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))}=\limsup_{n\to\infty}\frac{\|\varphi'\varphi^n\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_\alpha}=0.$

Proof Let $\alpha>0$ and $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ be compact. It can be easily seen that ψ_0 is compact. For showing the compactness of $T_{\psi'},T_{\varphi'\psi}:X\to\mathcal{H}^\infty_\beta(Y)$, let $\{x_n\}$ be a bounded sequence in X with $\|x_n\|_X\le 1$. We consider $f_n(z)=x_n$ and $g_n(z)=x_nz$ for each positive integer n and every $z\in\mathbb{D}$. Then $\|f_n\|_{\mathcal{B}_\alpha(X)}=\|g_n\|_{\mathcal{B}_\alpha(X)}=\|x_n\|_X$. Using the compactness of $W_{\psi,\varphi}$ and passing to subsequences of $\{f_n\}$ and $\{g_n\}$ if necessary, we may assume that $\{W_{\psi,\varphi}f_n\}$ and $\{W_{\psi,\varphi}g_n\}$ are Cauchy sequences in $\mathcal{B}_\beta(Y)$. Then

$$\begin{aligned}\|T_{\psi'}x_n-T_{\psi'}x_m\|_{\beta,Y}&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|\psi'_zx_n-\psi'_zx_m\|_Y\\&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|(W_{\psi,\varphi}(f_n-f_m))'(z)\|_Y\to 0,\end{aligned}$$

as $n,m\to\infty$. Thus, $\{T_{\psi'}x_n\}$ is a Cauchy sequence in $\mathcal{H}^\infty_\beta(Y)$, which implies that $T_{\psi'}:X\to\mathcal{H}^\infty_\beta(Y)$ is compact. Similarly,

$$\begin{aligned}\|T_{\varphi'\psi}x_n-T_{\varphi'\psi}x_m\|_{\beta,Y}=&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|\varphi'(z)\psi_z(x_n-x_m)\|_Y\\\le&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|(\varphi'(z)\psi_z+\varphi(z)\psi'_z)(x_n-x_m)\|_Y\\&+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi(z)|\|\psi'_z(x_n-x_m)\|_Y\\\le&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|(W_{\psi,\varphi}(g_n-g_m))'(z)\|_Y\\&+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\|(W_{\psi,\varphi}(f_n-f_m))'(z)\|_Y\to 0,\end{aligned}$$

as $n,m\to\infty$. Thus, $\{T_{\varphi'\psi}x_n\}$ is a Cauchy sequence in $\mathcal{H}^\infty_\beta(Y)$, and therefore $T_{\varphi'\psi}:X\to\mathcal{H}^\infty_\beta(Y)$ is compact.

Suppose that $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ is compact and (ii) does not hold. Then by Corollary 2.6 and Lemma 3.1, for some $\varepsilon>0$ there exists a sequence $\{z_n\}$ of $\mathbb{D}$ such that $|\varphi(z_n)|>\frac{1}{2}$, $|\varphi(z_n)|\to 1$ and

$$\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha}|\varphi'(z_n)|\|\psi_{z_n}\|_{X\to Y}\ge\varepsilon,$$

for each n. Let $\{x_n\}$ be a sequence in X such that $\|x_n\|_X\le 1$ and $\frac{n}{n+1}\|\psi_{z_n}\|_{X\to Y}<\|\psi_{z_n}(x_n)\|_Y$. Defining $f_n=K^{2,\alpha+1}_{z_n}-K^{1,\alpha}_{z_n}$, for every n, we see that

$$f_n(\varphi(z_n))=0\quad\text{and}\quad f'_n(\varphi(z_n))=\frac{\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^\alpha}.$$

Moreover, $\{f_n\}$ is a bounded sequence in $\mathcal{B}^0_\alpha$ converging to zero on compact subsets. Hence, $\{f_nx_n\}$ converges to zero uniformly on compact subsets. Lemma 3.4 implies that $\{f_nx_n\}$ converges weakly to zero, and since

$W_{\psi,\varphi}$ is compact, $\|W_{\psi,\varphi}(f_n x_n)\|_{\mathcal{B}_\beta(Y)} \to 0$, as $n \to \infty$. On the other hand,

$$\begin{aligned}\lim_{n\to\infty} \|W_{\psi,\varphi}(f_n x_n)\|_{\mathcal{B}_\beta(Y)} &\geq \lim_{n\to\infty} (1-|z_n|^2)^\beta \|(W_{\psi,\varphi}(f_n x_n))'(z_n)\|_Y \\ &= \lim_{n\to\infty} (1-|z_n|^2)^\beta \|\varphi'(z_n) f_n'(\varphi(z_n))\psi_{z_n}(x_n)\|_Y \\ &= \lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |\varphi(z_n)||\varphi'(z_n)|\|\psi_{z_n}(x_n)\|_Y \\ &\geq \lim_{n\to\infty} \frac{n}{n+1} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |\varphi(z_n)||\varphi'(z_n)|\|\psi_{z_n}\|_{X\to Y} \\ &\geq \varepsilon/2,\end{aligned}$$

which is impossible. Thus, (ii) holds.

For the converse, let (i) and (ii) hold. For each $f \in \mathcal{B}_\alpha(X)$, we consider $W_{\psi,\varphi}f(0) \in Y$ as a constant function in $\mathcal{B}_\beta(Y)$ and define the bounded operator $\Lambda : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ by $\Lambda f = 1 W_{\psi,\varphi}f(0)$, and we have

$$\begin{aligned}\|W_{\psi,\varphi} - \Lambda\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)} &= \inf \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} (\|Kf(0)\|_Y + \|D(W_{\psi,\varphi} - \Lambda - K)f\|_{\beta,Y}) \\ &= \inf \sup_{\|f\|_{\mathcal{B}_\alpha(X)}\leq 1} \|D(W_{\psi,\varphi} - \Lambda - K)f\|_{\beta,Y} \\ &= \inf \|D(W_{\psi,\varphi} - \Lambda - K)\|_{\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)},\end{aligned}$$

where the infimum is taken over all $K \in \mathcal{K}(\mathcal{B}_\alpha(X), \mathcal{B}_\beta(Y))$. Arguing as at the beginning of Section 2, we obtain that

$$\begin{aligned}\|W_{\psi,\varphi} - \Lambda\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)} &= \|D(W_{\psi,\varphi} - \Lambda)\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)} \\ &= \|DW_{\psi,\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)} \\ &\leq \|W_{\varphi'\psi,\varphi}\|_{e;\mathcal{H}_\alpha^\infty(X)\to\mathcal{H}_\beta^\infty(Y)} + \|W_{\psi',\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)}.\end{aligned} \tag{3.7}$$

Since $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is bounded, by Corollary 2.6, the operators $W_{\psi',\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{H}_\beta^\infty(Y)$ and $W_{\varphi'\psi,\varphi} : \mathcal{H}_\alpha^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ are bounded. Since $T_{\psi'}$ is compact, by Theorem 3.3 we have

$$\|W_{\psi',\varphi}\|_{e;\mathcal{B}_\alpha(X)\to\mathcal{H}_\beta^\infty(Y)} = 0.$$

Since $T_{\varphi'\psi}$ is compact and (ii) holds, by Theorem 3.2 we deduce that $W_{\varphi'\psi,\varphi} : \mathcal{H}_\alpha^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ is compact. The relation (3.7) ensures that

$$W_{\psi,\varphi} - \Lambda : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$$

is compact. It is straightforward to see that the compactness of $\psi_0 : X \to Y$ implies the compactness of Λ. Therefore, $W_{\psi,\varphi}$ is compact and the proof is complete. □

Theorem 3.6 *The bounded weighted composition operator $W_{\psi,\varphi} : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ is compact if and only if*

(i) $T_{\psi'}, T_{\varphi'\psi} : X \to \mathcal{H}_\beta^\infty(Y)$ and $\psi_0 : X \to Y$ are compact,

(ii) $\limsup_{n\to\infty} \|I_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} = \limsup_{n\to\infty} \frac{\|\varphi' \varphi^n \psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_1} = 0,$

(iii) $\limsup_{n\to\infty} \log n \|J_\psi \varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} = \limsup_{n\to\infty} \frac{\|\varphi^n \psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\log}} = 0.$

Proof The necessity of (i) and (ii) can be proved in the same way as in the proof of Theorem 3.5. Assume that (iii) does not hold. Then by Corollary 2.6 and Lemma 3.1, for some $\varepsilon > 0$ we may choose $\{z_n\}$ such that $|\varphi(z_n)| > \frac{1}{2}$, $|\varphi(z_n)| \to 1$, and

$$(1-|z_n|^2)^\beta \|\psi'_{z_n}\|_{X\to Y} \log \frac{2}{1-|\varphi(z_n)|^2} \geq \varepsilon,$$

for each n. Let $\{x_n\}$ be a sequence in X such that $\|x_n\|_X \leq 1$ and $\frac{n}{n+1}\|\psi'_{z_n}\|_{X\to Y} < \|\psi'_{z_n}(x_n)\|_Y$. Defining

$$g_n = \frac{3\lambda_{z_n}^2}{\lambda_{z_n}(\varphi(z_n))} - \frac{2\lambda_{z_n}^3}{(\lambda_{z_n}(\varphi(z_n)))^2},$$

we observe that $\{g_n\}$ is a bounded sequence in $\mathcal{B}^0$ converging to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.4, $\{g_n x_n\}$ converges to zero, weakly as $n \to \infty$. Furthermore,

$$g_n(\varphi(z_n)) = \log \frac{2}{1-|(\varphi(z_n))|^2} \quad \text{and} \quad g_n'(\varphi(z_n)) = 0.$$

Therefore,

$$\begin{aligned}
\varepsilon &\leq \limsup_{n\to\infty} (1-|z_n|^2)^\beta \|\psi'_{z_n}\|_{X\to Y} \log \frac{2}{1-|\varphi(z_n)|^2} \\
&\leq \limsup_{n\to\infty} (1-|z_n|^2)^\beta \|\psi'_{z_n} x_n\|_Y |g_n(\varphi(z_n))| \\
&= \limsup_{n\to\infty} (1-|z_n|^2)^\beta \|(W_{\psi,\varphi}(g_n x_n))'(z_n)\|_Y \\
&\leq \limsup_{n\to\infty} \|W_{\psi,\varphi}(g_n x_n)\|_{\mathcal{B}_\beta(Y)} = 0,
\end{aligned}$$

which is a contradiction. Thus, (iii) holds.

Conversely, let (i)–(iii) hold. By Corollary 2.6, the operators $W_{\psi',\varphi} : \mathcal{B}(X) \to \mathcal{H}_\beta^\infty(Y)$ and $W_{\varphi'\psi,\varphi} : \mathcal{H}_1^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ are bounded. Since $T_{\varphi'\psi}$ is compact and (ii) holds, by Theorem 3.2 we deduce that $W_{\varphi'\psi,\varphi} : \mathcal{H}_1^\infty(X) \to \mathcal{H}_\beta^\infty(Y)$ is compact. The compactness of $T_{\psi'}$, relation (iii), and Theorem 3.3 imply that $W_{\psi',\varphi} : \mathcal{B}(X) \to \mathcal{H}_\beta^\infty(Y)$ is compact. Therefore, we can see that in a similar way to the proof of Theorem 3.5, by relation (3.7) we have that $W_{\psi,\varphi} - \Lambda : \mathcal{B}(X) \to \mathcal{B}_\beta(Y)$ is compact. By the compactness of $\psi_0 : X \to Y$ we get that Λ is compact and hence $W_{\psi,\varphi}$ is compact, too. □

Theorem 3.7 *For $\alpha > 1$, the bounded weighted composition operator $W_{\psi,\varphi} : \mathcal{B}_\alpha(X) \to \mathcal{B}_\beta(Y)$ is compact if and only if*

(i) $T_{\psi'}, T_{\varphi'\psi} : X \to \mathcal{H}_\beta^\infty(Y)$ and $\psi_0 : X \to Y$ are compact,

(ii) $\limsup_{n\to\infty} n^{\alpha-1}\|I_\psi\varphi^n\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} = \limsup_{n\to\infty} \frac{\|\varphi'\varphi^n\psi\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_\alpha} = 0,$

(iii) $\limsup_{n\to\infty} n^{\alpha-1}\|J_\psi\varphi^{n-1}\|_{\mathcal{B}_\beta(\mathcal{L}(X,Y))} = \limsup_{n\to\infty} \frac{\|\varphi^n\psi'\|_{\beta,\mathcal{L}(X,Y)}}{\|z^n\|_{\alpha-1}} = 0.$

Proof We prove the necessity of (iii). Let $W_{\psi,\varphi}:\mathcal{B}_\alpha(X)\to\mathcal{B}_\beta(Y)$ be compact and assume that (iii) does not hold. Then for some $\varepsilon>0$, we may choose sequences $\{z_n\}$ and $\{x_n\}$ as in the proof of Theorem 3.6 such that $\frac{n}{n+1}\|\psi'_{z_n}\|_{X\to Y} < \|\psi'_{z_n}(x_n)\|_Y$ and

$$\frac{(1-|z_n|^2)^\beta\|\psi'_{z_n}\|_{X\to Y}}{(1-|\varphi(z_n)|^2)^{\alpha-1}} \geq \varepsilon.$$

Defining

$$h_n = \alpha K^{0,\alpha-1}_{z_n} - (\alpha-1)K^{1,\alpha}_{z_n},$$

we observe that $\{h_n\}$ is a bounded sequence in $\mathcal{B}^0_\alpha$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Therefore, $\{h_n x_n\}$ converges to zero, weakly as $n\to\infty$. Since $h_n(\varphi(z_n)) = \frac{1}{(1-|(\varphi(z_n))|^2)^{\alpha-1}}$ and $h'_n(\varphi(z_n))=0$, we have

$$\begin{aligned}
\varepsilon &\leq \limsup_{n\to\infty} \frac{(1-|z_n|^2)^\beta\|\psi'_{z_n}\|_{X\to Y}}{(1-|\varphi(z_n)|^2)^{\alpha-1}} \\
&\leq \limsup_{n\to\infty} (1-|z_n|^2)^\beta\|\psi'_{z_n}x_n\|_Y|h_n(\varphi(z_n))| \\
&= \limsup_{n\to\infty} (1-|z_n|^2)^\beta\|(W_{\psi,\varphi}(h_nx_n))'(z_n)\|_Y \\
&\leq \limsup_{n\to\infty} \|W_{\psi,\varphi}(h_nx_n)\|_{\mathcal{B}_\beta(Y)} = 0,
\end{aligned}$$

which is a contradiction. Thus, (iii) holds.

Conversely, we can proceed in the same way as in the proof of Theorem 3.6. □

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