[Turkish Journal of Mathematics](https://dctubitak.researchcommons.org/math)

Volume 43 | Number 2 Article 33

1-1-2019

Conformal changes of generalized paracontact pseudometric structures

FERESHTEH MALEK

NILOUFAR HOSSEINPOUR KASHANI

Follow this and additional works at: https://dctubitak.researchcommons.org/math

Part of the Mathematics Commons

Recommended Citation

MALEK, FERESHTEH and KASHANI, NILOUFAR HOSSEINPOUR (2019) "Conformal changes of generalized paracontact pseudometric structures," *Turkish Journal of Mathematics*: Vol. 43: No. 2, Article 33.
https://doi.org/10.3906/mat-1809-123
Available at: https://dctubitak.researchcommons.org/math/vol43/iss2/33

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1018 – 1035
© TÜBİTAK
doi:10.3906/mat-1809-123

Conformal changes of generalized paracontact pseudometric structures

Fereshteh MALEK, **Niloufar HOSSEINPOUR KASHANI***
Department of Pure Mathematics, Faculty of Mathematical Science, K. N. Toosi University of Technology, Tehran, Iran

Received: 28.09.2018 • **Accepted/Published Online:** 02.03.2019 • **Final Version:** 27.03.2019

Abstract: We characterize generalized almost paracontact pseudometric structures after a conformal change and investigate invariant components of these structures under conformal changes. Then we characterize the conformal changes of normal generalized paracontact pseudometric structures, generalized almost para-co-Kähler structures, and generalized para-co-Kähler structures. We also give examples of a generalized almost para-co-Kähler structure and a normal generalized paracontact pseudometric structure that remains invariant under a nonhomothety conformal change.

Key words: Conformal change, generalized almost paracontact pseudometric structure, generalized almost para-co-Kähler structure, generalized para-co-Kähler structure, normal generalized paracontact pseudometric structure

1. Introduction

Hitchin in [3] introduced the notion of generalized complex structures and Gualtieri developed it into generalized Kähler structures [2].

On the other hand, in odd dimensions, Vaisman introduced generalized almost contact structures and defined generalized Sasakian structures from the viewpoint of generalized Kähler structures [8,11]. He also defined conformal changes of generalized complex structures and investigated invariant generalized geometry under conformal changes [9]. Poon and Wade expanded the study of these structures and investigated integrability conditions of generalized almost contact structures [6,7].

With the definition of generalized almost paracontact structures by Sahin and Sahin in [7], and the definition of generalized almost para-Hermitian structures and generalized para-Kähler structures by Vaisman in [11], these discussions entered the realm of generalized para-structures. Then it was natural that the authors introduced the odd-dimensional analogs of generalized almost para-Hermitian structures and generalized para-Kähler structures, which are called normal generalized paracontact pseudometric structures and generalized para-co-Kähler structures [4]. They showed that generalized almost paracontact pseudometric structures $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ are in a one-to-one correspondence with the quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, where γ is a pseudo-Riemannian metric of M, ψ is a 2-form, and φ is a $(1,1)$-tensor field such that $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$, $\gamma(\varphi X, Y) = -\gamma(X, \varphi Y)$, $\gamma^\sharp(\eta) = \xi$, and $\psi^\flat(\xi) = 0$. They also gave equivalent conditions by which H and $\bar{H}$, the eigenspaces of $\mathcal{H}$, are closed under the Courant bracket [4].

Accordingly, our purpose in this paper is to use these correspondences and investigate odd-dimensional

*Correspondence: nkashani@mail.kntu.ac.ir
2010 *AMS Mathematics Subject Classification:* 53D18, 53D15

This work is licensed under a Creative Commons Attribution 4.0 International License.

generalized parastructures under conformal changes.

This paper is divided into four sections. In Section 2, we recall the needed background, including definitions and theorems about generalized structures. In Section 3, we characterize generalized almost paracontact pseudometric structures after a conformal change and investigate invariant components of these structures under conformal changes. In Section 4, we characterize conformal changes of normal generalized paracontact pseudometric structures, generalized almost para-co-Kähler structures, and generalized para-co-Kähler structures, respectively. Also, we give examples of a generalized almost para-co-Kähler structure and a normal generalized paracontact pseudometric structure that remains invariant under a nonhomothety conformal change.

2. Preliminaries

To study the big tangent bundle $\mathbb{T}M = TM \oplus TM^*$ on a smooth manifold M, a natural inner product needs to be taken on $\mathbb{T}M = TM \oplus TM^*$, defined by

$$\langle X+\alpha, Y+\beta\rangle = \mathrm{g}(X+\alpha, Y+\beta) = \frac{1}{2}(\beta(X)+\alpha(Y)), \tag{1}$$

and the Courant bracket by

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \$_X\beta - \$_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta), \tag{2}$$

where $X, Y \in TM$ and $\alpha, \beta \in TM^*$.

A generalized almost complex structure on M is defined by an endomorphism $\mathcal{J}$ of $TM \oplus TM^*$ such that $\mathcal{J} + \mathcal{J}^* = 0$ and $\mathcal{J}^2 = -\mathrm{Id}$. Furthermore, $\mathcal{J}$ is called a generalized complex structure, or $\mathcal{J}$ is integrable, if the Nijenhuis torsion of $\mathcal{J}$ vanishes, i.e.

$$\begin{aligned} N_{\mathcal{J}}(X+\alpha, Y+\beta) :=& [\![\mathcal{J}(X+\alpha), \mathcal{J}(Y+\beta)]\!] + \mathcal{J}^2[\![X+\alpha, Y+\beta]\!] \\ &- \mathcal{J}[\![X+\alpha, \mathcal{J}(Y+\beta)]\!] - \mathcal{J}[\![\mathcal{J}(X+\alpha), Y+\beta]\!] = 0, \end{aligned}$$

where the brackets are Courant brackets.

The analog of the generalized almost complex structures of odd-dimensional spaces is generalized almost contact structures. For defining almost contact metric normal structures in generalized cases, we need to recall some important formal definitions.

A smooth manifold M^{2n+1} has an almost contact metric structure (φ, ξ, η, g) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η and a Riemannian metric g, satisfying the following conditions:

$$\begin{gathered} \varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \quad \eta(\xi) = 1, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{gathered} \tag{3}$$

for vector fields X and Y on M. The associated fundamental 2-form is defined by $\Theta(X,Y) = g(X, \varphi Y)$. One can now define a contact metric manifold as an almost contact metric structure (φ, ξ, η, g) such that $\Theta = \mathrm{d}\eta$. Furthermore, an almost contact metric structure on M is called normal if the Nijenhuis tensor of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] - \varphi[X,\varphi Y] - \varphi[\varphi X, Y] + [\varphi X, \varphi Y],$$

satisfies $N_\varphi = -2\xi\otimes\mathrm{d}\eta$ [1].

We can now give the definition of generalized structures on odd-dimensional spaces.

As in [6], a pair $(\Phi, Z+\eta)$ defines a generalized almost contact structure if it consists of a bundle endomorphism Φ from $TM \oplus TM^*$ to itself and a section $Z+\eta$ of $TM \oplus TM^*$ such that

$$\Phi + \Phi^* = 0, \ \ \Phi^2 = -\mathrm{Id} + Z \odot \eta,$$
$$\eta(Z) = 1, \ \ \Phi(Z) = 0 \ \ and \ \Phi(\eta) = 0, \tag{4}$$

and $Z \odot \eta(X+\alpha) := \eta(X)Z + \alpha(Z)\eta$, for any $X+\alpha \in \Gamma(\mathbb{TM})$.

Since Φ satisfies $\Phi^3 + \Phi = 0$ and is linearly extended to the complexities bundle $\mathbb{T}M \otimes \mathbb{C}$, one can see that Φ has 0 as well as $\pm \mathrm{i}$ as eigenvalues. For identifying the corresponding eigenbundles of Φ, consider $E^{(1,0)}$ and $E^{(0,1)}$ as follows:

$$E^{(1,0)} = \{X+\alpha - \mathrm{i}\Phi(X+\alpha) | X+\alpha \in \ker\eta \oplus \ker Z\},$$
$$E^{(0,1)} = \{X+\alpha + \mathrm{i}\Phi(X+\alpha) | X+\alpha \in \ker\eta \oplus \ker Z\}. \tag{5}$$

Then the corresponding eigenbundles of Φ are

$$L_Z \oplus L_\eta, \ E^{(1,0)}, \ \ and \ \ E^{(0,1)}, \tag{6}$$

respectively, where L_Z and L_η are the complex vector bundles of rank 1 generated by Z and η. From (6), we define

$$L := L_Z \oplus E^{(1,0)}, \quad L^* := L_\eta \oplus E^{(0,1)},$$
$$\bar{L} := L_Z \oplus E^{(0,1)}, \quad \bar{L}^* := L_\eta \oplus E^{(1,0)}. \tag{7}$$

The generalized almost contact pair $(\Phi, Z+\eta)$ is called a generalized contact structure, or $(\Phi, Z+\eta)$ is integrable, if L is involutive, which means that the space sections of the subbundle L are closed under the Courant bracket. Moreover, if both L and L^* are involutive, the pair $(\Phi, Z+\eta)$ is called a strong generalized contact structure and the strong generalized contact structure $(\Phi, Z+\eta)$ is called a normal generalized contact structure if $\$_Z\eta = 0$ [6].

If the bundle map $\Phi : \mathbb{T}M \to \mathbb{T}M$ is given by

$$\Phi = \begin{pmatrix} P & \phi^\sharp \\ \theta^\flat & -P^* \end{pmatrix},$$

then in terms of components, one sees that a generalized almost contact pair is equivalent to a quintuple $(P, \phi^\sharp, \theta^\flat, Z, \eta)$, where Z is a vector field, η a 1-form, P a $(1,1)$-tensor field, π a bivector field, and θ a 2-form and according to (4) they satisfy the following relations:

$$i)\ P^2 + \phi^\sharp\theta^\flat = -\mathrm{Id} + Z \otimes \eta, \ \ ii)\ P^{*2} + \theta^\flat\phi^\sharp = -\mathrm{Id} + \eta \otimes Z, \tag{8}$$

$$i)\ \theta(PX, Y) = \theta(X, PY), \ \ ii)\ \phi(\alpha, P^*\beta) = \phi(P^*\alpha, \beta), \tag{9}$$

$$i)\ \eta \circ P = 0, \ \ ii)\ \eta \circ \phi^\sharp = 0 \ \ iii)\ i_Z P = 0, \ \ iv)\ i_Z\theta = 0, \ \ v)\ i_Z\eta = 1. \tag{10}$$

In this classical form, the normalization conditions of $(P, \phi^\sharp, \theta^\flat, Z, \eta)$ are stated in the following theorem.

Theorem 2.1 *[10] A generalized almost contact pair corresponding to the quintuple $(P, \phi^\sharp, \theta^\flat, \xi, \eta)$ is normal if and only if the following relations are satisfied:*

$$(A_1)\ \frac{1}{2}[\phi, \phi] = \xi \wedge (\phi^\sharp \otimes \phi^\sharp)\mathrm{d}\eta, \quad [\xi, \phi] = -\xi \wedge \phi^\sharp \$_\xi \eta,$$

$$(A_2)\ P^*\{\alpha, \beta\}_\phi = \$_{\phi^\sharp\alpha} P^*\beta - \$_{\phi^\sharp\beta} P^*\alpha - \mathrm{d}\phi(P^*\alpha, \beta),$$

$$(A_3)\ N_P(X, Y) + \mathrm{d}\eta(PX, PY)\xi = \phi^\sharp(i_{X\wedge Y}\mathrm{d}\theta),$$

$$(A_4)\ \mathrm{d}\theta_P(X, Y, Z) = \mathrm{d}\theta(PX, Y, Z) + \mathrm{d}\theta(X, PY, Z) + \mathrm{d}\theta(X, Y, PZ),$$

$$(A_5)\ \$_\xi P = 0, \quad \$_\xi \theta = 0,$$

$$(A_6) \$_\xi \eta = 0, \quad \$_{\phi^\sharp\alpha}\eta = 0,$$

$$(A_7)\ \mathrm{d}\eta(PX, Y) - \mathrm{d}\eta(PY, X) = 0,$$

where the bracket is the Schouten–Nijenhuis bracket, as explained in [8], $\{\alpha, \beta\}_\phi = \$_{\phi^\sharp\alpha}\beta - \$_{\phi^\sharp\beta}\alpha - \mathrm{d}\phi(\alpha, \beta)$, $i_{X\wedge Y}\mathrm{d}\theta = \$_X\theta^\flat(Y) - \$_Y\theta^\flat(X) - \mathrm{d}\theta(X, Y)$, and $\theta_P(X, Y) = \theta(PX, Y)$.

Now let us recall and discuss generalized almost paracontact pseudometric structures.

Definition 2.1 *[7] For an odd-dimensional manifold M, a generalized almost paracontact structure is a pair $(\mathcal{A}, Z + \eta)$, where $\mathcal{A}$ is an endomorphism of $TM \oplus TM^*$, and $Z + \eta$ is a section of $TM \oplus TM^*$, satisfying*

$$\mathcal{A} + \mathcal{A}^* = 0, \ \ \mathcal{A}^2 = \mathrm{Id} - Z \odot \eta,$$
$$\eta(Z) = 1, \ \ \mathcal{A}(Z) = 0 \ \text{and}\ \mathcal{A}(\eta) = 0, \tag{11}$$

and $Z \odot \eta(X + \alpha) := \eta(X)Z + \alpha(Z)\eta$, for any $X + \alpha \in \Gamma(\mathbb{T}M)$.

The endomorphism $\mathcal{A}$ has three eigenvalues, namely 0, 1, and -1, because of $\mathcal{A}^3 = \mathcal{A}$. Identify the corresponding eigenbundles of $\mathcal{A}$ by $E^{(1,0)}$ and $E^{(0,1)}$ as follows:

$$E^{(1,0)}_{\mathcal{A}} = \{X + \alpha - \mathcal{A}(X + \alpha) | X + \alpha \in \ker\eta \oplus \ker Z\},$$
$$E^{(0,1)}_{\mathcal{A}} = \{X + \alpha + \mathcal{A}(X + \alpha) | X + \alpha \in \ker\eta \oplus \ker Z\}.$$

Therefore, the corresponding eigenbundles are $L_Z \oplus L_\eta$, $E^{(1,0)}_{\mathcal{A}}$, and $E^{(0,1)}_{\mathcal{A}}$, where L_Z and L_η are vector bundles of rank 1 generated by Z and η, respectively. Define

$$L_{\mathcal{A}} := L_Z \oplus E^{(1,0)}_{\mathcal{A}}, \quad L^*_{\mathcal{A}} := L_\eta \oplus E^{(0,1)}_{\mathcal{A}},$$
$$\bar{L}_{\mathcal{A}} := L_Z \oplus E^{(0,1)}_{\mathcal{A}}, \quad \bar{L}^*_{\mathcal{A}} := L_\eta \oplus E^{(1,0)}_{\mathcal{A}}.$$

The involutivity of $L_{\mathcal{A}}$ in the generalized almost paracontact pair $(\mathcal{A}, Z + \eta)$ defines a generalized paracontact structure and the involutivity of both $L_{\mathcal{A}}$ and $L^*_{\mathcal{A}}$ define a strong generalized paracontact structure. The strong generalized paracontact structure $(\mathcal{A}, Z + \eta)$ is called a normal generalized paracontact structure if $\$_Z\eta = 0$.

Since $\mathcal{A}$ has a matrix form as $\mathcal{A} = \begin{pmatrix} A & \pi^\sharp \\ \sigma^\flat & -A^* \end{pmatrix}$, one can see that a generalized almost paracontact pair is equivalent to a quintuple $(A, \pi^\sharp, \sigma^\flat, Z, \eta)$, where Z is a vector field, η a 1-form, A a $(1,1)$-tensor field,

π a bivector field, and σ a 2-form and, according to (11), they satisfy the following relations:

$$i)\ A^2+\pi^\sharp\sigma^\flat=\mathrm{Id}-Z\otimes\eta,\ \ ii)\ A^{*2}+\sigma^\flat\pi^\sharp=\mathrm{Id}-\eta\otimes Z,$$
$$i)\ \sigma(AX,Y)=\sigma(X,AY),\ \ ii)\ \pi(\alpha,A^*\beta)=\pi(A^*\alpha,\beta),$$
$$i)\ \eta\circ A=0,\ \ ii)\ \eta\circ\pi^\sharp=0\ \ iii)\ i_ZA=0,\ \ iv)\ i_Z\sigma=0,\ \ v)\ i_Z\eta=1.$$

By using tensors $A,\pi^\sharp$, and $\sigma^\flat$ instead of tensors $P,\phi^\sharp$, and $\theta^\flat$, the normalization conditions of the generalized almost paracontact structure $(A,\pi^\sharp,\sigma^\flat,\xi,\eta)$ are stated similarly to Theorem 2.1 in [7].

2.1. Generalized almost paracontact pseudometric structures

A generalized almost paracontact pseudometric structure is a quadruple $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$, where $(\Phi,\xi+\eta)$ is a generalized almost contact structure, $\mathcal{H}$ is a g-symmetric matrix with $\mathcal{H}^2=-\mathrm{Id}$, $\mathcal{H}(\eta)=\xi$, $\Phi\circ\mathcal{H}=\mathcal{H}\circ\Phi=\mathcal{A}$, and $(\mathcal{A},\xi+\eta)$ is a generalized almost paracontact structure, and

$$\gamma(\alpha,\beta):=2\langle\mathcal{H}(0,\alpha),(0,\beta)\rangle,$$
$$\nu(X,Y):=2\langle\mathcal{H}(X,0),(Y,0)\rangle$$

are nondegenerate [4].

The matrix representations of $\Phi,\mathcal{H},\mathcal{A}$ are

$$\Phi=\begin{pmatrix} P & \phi^\sharp\\ \theta^\flat & -P^*\end{pmatrix},\quad \mathcal{H}=\begin{pmatrix} Q & \gamma^\sharp\\ \nu^\flat & Q^*\end{pmatrix},\quad \mathcal{A}=\begin{pmatrix} A & \pi^\sharp\\ \sigma^\flat & -A^*\end{pmatrix}, \tag{12}$$

where ϕ,θ,π,σ are skew-symmetric, and $\gamma\in End(TM)$ and $\nu\in End(T^*M)$ are the symmetric 2-vector field and symmetric 2-covector field, respectively. Thus, according to the conditions of the above definition, we get

$$\begin{aligned}
&P^2=-\mathrm{Id}-\phi^\sharp\circ\theta^\flat+\eta\otimes\xi,\ \ \phi^\sharp\circ P^*=P\circ\phi^\sharp,\ \ \theta^\flat\circ P=P^*\circ\theta^\flat,\\
&A^2=\mathrm{Id}-\pi^\sharp\circ\sigma^\flat-\eta\otimes\xi,\ \ \pi^\sharp\circ A^*=A\circ\pi^\sharp,\ \ \sigma^\flat\circ A=A^*\circ\sigma^\flat,\\
&Q^2=-\mathrm{Id}-\gamma^\sharp\circ\nu^\flat,\ \ \gamma^\sharp\circ Q^*=-Q\circ\gamma^\sharp,\ \ \nu^\flat\circ Q=-Q^*\circ\nu^\flat,\\
&A=P\circ Q+\phi^\sharp\circ\nu^\flat=Q\circ P+\gamma^\sharp\circ\theta^\flat,\\
&\pi^\sharp=P\circ\gamma^\sharp+\phi^\sharp\circ Q^*=Q\circ\phi^\sharp-\gamma^\sharp\circ P^*,\\
&\sigma^\flat=\nu^\flat\circ P+Q^*\circ\theta^\flat=\theta^\flat\circ Q-P^*\circ\nu^\flat.
\end{aligned}\tag{13}$$

By defining $\tau:T^cM\to T^cM\oplus T^cM^*$ as

$$\tau(X'+\mathrm{i}X''):=(X'+\mathrm{i}X'',-\gamma^\flat(X''+QX')+\mathrm{i}\gamma^\flat(X'-QX'')), \tag{14}$$

one can imply $\mathcal{H}(\tau(X'+\mathrm{i}X''))=\mathrm{i}\tau(X'+\mathrm{i}X'')$, and thus $\tau:T^cM\to H$ is an isomorphism [4], where $H=im(\mathrm{Id}-\mathrm{i}\mathcal{H})$ is the i-eigenspace of $\mathcal{H}$. We define the 2-form ψ as $\psi^\flat:=-\gamma^\flat\circ Q$, and then from (14) we get

$$\tau(X)=(X,(\psi^\flat+\mathrm{i}\gamma^\flat)X). \tag{15}$$

The following property is a consequence of formula (15):

$$\gamma(X,Y) = -\mathrm{i}\,\langle \tau X, \tau Y\rangle, \tag{16}$$

where $X,Y \in T^cM$ (we should have written γ^{-1}, but we follow the custom of Riemannian geometry).

Similarly, if we define $\bar{\tau} : T^cM \to T^cM \oplus T^cM^*$ by

$$\bar{\tau}(X' + \mathrm{i}X'') := (X' + \mathrm{i}X'', -\gamma^\flat(QX' - X'') - \mathrm{i}\gamma^\flat(X' + QX'')),$$

one can see $\bar{\tau}(X) = (X, (\psi^\flat - \mathrm{i}\gamma^\flat)X)$, and that $\bar{\tau} : T^cM \to \bar{H}$ is an isomorphism in which $\bar{H} = im(\mathrm{Id} + \mathrm{i}\mathcal{H})$ is the $(-\mathrm{i})$-eigenspace of $\mathcal{H}$ and $\gamma(X,Y) = \mathrm{i}\,\langle \bar{\tau} X, \bar{\tau} Y\rangle$.

From what has already been given in the above relations, we have the following theorem.

Theorem 2.2 *[4] The generalized almost paracontact pseudometric structures $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$, introduced by the matrices in (12), are in a one-to-one correspondence with the quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, where $(\varphi, \xi, \eta, \gamma)$ is an almost contact metric structure on M, γ is a pseudo-Riemannian metric of M, and ψ is a 2-form such that $\psi^\flat(\xi) = 0$.*

Using the quintuple associated to the generalized almost paracontact pseudometric structure, the authors gave a necessary and sufficient condition for H and $\bar{H}$ to be closed under the Courant bracket in the following theorem.

Theorem 2.3 *[4] If a generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is associated to a quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, then H and $\bar{H}$ are closed under the Courant bracket if and only if*

$$\gamma(\varphi(\nabla^\gamma_Z \varphi)Y, X) = \frac{1}{2}\{\mathrm{d}\psi(\varphi^2 X, \varphi Y, Z) - \mathrm{d}\psi(\varphi X, Y, Z)\}, \tag{17}$$

where ∇^γ is the Levi-Civita connection of the metric γ and $X, Y, Z \in \Gamma(M)$.

3. Conformal changes of generalized almost paracontact pseudometric structures

The automorphism $C_\epsilon : \mathbb{T}M \longrightarrow \mathbb{T}M$ [9] defined by $C_\epsilon(X, \alpha) := (X, \mathrm{e}^\epsilon \alpha)$ for $\epsilon \in C^\infty(M)$ is called a conformal change of $\mathbb{T}M$ because it produces a conformal change of the natural inner product $\langle , \rangle$ such that

$$\langle C_\epsilon(X + \alpha), C_\epsilon(Y + \beta)\rangle = \mathrm{e}^\epsilon \langle X + \alpha, Y + \beta\rangle.$$

Moreover, if ϵ is locally constant, the change is called a homothety [9].

Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ be a generalized almost paracontact pseudometric structure. Applying the conformal change C_ϵ results in

$$\Phi \mapsto \tilde{\Phi} = C_{-\epsilon} \circ \Phi \circ C_\epsilon, \quad \mathcal{A} \mapsto \tilde{\mathcal{A}} = C_{-\epsilon} \circ \mathcal{A} \circ C_\epsilon.$$

With matrix representation as in (12), this clearly forces

$$\tilde{\Phi} = \begin{pmatrix} P & \mathrm{e}^\epsilon \phi^\sharp \\ \mathrm{e}^{-\epsilon}\theta^\flat & -P^* \end{pmatrix}, \quad \tilde{\mathcal{H}} = \begin{pmatrix} Q & \mathrm{e}^\epsilon \gamma^\sharp \\ \mathrm{e}^{-\epsilon}\nu^\flat & Q^* \end{pmatrix}, \quad \tilde{\mathcal{A}} = \begin{pmatrix} A & \mathrm{e}^\epsilon \pi^\sharp \\ \mathrm{e}^{-\epsilon}\sigma^\flat & -A^* \end{pmatrix}.$$

To communicate between the conformal change of the generalized almost paracontact pseudometric structure $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ and conformal change of the almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma}, \tilde{\psi})$ in the next theorem, we define $\tau_\epsilon : T^cM \to T^cM \oplus T^cM^*$ by

$$\tau_\epsilon(X) = (X, (\tilde{\psi}^\flat + \mathrm{i}\tilde{\gamma}^\flat)X), \tag{18}$$

where $\tilde{\psi}^\flat = \mathrm{e}^{-\epsilon}\psi^\flat$ and $\tilde{\gamma}^\flat = \mathrm{e}^{-\epsilon}\gamma^\flat$. Thus, $im\tau_\epsilon \subset H_\epsilon$, in which $H_\epsilon = im(\mathrm{Id} - \mathrm{i}\tilde{\mathcal{H}})$ is the (i)-eigenspace of $\tilde{\mathcal{H}}$. Also, we define $\bar{\tau}_\epsilon : T^cM \to T^cM \oplus T^cM^*$ by

$$\bar{\tau}_\epsilon(X) = (X, (\tilde{\psi}^\flat - \mathrm{i}\tilde{\gamma}^\flat)X). \tag{19}$$

Similarly, one can see $im\bar{\tau}_\epsilon \subset \bar{H}_\epsilon$, in which $\bar{H}_\epsilon = im(\mathrm{Id} + \mathrm{i}\tilde{\mathcal{H}})$ is the $(-\mathrm{i})$-eigenspace of $\tilde{\mathcal{H}}$. We use (16) and give an interesting utilizable consequence of formulas (18) and (19) as follows:

$$\tilde{\gamma}(X, Y) = -\mathrm{i}\,\langle \tau_\epsilon X, \tau_\epsilon Y\rangle, \quad \tilde{\gamma}(X, Y) = -\mathrm{i}\,\langle \bar{\tau}_\epsilon X, \bar{\tau}_\epsilon Y\rangle, \tag{20}$$

where $X, Y \in T^cM$.

Proposition 3.1 *If the generalized almost paracontact pseudometric structure* $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ *is related to* $(\varphi, \xi, \eta, \gamma, \psi)$, *then its conformal change* $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ *is a generalized almost paracontact pseudometric structure related to* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma}, \tilde{\psi})$ *in which* $\tilde{\xi} = \mathrm{e}^{\frac{\epsilon}{2}}\xi, \tilde{\eta} = \mathrm{e}^{\frac{-\epsilon}{2}}\eta, \tilde{\gamma} = \mathrm{e}^{-\epsilon}\gamma$, *and* $\tilde{\psi} = \mathrm{e}^{-\epsilon}\psi$.

Proof We consider the conformal change $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ of the generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$. An easy consequence of definitions together with (13) show that $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is again a generalized almost paracontact pseudometric structure. Now we show that its related quintuple $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{\psi}, \tilde{\gamma})$ satisfies the desired conditions. From $\tilde{\mathcal{H}}(\tilde{\eta}) = \tilde{\xi}$, we have $\tilde{\gamma}^\flat(\tilde{\eta}) = \tilde{\xi}$ and $\tilde{\psi}^\flat(\tilde{\xi}) = 0$, in which $\tilde{\psi}^\flat := -\mathrm{e}^{-\epsilon}\gamma^\flat \circ Q$. Since $\tilde{\Phi} \circ \tilde{\mathcal{H}} = \tilde{\mathcal{H}} \circ \tilde{\Phi}$, then $\tilde{\Phi}$ preserves $\tilde{\mathcal{H}}$ and leads to a tensor $\tilde{\varphi} \in End(T^cM)$ given by

$$\tilde{\varphi}|_{H_\epsilon} := \tau_\epsilon^{-1}\tilde{\Phi}\tau_\epsilon \quad \textit{and} \quad \tilde{\varphi}|_{\bar{H}_\epsilon} := \bar{\tau}_\epsilon^{-1}\tilde{\Phi}\bar{\tau}_\epsilon. \tag{21}$$

Since $\tau_\epsilon^{-1}\tilde{\Phi}\tau_\epsilon = \tau^{-1}\Phi\tau$, we get $\tilde{\varphi}|_{H_\epsilon} = \varphi|_H$ and $\tilde{\varphi}|_{\bar{H}_\epsilon} = \varphi|_{\bar{H}}$. Then, by (15) and (18), we get

$$\tilde{\varphi}^2 = \varphi^2 = -\mathrm{Id} + \eta \otimes \xi = -\mathrm{Id} + \tilde{\eta} \otimes \tilde{\xi}$$

on H_ϵ and $\bar{H}_\epsilon$. Finally, by using (20) for $X \in H_\epsilon$ or $\bar{H}_\epsilon$, we get $\tilde{\gamma}(\tilde{\varphi}X, Y) = \tilde{\gamma}(X, \tilde{\varphi}Y)$.

□

Now we consider properties of generalized almost paracontact pseudometric structures under conformal changes.

Let us use the Weyl connection, $\tilde{\nabla}^\gamma$, defined by a pseudo-Riemannian metric γ and a 1-form ϖ [5] as follows:

$$\tilde{\nabla}^\gamma_X Y = \nabla^\gamma_X Y - \frac{1}{2}\varpi(X)Y - \frac{1}{2}\varpi(Y)X + \frac{1}{2}\gamma(X, Y)\varpi^\sharp, \tag{22}$$

where ∇^γ is the Levi-Civita connection of γ. The Weyl connection is the Levi-Civita connection of $\mathrm{e}^{-\epsilon}\gamma$ for the smooth function ϵ that satisfies $\mathrm{d}\epsilon = \varpi$, and it is the unique torsionless connection such that $\tilde{\nabla}^\gamma_X\gamma = \varpi(X)\gamma$.

Now we investigate a necessary and sufficient condition with which H_ϵ and $\bar{H}_\epsilon$ are closed under the Courant bracket for conformal change $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$. First we have the following proposition.

Proposition 3.2 *If $(\varphi, \xi, \eta, \gamma, \psi)$ is the quintuple related to the generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$, then H_ϵ and $\bar{H}_\epsilon$, the eigenspaces of $\tilde{\mathcal{H}}$ of its conformal change $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$, are closed under the Courant bracket if and only if the Weyl connection $\tilde{\nabla}^\gamma$ satisfies the following condition:*

$$\begin{aligned}\gamma((\tilde{\nabla}^\gamma_Z \varphi)Y, \varphi X) =& \frac{1}{2}\{\mathrm{d}\psi(\varphi^2 X, \varphi Y, Z) - \mathrm{d}\psi(\varphi X, Y, Z) \\ & - \varpi \wedge \psi(\varphi^2 X, \varphi Y, Z) + \varpi \wedge \psi(\varphi X, Y, Z)\}.\end{aligned} \tag{23}$$

Proof Using Proposition 3.1, the conformal change $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is related to the quintuple $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{\psi}, \tilde{\gamma})$, and thus by considering $\tilde{\varphi}|_{H_\epsilon} = \varphi|_H$, $\tilde{\varphi}|_{\bar{H}_\epsilon} = \varphi|_{\bar{H}}$, and (17), we get

$$\begin{aligned}2\tilde{\gamma}((\tilde{\nabla}^\gamma_Z \varphi)Y, \varphi X) =& \mathrm{d}\tilde{\psi}(\varphi^2 X, \varphi Y, Z) - \mathrm{d}\tilde{\psi}(\varphi X, Y, Z) \\ =& \mathrm{d}(\mathrm{e}^{-\epsilon}\psi)(\varphi^2 X, \varphi Y, Z) - \mathrm{d}(\mathrm{e}^{-\epsilon}\psi)(\varphi X, Y, Z) \\ =& \mathrm{e}^{-\epsilon}\{\mathrm{d}\psi(\varphi^2 X, \varphi Y, Z) - \mathrm{d}\psi(\varphi X, Y, Z) - \varpi \wedge \psi(\varphi^2 X, \varphi Y, Z) \\ & + \varpi \wedge \psi(\varphi X, Y, Z)\}.\end{aligned}$$

By removing $\mathrm{e}^{-\epsilon}$ from both sides of the above relation, we get (23). □

Theorem 3.1 *Let M be a generalized almost paracontact pseudometric manifold $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ associated to a quintuple $(\varphi, \xi, \eta, \gamma, \psi)$ such that H and $\bar{H}$ are closed under the Courant bracket, and let $\epsilon \in C^\infty(M)$ be such that $\varpi = d\epsilon = \alpha\eta$ for $\alpha \in C^\infty(M)$. Then H_ϵ and $\bar{H}_\epsilon$, the eigenspaces of $\tilde{\mathcal{H}}$ of its conformal change $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$, are closed under the Courant bracket if and only if the conformal change is a homothety.*

Proof With the notations of Theorem 2.3 and by using (23), it is easy to see that the following required condition holds:

$$\varpi \wedge \psi(\varphi^2 X, \varphi Y, Z) - \varpi \wedge \psi(\varphi X, Y, Z) = 0. \tag{24}$$

Replacing X, Y by $\varphi X, \varphi Y$ in (24) respectively gives

$$-\varpi \wedge \psi(\varphi X, \varphi^2 Y, Z) - \varpi \wedge \psi(\varphi^2 X, \varphi Y, Z) = 0. \tag{25}$$

By summing up the two relations (24) and (25), we get

$$\begin{aligned} & \varpi \wedge \psi(\varphi X, \varphi^2 Y, Z) + \varpi \wedge \psi(\varphi X, Y, Z) = 0 \\ \Rightarrow \quad & \varpi \wedge \psi(\varphi X, \varphi^2 Y + Y, Z) = 0 \\ \Rightarrow \quad & \varpi \wedge \psi(\varphi X, \xi, Z) = 0. \end{aligned} \tag{26}$$

Since $\psi^\flat(\xi) = 0$, we get $\varpi(\xi)\psi(\varphi X, Z) = 0$, for $X, Z \in \Gamma(M)$. Thus,

$$\varpi(\xi) = 0. \tag{27}$$

From $\varpi = \alpha\eta$ and (27), we get $\alpha = 0$. Therefore, we conclude the proof. □

4. Conformal changes of generalized para-coKähler structures

In this section we consider the conditions with which the conformal change of generalized structures are normal.

Definition 4.1 *A generalized almost contact structure* $(\Phi, \xi+\eta)$ *is conformal normal if there exists a conformal change* $(\tilde{\Phi}, \tilde{\xi}+\tilde{\eta})$ *that is normal.*

We have the following definition:

Definition 4.2 *[4] In the generalized almost paracontact pseudometric structure* $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$*, if* $(\mathcal{A}, \xi+\eta)$ *is a normal generalized paracontact structure, the structure is called normal generalized paracontact pseudometric. If* $(\Phi, \xi+\eta)$ *is a normal generalized contact structure, the structure is called generalized almost para-co-Kähler. The normalization of both* $(\Phi, \xi+\eta)$ *and* $(\mathcal{A}, \xi+\eta)$ *in the generalized almost paracontact pseudometric structure defines the generalized para-co-Kähler structure.*

It is interesting to emphasize that there may be a generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ such that $(\Phi, \xi+\eta)$ is normal but $(\mathcal{A}, \xi+\eta)$ is not normal, as the following example shows.

Example 1 *Let* $M=\mathbb{R}^5$ *and choose a local frame* $\{X_1, X_2, X_3, X_4, X_5\}$ *and its dual local frame* $\{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$ *such that*

$$[X_5, X_1] = X_4,\ [X_5, X_2] = -X_3,\ [X_5, X_3] = -X_2,$$
$$[X_5, X_4] = X_1, \quad [X_i, X_j] = 0,\ o.w.$$

Thus, we have

$$\mathrm{d}\mu^1 = \mu^4 \wedge \mu^5, \quad \mathrm{d}\mu^2 = -\mu^3 \wedge \mu^5, \quad \mathrm{d}\mu^3 = -\mu^2 \wedge \mu^5, \quad \mathrm{d}\mu^4 = \mu^1 \wedge \mu^5,$$

and μ^5 *is closed. To construct a normal generalized contact structure, one takes generalized almost contact structure components with* $P = X_2 \otimes \mu^1 - X_1 \otimes \mu^2 + X_4 \otimes \mu^3 - X_3 \otimes \mu^4$*,* $\Phi = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}$*,* $\xi = X_5$*, and* $\eta = \mu^5$*, where* $(P^*\alpha)X = \alpha(PX)$ *and* $X + \alpha \in \mathbb{T}M$*. One computes easily that*

$$L = span\{X_5, X_1 - \mathrm{i}X_2, X_3 - \mathrm{i}X_4, \mu^1 - \mathrm{i}\mu^2, \mu^3 - \mathrm{i}\mu^4\},$$
$$L^* = span\{\mu^5, X_1 + \mathrm{i}X_2, X_3 + \mathrm{i}X_4, \mu^1 + \mathrm{i}\mu^2, \mu^3 + \mathrm{i}\mu^4\}.$$

For L*, the relevant Courant brackets give*

$$[\![X_5, X_1 - \mathrm{i}X_2]\!] = \mathrm{i}(X_3 - \mathrm{i}X_4), \quad [\![X_5, X_3 - \mathrm{i}X_4]\!] = -\mathrm{i}(X_1 - \mathrm{i}X_2),$$
$$[\![X_5, \mu^1 - \mathrm{i}\mu^2]\!] = -\mathrm{i}(\mu^3 - \mathrm{i}\mu^4), \quad [\![X_5, \mu^3 - \mathrm{i}\mu^4]\!] = \mathrm{i}(\mu^1 - \mathrm{i}\mu^2),$$

and the rest of the brackets are equal to zero. Similarly, for L^* *we compute the Courant brackets and we see that all of them are equal to zero as well as* $\$_{X_5}\mu^5 = \mathrm{d}\mu^5(X_5) = 0$*. Thus,* $(\Phi, F+\eta)$ *is a normal generalized contact structure.*

Now we define a g*-symmetric matrix* $\mathcal{H}$ *on* $TM \oplus TM^*$ *by* $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ *in which* $\gamma(X_i, X_j) = \delta_{ij}$ *for* $i = 1, \cdots, 5$*. Then we have* $\mathcal{H}(X_i) = -\mu^i$ *and* $\mathcal{H}(\mu^i) = X_i$*, for* $i = 1, \cdots, 5$*, and so* $(\Phi \circ \mathcal{H} = \mathcal{A}, \xi+\eta)$ *defines a generalized almost paracontact structure. Now it is evident that*

$$L_{\mathcal{A}} = span\{X_5, X_1 + \mu^2, X_3 + \mu^4, \mu^1 - X_2, \mu^3 - X_4\}.$$

Naturally, the expression $[\![X_5, \mu^1 - X_2]\!] = X_3 - \mu^4 \notin L_{\mathcal{A}}$ *shows that* $(\mathcal{A}, \xi + \eta)$ *is not integrable; thus,* $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ *is a generalized almost para-co-Kähler structure.*

Moreover, we emphasize that there may be a generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ such that $(\mathcal{A}, \xi + \eta)$ is normal but $(\Phi, \xi + \eta)$ is not normal, as the following example shows.

Example 2 *Let* $M = \mathbb{R}^3$ *and choose a local frame* $\{X_1, X_2, X_3\}$ *and a dual basis* $\{\mu^1, \mu^2, \mu^3\}$ *such that* $[X_1, X_3] = X_2$, $[X_1, X_2] = X_3$, *and* $[X_3, X_2] = 0$*; thus,* $\mathrm{d}\mu^2 = \mu^3 \wedge \mu^1$, $\mathrm{d}\mu^3 = \mu^2 \wedge \mu^1$, *and* $\mathrm{d}\mu^1 = 0$*. We form a generalized contact structure associated to an almost paracontact structure* $(F, \xi, \eta = \mu^1)$ *given by* $\xi = X_1$ *and* $F = -\mu^3 \otimes X_2 - \mu^2 \otimes X_3$ *by setting* $\gamma(X_i, X_j) = \delta_{ij}$ *and*

$$\Phi = \begin{pmatrix} 0 & -F \circ \gamma^\sharp \\ \gamma^\flat \circ F & 0 \end{pmatrix}.$$

Having

$$L = span\{X_1, X_2 + \mathrm{i}\mu^3, X_3 + \mathrm{i}\mu^2\}, \quad L^* = span\{\mu^1, X_2 - \mathrm{i}\mu^3, X_3 - \mathrm{i}\mu^2\},$$

the expression $[\![X_1, X_2 + \mathrm{i}\mu^3]\!] = X_3 - \mathrm{i}\mu^2 \notin L$ *shows that* $(\Phi, \xi + \eta)$ *is not normal.*

To form a tensor $\mathcal{A} = \Phi \circ \mathcal{H}$*, we define a matrix* $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ *that gives* $\mathcal{H}(X_i) = -\mu^i$ *and* $\mathcal{H}(\mu^i) = X_i$*. Thus,* $(\Phi \circ \mathcal{H} = \mathcal{A}, \xi + \eta)$ *defines a generalized almost paracontact structure. Now we have*

$$L_{\mathcal{A}} = span\{X_1, X_2 + X_3\} \text{ and } L^*_{\mathcal{A}} = span\{\mu^1, X_2 - X_3\},$$

such that their relevant Courant brackets give

$$[\![X_1, X_2 + X_3]\!] = X_2 + X_3, \quad [\![\mu^1, X_2 - X_3]\!] = 0,$$

and $\$_{X_1}\mu^1 = \mathrm{d}\mu^1(X_1) = 0$*. Therefore,* $(\mathcal{A}, \xi+\eta)$ *is a normal generalized paracontact structure and* $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ *is a normal generalized paracontact pseudometric structure.*

Now we give an example of a generalized para-co-Kähler structure.

Example 3 *Suppose* $M = \mathbb{R}^3$ *and take a local frame* $\{X_1, X_2, X_3\}$ *and a dual basis* $\{\mu^1, \mu^2, \mu^3\}$ *such that* $[X_1, X_3] = -X_2$, $[X_1, X_2] = X_3$, *and* $[X_3, X_2] = 0$*; thus,* $\mathrm{d}\mu^2 = \mu^1 \wedge \mu^3$, $\mathrm{d}\mu^3 = \mu^2 \wedge \mu^1$, *and* $\mathrm{d}\mu^1 = 0$*. One can simply construct a generalized contact structure associated to an almost contact structure* $(\varphi, \xi, \eta = \mu^1)$ *given by*

$$\xi = X_1, \quad \varphi = \mu^3 \otimes X_2 - \mu^2 \otimes X_3, \quad \text{and} \quad \Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix},$$

and then we have

$$L = span\{X_1, X_3 - \mathrm{i}X_2, \mu^3 - \mathrm{i}\mu^2\}, \quad L^* = span\{\mu^1, X_3 + \mathrm{i}X_2, \mu^3 + \mathrm{i}\mu^2\}.$$

Applying the Courant brackets for subbundles L and L^*, we find that

$$[\![X_1, X_3 - \mathrm{i}X_2]\!] = -\mathrm{i}(X_3 - \mathrm{i}X_2), \quad [\![X_1, \mu^3 - \mathrm{i}\mu^2]\!] = -\mathrm{i}(\mu^3 - \mathrm{i}\mu^2),$$
$$[\![X_3 - \mathrm{i}X_2, \mu^3 - \mathrm{i}\mu^2]\!] = 0,$$
$$[\![X_3 + \mathrm{i}X_2, \mu^3 + \mathrm{i}\mu^2]\!] = 0 = [\![X_3 + \mathrm{i}X_2, \mu^1]\!],$$

and $\$_\xi\eta = 0$. Therefore, we obtain a normal generalized contact structure $(\Phi, \xi + \eta)$.

By defining $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$, in which $\gamma(X_i, X_j) = \delta_{ij}$ for $i = 1, 2, 3$, we have $\mathcal{H}(X_i) = -\mu^i$ and $\mathcal{H}(\mu^i) = X_i$. Then $(\Phi \circ \mathcal{H} = \mathcal{A}, \xi + \eta)$ defines a generalized almost paracontact structure. Now it is evident that

$$L_{\mathcal{A}} = span\{X_1, -X_2 + \mu^3, X_3 + \mu^2\}, \quad L^*_{\mathcal{A}} = span\{\mu^1, X_2 + \mu^3, X_3 - \mu^2\}.$$

For $L_{\mathcal{A}}$, we compute the Courant brackets,

$$[\![X_1, -X_2 + \mu^3]\!] = -(X_3 + \mu^2), \quad [\![X_1, X_3 + \mu^2]\!] = -X_2 + \mu^3,$$
$$[\![X_3 + \mu^2, \mu^3 - X_2]\!] = 0,$$

and thus $L_{\mathcal{A}}$ is involutive. Finally, for $L^*_{\mathcal{A}}$, we compute the Courant brackets and we see that all of them are equal to zero. Therefore, $(\mathcal{A}, \xi + \eta)$ is a strong structure and $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is a generalized para-co-Kähler structure.

We now give a consideration for the normalization of generalized almost paracontact pseudometric structures after conformal changes.

Definition 4.3 *Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ be a generalized almost paracontact pseudometric structure and $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi}+\tilde{\eta})$ be its conformal change by C_ϵ. That means $\tilde{\Phi} = C_{-\epsilon}\Phi C_\epsilon$, $\tilde{\mathcal{A}} = C_{-\epsilon}\mathcal{A}C_\epsilon$, $\tilde{\mathcal{H}} = C_{-\epsilon}\mathcal{H}C_\epsilon$, $\tilde{\xi} = \mathrm{e}^{\frac{\epsilon}{2}}\xi$, and $\tilde{\eta} = \mathrm{e}^{\frac{-\epsilon}{2}}\eta$. Then $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is called conformal Φ-normal if $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ is normal; it is called conformal paranormal if $(\tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is normal; and the structure is called conformal para-co-Kähler if both $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ and $(\tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ are normal.*

We obtain conditions of conformal normalization by applying conditions (A_1)–(A_7) in Theorem 2.1 to the conformal change $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ as in the following proposition.

Proposition 4.1 *The generalized almost paracontact pseudometric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$, with matrix representations as in (12), is conformal Φ-normal by C_ϵ if $\varpi = \mathrm{d}\epsilon$ satisfies the following conditions:*

(C_1) $[\phi,\phi]-2\xi\wedge(\phi^\sharp\otimes\phi^\sharp)\mathrm{d}\eta=-2\phi^\sharp\varpi\wedge\phi,$

and $[\xi,\phi]+\xi\wedge\phi^\sharp \$_\xi\eta=-\varpi(\xi)\phi;$

(C_2) $P^*\{\alpha,\beta\}_\phi-(\$_{\phi^\sharp\alpha}P^*\beta-\$_{\phi^\sharp\beta}P^*\alpha-\mathrm{d}\phi(P^*\alpha,\beta))$

$=-\phi(\alpha,\beta)P^*\varpi+\phi(P^*\alpha,\beta)\varpi;$

(C_3) $N_P(X,Y)+\mathrm{d}\eta(PX,PY)\xi-\phi^\sharp(i_{X\wedge Y}\mathrm{d}\theta)$

$=\theta(X,Y)\phi^\sharp\varpi-\varpi(X)\phi^\sharp\theta^\flat(Y)+\varpi(Y)\phi^\sharp\theta^\flat(X);$

(C_4) $\mathrm{d}\theta_P(X,Y,Z)-\sum_{cycle(X,Y,Z)}\mathrm{d}\theta(PX,Y,Z)$

$=-\sum_{cycle(X,Y,Z)}(\varpi\wedge\theta_P+(\varpi\circ P)\wedge\theta)(X,Y,Z);$

(C_5) $2(\$_\xi P)=P^*(\varpi)\otimes\xi,\quad (\$_\xi\theta^\flat)=\varpi(\xi)\theta^\flat;$

(C_6) $2(\$_{\phi^\sharp\alpha}\eta)=\phi(\varpi,\alpha)\eta,\quad 2(\$_\xi\eta)=(\varpi\wedge\eta)\xi;$

(C_7) $2\{\mathrm{d}\eta(PX,Y)-\mathrm{d}\eta(PY,X)\}=(P^*\varpi\wedge\eta)(X,Y).$

Here, the bracket is the Schouten–Nijenhuis bracket as explained in [8], $\{\alpha,\beta\}_\phi=\$_{\phi^\sharp\alpha}\beta-\$_{\phi^\sharp\beta}\alpha-\mathrm{d}\phi(\alpha,\beta)$, $i_{X\wedge Y}\mathrm{d}\theta=\$_X\theta^\flat(Y)-\$_Y\theta^\flat(X)-\mathrm{d}\theta(X,Y)$, $\theta_P(X,Y)=\theta(PX,Y)$, *and* $X+\alpha,Y+\beta\in\mathbb{T}M$.

Proof Let $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$ be a generalized almost paracontact pseudometric structure and its conformal change by C_ϵ, $(\tilde\Phi,\tilde{\mathcal{H}},\tilde{\mathcal{A}},\tilde\xi+\tilde\eta)$, is Φ-normal. Thus, the generalized almost contact structure $(P,\phi^\sharp,\theta^\flat,\xi+\eta)$ is conformal normal. The crucial fact is that the first part of condition (A_1) in Theorem 2.1 is satisfied and we have $[\tilde\phi,\tilde\phi]=2\tilde\xi\wedge(\tilde\phi^\sharp\otimes\tilde\phi^\sharp)\mathrm{d}\tilde\eta$. On the other hand, we have

$$\begin{aligned}[\tilde\phi,\tilde\phi]=&D(\mathrm{e}^\epsilon\phi\wedge\mathrm{e}^\epsilon\phi)-2D(\mathrm{e}^\epsilon\phi)\wedge(\mathrm{e}^\epsilon\phi)\\=&\mathrm{e}^{2\epsilon}[\phi,\phi]+2\mathrm{e}^{2\epsilon}((\phi\wedge\phi)(\varpi)-\phi^\sharp(\varpi)\wedge\phi)\\=&\mathrm{e}^{2\epsilon}([\phi,\phi]+2(\phi^\sharp(\varpi)\wedge\phi)),\end{aligned}$$

where D is the generalized divergence that generates the Schouten bracket [8]. Also, we have $\tilde\xi\wedge(\tilde\phi^\sharp\otimes\tilde\phi^\sharp)\mathrm{d}\tilde\eta=\mathrm{e}^{2\epsilon}\xi\wedge(\phi^\sharp\otimes\phi^\sharp)\mathrm{d}\eta$. What is left is to compare these relations and get the first part of (C_1). The second part of (C_1) follows by the same method. We use the second part of (A_1) and obtain

$$\begin{aligned}[\tilde\xi,\tilde\phi]=&D(\mathrm{e}^{\frac{3\epsilon}{2}}\xi\wedge\phi)-D(\mathrm{e}^{\frac{\epsilon}{2}}\xi)\wedge(\mathrm{e}^\epsilon\phi)-D(\mathrm{e}^\epsilon\phi)\wedge(\mathrm{e}^{\frac{\epsilon}{2}}\xi)\\=&\mathrm{e}^{\frac{3\epsilon}{2}}([\xi,\phi]+\frac{3}{2}(\xi\wedge\phi)(\varpi)-\frac{1}{2}(\xi\varpi)\wedge\phi-\phi^\sharp(\varpi)\wedge\xi)\\=&\mathrm{e}^{\frac{3\epsilon}{2}}([\xi,\phi]+\varpi(\xi)\phi)+\frac{1}{2}\phi^\sharp(\varpi)\wedge\xi).\end{aligned}$$

Since $[\tilde\xi,\tilde\phi]=-\tilde\xi\wedge\tilde\phi^\sharp\$_{\tilde\xi}\tilde\eta$ and $\tilde\xi\wedge\tilde\phi^\sharp\$_{\tilde\xi}\tilde\eta=\mathrm{e}^{\frac{3\epsilon}{2}}(\xi\wedge\phi^\sharp\$_\xi\eta+\frac{1}{2}\xi\wedge\phi^\sharp\varpi)$, the result is obtained. Also, by (A_2), we have $P^*\{\alpha,\beta\}_{\tilde\phi}=(\$_{\tilde\phi^\sharp\alpha}P^*\beta-\$_{\tilde\phi^\sharp\beta}P^*\alpha-\mathrm{d}\tilde\phi(P^*\alpha,\beta))$, and then a direct computation gives (C_2). Our next step is to evaluate (C_3), use (A_3) and $N_P(X,Y) = -\mathrm{d}\tilde{\eta}(PX,PY)\tilde{\xi} + \tilde{\phi}^\sharp(i_{X\wedge Y}\mathrm{d}\tilde{\theta})$, and get

$$\begin{aligned}
N_P(X,Y) =& -\mathrm{d}\eta(PX,PY)\xi + \tilde{\phi}^\sharp(\$_X\tilde{\theta}^\flat(Y) - \$_Y\tilde{\theta}^\flat(X) - \mathrm{d}\tilde{\theta}(X,Y)) \\
=& -\mathrm{d}\eta(PX,PY)\xi + \phi^\sharp(\$_X\theta^\flat(Y) - \$_Y\theta^\flat(X) - \mathrm{d}\theta(X,Y)) \\
&- \phi^\sharp(\varpi(X)\theta^\flat(Y) - \varpi(Y)\theta^\flat(X) - \theta(X,Y)\varpi),
\end{aligned}$$

where $\mathrm{d}\eta(PX,PY)\xi = \mathrm{d}\tilde{\eta}(PX,PY)\tilde{\xi}$, and thus (C_3) is proved. Replacing θ by $\tilde{\theta}$ in (A_4), a direct computation gives (C_4). It follows from the first part of (A_5) that $\$_{\tilde{\xi}}P = 0$, and then from (10), we get

$$\begin{aligned}
0 =& (\$_{\tilde{\xi}}P)X = \$_{\tilde{\xi}}PX - P(\$_{\tilde{\xi}}X) = [\mathrm{e}^{\frac{\epsilon}{2}}\xi, PX] - P[\mathrm{e}^{\frac{\epsilon}{2}}\xi, X] \\
=& \mathrm{e}^{\frac{\epsilon}{2}}(([\xi,PX] - P[\xi,X]) - \frac{1}{2}\varpi(PX)\xi) \\
=& \mathrm{e}^{\frac{\epsilon}{2}}(\$_\xi P)X - \frac{1}{2}\varpi(PX)\xi),
\end{aligned}$$

which gives the first part of (C_5). Taking the second part of (A_5) and (10ii), we get

$$\begin{aligned}
0 =& (\$_{\tilde{\xi}}\tilde{\theta})X = \$_{\tilde{\xi}}\tilde{\theta}^\flat X - \tilde{\theta}^\flat(\$_{\tilde{\xi}}X) \\
=& i_{\mathrm{e}^{\frac{\epsilon}{2}}\xi} \circ \mathrm{d}(\mathrm{e}^{-\epsilon}\theta^\flat)X + \mathrm{d} \circ i_{\mathrm{e}^{\frac{\epsilon}{2}}\xi}(\mathrm{e}^{-\epsilon}\theta^\flat)X - \mathrm{e}^{-\epsilon}\theta^\flat([\mathrm{e}^{\frac{\epsilon}{2}}\xi, X]) \\
=& \mathrm{e}^{\frac{-\epsilon}{2}}((i_\xi \circ \mathrm{d})\theta^\flat(X) - \varpi(\xi)\theta^\flat(X) + \mathrm{d} \circ i_\xi\theta^\flat(X) - \theta^\flat(\$_\xi X)) \\
=& \mathrm{e}^{\frac{-\epsilon}{2}}((\$_\xi\theta^\flat)X - \varpi(\xi)\theta^\flat(X)).
\end{aligned}$$

For the first part of (C_6), from $\$_{\tilde{\xi}}\tilde{\eta} = 0$, we get

$$\begin{aligned}
0 = (\$_{\tilde{\xi}}\tilde{\eta}) &= i_{\mathrm{e}^{\frac{\epsilon}{2}}\xi} \circ \mathrm{d}(\mathrm{e}^{-\frac{\epsilon}{2}}\eta) + \mathrm{d} \circ i_{\mathrm{e}^{\frac{\epsilon}{2}}\xi}\mathrm{e}^{\frac{-\epsilon}{2}}\eta \\
&= i_{\mathrm{e}^{\frac{\epsilon}{2}}\xi}(-\frac{\mathrm{e}^{-\frac{\epsilon}{2}}}{2}\varpi\wedge\eta + \mathrm{e}^{\frac{-\epsilon}{2}}\mathrm{d}\eta) + \mathrm{d} \circ i_\xi\eta \\
&= \frac{-1}{2}(\varpi\wedge\eta)\xi + \$_\xi\eta.
\end{aligned}$$

Using the second part of (A_6) gives $\$_{\tilde{\phi}^\sharp\alpha}\tilde{\eta} = 0$. Thus, by (10ii), the second part of (C_6) holds as follows:

$$\begin{aligned}
0 = (\$_{\tilde{\phi}^\sharp\alpha}\tilde{\eta}) &= i_{\mathrm{e}^\epsilon\phi^\sharp\alpha} \circ \mathrm{d}(\mathrm{e}^{\frac{-\epsilon}{2}}\eta) + \mathrm{d} \circ i_{\mathrm{e}^\epsilon\phi^\sharp\alpha}\mathrm{e}^{\frac{-\epsilon}{2}}\eta \\
&= i_{\mathrm{e}^\epsilon\phi^\sharp\alpha}(\frac{-\mathrm{e}^{\frac{-\epsilon}{2}}}{2}\varpi\wedge\eta + \mathrm{e}^{\frac{-\epsilon}{2}}\mathrm{d}\eta) + \mathrm{e}^{\frac{\epsilon}{2}}\mathrm{d} \circ i_{\phi^\sharp\alpha}\eta \\
&= \frac{-\mathrm{e}^{\frac{\epsilon}{2}}}{2}(\varpi\wedge\eta)(\phi^\sharp\alpha) + \mathrm{e}^{\frac{\epsilon}{2}}\$_{\phi^\sharp\alpha}\eta \\
&= \mathrm{e}^{\frac{\epsilon}{2}}(-\frac{1}{2}\alpha\phi^\sharp(\varpi)\eta + \$_{\phi^\sharp\alpha}\eta).
\end{aligned}$$

Finally, (A_7) gives $\mathrm{d}\tilde{\eta}(PX,Y)-\mathrm{d}\tilde{\eta}(PY,X)=0$. Then, by (10i), we get

$$\begin{aligned}0&=\mathrm{d}\tilde{\eta}(PX,Y)-\mathrm{d}\tilde{\eta}(PY,X)=\mathrm{d}(\mathrm{e}^{-\frac{\epsilon}{2}}\eta)(PX,Y)-\mathrm{d}(\mathrm{e}^{-\frac{\epsilon}{2}}\eta)(PY,X)\\&=(\frac{-\mathrm{e}^{-\frac{\epsilon}{2}}}{2}\varpi\wedge\eta+\mathrm{e}^{\frac{-\epsilon}{2}}\mathrm{d}\eta)(PX,Y)-(\frac{-\mathrm{e}^{-\frac{\epsilon}{2}}}{2}\varpi\wedge\eta+\mathrm{e}^{\frac{-\epsilon}{2}}\mathrm{d}\eta)(PY,X)\\&=\frac{-\mathrm{e}^{\frac{\epsilon}{2}}}{2}\{\varpi(PX)\eta(Y)-\varpi(PY)\eta(X)+2(\mathrm{d}\eta(PY,X)-\mathrm{d}\eta(PX,Y))\}.\end{aligned}$$

Therefore, the proof is completed. □

With the same procedure and the same notations as in Proposition 4.1, we can state the following proposition.

Proposition 4.2 *The generalized almost paracontact pseudometric structure $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$, with matrix representations as in (12), is conformal paranormal by C_ϵ if $\varpi=\mathrm{d}\epsilon$ satisfies the following conditions:*

$$\begin{aligned}&(D_1)\ i)\ [\pi,\pi]-2\xi\wedge(\pi^\sharp\otimes\pi^\sharp)\mathrm{d}\eta=-2\pi^\sharp\varpi\wedge\pi,\\&\qquad ii)\ [\xi,\pi]+\xi\wedge\pi^\sharp\$_\xi\eta=-\varpi(\xi)\pi;\\&(D_2)\ i)\ A^*\{\alpha,\beta\}_\pi-(\$_{\pi^\sharp\alpha}A^*\beta-\$_{\pi^\sharp\beta}A^*\alpha-\mathrm{d}\pi(A^*\alpha,\beta))\\&\qquad=-\pi(\alpha,\beta)A^*\varpi+\pi(A^*\alpha,\beta)\varpi,\quad ii)\ A\pi^\sharp=\pi^\sharp A;\\&(D_3)\ N_A(X,Y)+\mathrm{d}\eta(AX,AY)\xi-\pi^\sharp(i_{X\wedge Y}\mathrm{d}\sigma)\\&\qquad=\sigma(X,Y)\pi^\sharp\varpi-\varpi(X)\pi^\sharp\sigma^\flat(Y)+\varpi(Y)\pi^\sharp\sigma^\flat(X);\\&(D_4)\ \mathrm{d}\sigma_A(X,Y,Z)-\sum_{cycle(X,Y,Z)}\mathrm{d}\sigma(AX,Y,Z)\\&\qquad=-\sum_{cycle(X,Y,Z)}(\varpi\wedge\sigma_A+(\varpi\circ A)\wedge\sigma)(X,Y,Z);\\&(D_5)\ 2(\$_\xi A)=-A^*(\varpi)\otimes\xi,\quad and\quad(\$_\xi\sigma^\flat)=-\varpi(\xi)\sigma^\flat;\\&(D_6)\ 2(\$_{\pi^\sharp\alpha}\eta)=\pi(\varpi,\alpha)\eta\ \ and\ \ 2(\$_\xi\eta)=-(\varpi\wedge\eta)\xi;\\&(D_7)\ 2\{\mathrm{d}\eta(AX,Y)-\mathrm{d}\eta(AY,X)\}=-(A^*\varpi\wedge\eta)(Y,X).\end{aligned}$$

Corollary 4.1 *The generalized almost paracontact pseudometric structure $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$ is conformal para-co-Kähler by C_ϵ if $\varpi=\mathrm{d}\epsilon$ satisfies conditions (C_1)–(C_7) and (D_1)–(D_7).*

We will investigate necessary and sufficient conditions with which $(\tilde{\Phi},\tilde{\mathcal{H}},\tilde{\mathcal{A}},\ \tilde{\xi}+\tilde{\eta})$, the conformal change of a generalized para-co-Kähler structure $(\Phi,\mathcal{H},\mathcal{A},\ \xi+\eta)$, is a generalized para-co-Kähler structure.

Theorem 4.1 *(a) Let $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$ be a normal generalized paracontact pseudometric structure on M with matrix representations as in (12), $\dim M>3$, and let $(\tilde{\Phi},\tilde{\mathcal{H}},\tilde{\mathcal{A}},\tilde{\xi}+\tilde{\eta})$ be its conformal change by C_ϵ. If $(\Phi,\xi+\eta)$ satisfies one of the following conditions,*

(1) rank $\phi>2$,

(2) P_x has no real eigenvalue, $\forall x\in M$,

then the structure, $(\Phi,\mathcal{H},\mathcal{A},\xi+\eta)$, is conformal Φ-normal if and only if the conformal change is a homothety.

(b) Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ be an almost para-co-Käler structure on M with the same representation and dimension. If $(\mathcal{A}, \xi + \eta)$ satisfies one of the following conditions,

(3) rank $\pi > 2$,

(4) A_x has no real eigenvalue, $\forall x \in M$,

then the structure, $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$, is conformal paranormal if and only if the conformal change is a homothety.

Proof (a) Since $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is a normal generalized paracontact pseudometric structure, then $(\Phi, \xi + \eta)$ is normal, and thus by Theorem 2.1, $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ is normal if and only if the right hand side of the equalities (C_1)–(C_7) vanishes. Taking into account (8ii), the vanishing of the second part of (C_1), and the first part of (C_5), the result is

$$\theta^\flat \phi^\sharp \varpi = -\varpi. \tag{28}$$

Moreover, we get the condition $\phi^\sharp \varpi \wedge \phi = 0$, which is obtained from the vanishing of the first part of (C_1), if and only if either $rank\ \phi = 2$ or $\phi^\sharp \varpi = 0$. Then, in case (1), we must have

$$\phi^\sharp \varpi = 0. \tag{29}$$

Thus, by (28) and (29), we get $\mathrm{d}\epsilon = \varpi = 0$.
Now turn to case (2) and assume that $d_x \epsilon \neq 0$ is on a neighborhood U_x. Since (C_2) holds for every 1-form β, its vanishing result is

$$(P^* \varpi) X \phi^\sharp(\alpha) = \varpi(X) \phi^\sharp(P^* \alpha) \tag{30}$$

for a vector field X on U_x. Then, since $d_x \epsilon \neq 0$, we rewrite (30) as follows:

$$f \phi^\sharp(\alpha) = P_x \phi^\sharp(\alpha), \tag{31}$$

in which $f = \frac{P^* \varpi_x X}{\varpi_x X} \in C^\infty(\mathbb{T}M)$. Then we replace α by the 1-form $\theta^\flat(Y)$ for any arbitrary vector field Y in (31). Thus, by taking into account (8) and (10), $P\,|_{U_x}$ satisfies

$$P^3 - f P^2 + P - f(\mathrm{Id} + \eta \otimes \xi) = 0,$$

and thus P must have a real eigenvalue. Therefore, the hypothesis of case (2) implies $\mathrm{d}\epsilon = \varpi = 0$.
Proof of (b) is similar to (a). □

In the following example, we show that if none of the conditions (1) and (2) of the above theorem are satisfied, then the conformal change is not necessarily a homothety.

Example 4 *Set $M = \mathbb{R}^5$ and choose a local frame $\{X_1, X_2, X_3, X_4, X_5\}$ and its dual local frame $\{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$ such that*

$$[X_5, X_1] = X_4,\ [X_5, X_2] = -X_3,\ [X_5, X_3] = -X_2,$$
$$[X_5, X_4] = X_1, \quad [X_i, X_j] = 0,\ o.w.$$

Thus, we have

$$\mathrm{d}\mu^1 = \mu^4 \wedge \mu^5, \quad \mathrm{d}\mu^2 = -\mu^3 \wedge \mu^5, \quad \mathrm{d}\mu^3 = -\mu^2 \wedge \mu^5, \quad \mathrm{d}\mu^4 = \mu^1 \wedge \mu^5,$$

and μ^5 is closed. To construct a normal generalized contact structure, one takes generalized almost contact structure components with

$$\Phi = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}, \quad \xi = X_5, \quad \text{and} \quad \eta = \mu^5,$$

where $P = X_2 \otimes \mu^1 - X_1 \otimes \mu^2 + X_4 \otimes \mu^3 - X_3 \otimes \mu^4$, $(P^*\alpha)X = \alpha(PX)$, and $X + \alpha \in \mathbb{T}M$. One can see in Example 3, with $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ and $\mathcal{A} = \mathcal{H} \circ \Phi$, that $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is a generalized almost para-co-Kähler structure.

Now let us consider the normality of conformal change $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ for nonconstant function ϵ such that $\mathrm{d}\epsilon = \varepsilon\mu^5$ for an arbitrary constant function ε. Then, for $\tilde{L}$ and $\tilde{L}^*$, the Courant brackets are given by

$$[\![\tilde{\xi}, X_1 - \mathrm{i}X_2]\!] = \mathrm{i}e^{\epsilon/2}(X_3 - \mathrm{i}X_4), \quad [\![\tilde{\xi}, X_3 - \mathrm{i}X_4]\!] = -\mathrm{i}e^{\epsilon/2}(X_1 - \mathrm{i}X_2),$$
$$[\![\tilde{\xi}, \mu^1 - \mathrm{i}\mu^2]\!] = -\mathrm{i}e^{\epsilon/2}(\mu^3 - \mathrm{i}\mu^4), \qquad [\![\tilde{\xi}, \mu^3 - \mathrm{i}\mu^4]\!] = \mathrm{i}e^{\epsilon/2}(\mu^1 - \mathrm{i}\mu^2),$$

and the others are equal to zero as well as $\$_{\tilde{\xi}}\tilde{\eta} = \mathrm{d}\mu^5(X_5) + \frac{1}{2}(\mathrm{d}\epsilon \wedge \mu^5)(X_5) = 0$. Thus, $(\tilde{\Phi}, \tilde{\xi} + \tilde{\eta})$ is a normal generalized contact structure and $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is a generalized almost para-co-Kähler structure. Therefore, $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is nonhomothety conformal Φ-normal.

In the following example, we show that in Theorem 4.1, when $\dim M = 3$, the conformal change does not necessarily need to be a homothety.

Example 5 *Set $M = \mathbb{R}^3$ and choose a local frame $\{X_1, X_2, X_3\}$ and a dual basis $\{\mu^1, \mu^2, \mu^3\}$ such that $[X_1, X_3] = X_2$, $[X_1, X_2] = X_3$, and $[X_3, X_2] = 0$. Thus, $\mathrm{d}\mu^2 = \mu^3 \wedge \mu^1$, $\mathrm{d}\mu^3 = \mu^2 \wedge \mu^1$, and $\mathrm{d}\mu^1 = 0$. These assumptions lead to a generalized contact structure associated to an almost paracontact structure $(\varphi, \xi, \eta = \mu^1)$ given by*

$$\xi = X_1, \quad F = -\mu^3 \otimes X_2 - \mu^2 \otimes X_3,$$

and $\Phi = \begin{pmatrix} 0 & -F \circ \gamma^\sharp \\ \gamma^\flat \circ F & 0 \end{pmatrix}$ in which $\gamma(X_i, X_j) = \delta_{ij}$. By $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ and $\mathcal{A} = \mathcal{H} \circ \Phi$, $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is the normal generalized almost paracontact pseudometric structure.

We only need to show the normality of conformal change $(\tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ for nonconstant function ϵ such that $\mathrm{d}\epsilon = \varepsilon\mu^1$ for an arbitrary constant function ε. For $\tilde{L}_{\mathcal{A}}$ and $\tilde{L}^*_{\mathcal{A}}$, the Courant brackets are given by

$$[\![\tilde{\xi}, X_1 + X_2]\!] = \mathrm{i}e^{\epsilon/2}(X_2 + X_3), \quad [\![\tilde{\eta}, X_2 - X_3]\!] = 0,$$

and $\$_{\tilde{\xi}}\tilde{\eta} = \mathrm{d}\mu^1(X_1) + \frac{1}{2}(\mathrm{d}\epsilon \wedge \mu^1)(X_1) = 0$.

Thus, $(\tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is a normal generalized paracontact structure and $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi} + \tilde{\eta})$ is a normal generalized almost paracontact pseudometric structure. Therefore, $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is nonhomothety conformal paranormal.

Let (P, ξ, η) be a normal almost contact structure on a manifold M^{2n+1}. It was shown in [6] that there is a normal generalized contact structure $(\Phi, \xi + \eta)$ in which

$$\Phi = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}.$$

Therefore, by Proposition 4.1, the structure is conformal Φ-normal if and only if $(\varpi \wedge \eta)\xi = 0$, and then we get $\varpi = \varpi(\xi)\eta$. Thus, the following proposition is valid.

Proposition 4.3 *Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ be a normal generalized paracontact pseudometric structure in which the normal generalized contact structure $(\Phi, \xi + \eta)$ is associated to a classical normal almost contact structure (P, ξ, η), and then the structure is conformal Φ-normal if and only if ϖ is a section of L_η.*

Proposition 4.4 *Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ be a generalized para-co-Kähler structure,* $\dim M > 3$*, and $(\tilde{\Phi}, \tilde{\mathcal{H}}, \tilde{\mathcal{A}}, \tilde{\xi}+\tilde{\eta})$ its conformal change. Assume that $(\Phi, \xi + \eta)$ satisfies one of the following conditions:*

(1) *rank $\phi > 2$ and* (2) *P_x has no real eigenvalue, $\forall x \in M$,*

and $(\mathcal{A}, \xi + \eta)$ satisfies one of the following conditions:

(3) *rank $\pi > 2$ and* (4) *A_x has no real eigenvalue, $\forall x \in M$. Then the structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is conformal para-co-Kähler if and only if the conformal change is a homothety.*

Proof The generalized para-co-Kähler structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is conformal para-co-Kähler if and only if $(\Phi, \xi + \eta)$ is conformal Φ-normal and $(\mathcal{A}, \xi + \eta)$ is conformal paranormal. Therefore, similar to Theorem 4.1, $(\Phi, \xi + \eta)$ is conformal Φ-normal and $(\mathcal{A}, \xi + \eta)$ is conformal paranormal if and only if the conformal change is a homothety and it completes the proof. □

Consider a generalized almost paracontact pseudometric structure
$(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$, which is associated to a classical normal almost contact structure (P, ξ, η) [6]. By using the matrix

$$\Phi = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}, \tag{32}$$

in which $\gamma(X_i, X_j) = \delta_{ij}$, we get

$$\mathcal{A} = \mathcal{H} \circ \Phi = \begin{pmatrix} 0 & -\gamma^\sharp \circ P^* \\ -\gamma^\flat \circ P & 0 \end{pmatrix}.$$

One can see that conditions (1) and (3) in Proposition 4.4 do not occur contemporaneously. Moreover, P_x and A_x have real eigenvalues because of $\Phi(\xi) = 0$ and $A = 0$. Thus, we can get the next corollary.

Corollary 4.2 *There is not any conformal para-co-Kähler structure $(\Phi, \mathcal{H}, \mathcal{A}, \ \xi + \eta)$ in which the normal generalized contact structure $(\Phi, \xi + \eta)$ is associated to a classical normal almost contact structure (P, ξ, η) as in (32).*

References

[1] Blair D. Riemannian Geometry of Contact and Symplectic Manifolds. New York, NY, USA: Birkhauser, 2002.

[2] Gualtieri M. Generalized complex geometry. Annals of Mathematics 2011; 174 (1): 75-123.

[3] Hitchin J. Generalized Calabi Yau manifolds. Quarterly Journal of Mathematics 2003; 54: 281-308.

[4] Kashani NH, Malek F. Generalized para-coKähler structures. Scientific Annals of the Alexandru Ioan Cuza University of Iaşi (New Series) (in press).

[5] Matsuzoe H. Geometry of semi-Weyl manifolds and Weyl manifolds. Kyushu Journal of Mathematics 2001; 55: 107-117.

[6] Poon YS, Wade A. Generalized contact structures. Journal of London Mathematical Society 2011; 83: 333-352.

[7] Sahin B, Sahin F. Generalized almost para-contact structures. International Journal of Geometric Methods in Modern Physics 2017; 14: 1-21.

[8] Vaisman I. Lectures on the Geometry of Poisson Manifolds. Progress in Mathematics, Vol. 118. Basel, Switzerland: Birkhauser, 1994.

[9] Vaisman I. Conformal changes of generalized complex structures. Scientific Annals of the Alexandru Ioan Cuza University of Iaşi (New Series) 2008; 54: 1-14.

[10] Vaisman I. From generalized Kähler to generalized Sasakian structures. Journal of Geometry and Symmetry in Physics 2010; 18: 63-86.

[11] Vaisman I. Generalized para-Kähler manifolds. Differential Geometry and Its Applications 2015; 42: 84-103.