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## On the divisors of shifted primes

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**Abstract:** Let  $\tau(n)$  stand for the number of positive divisors of  $n$ . Given an additive function  $f$  and a real number  $\alpha \in [0, 1)$ , let  $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\} < \alpha}} 1$ , where  $\{y\}$  stands for the fractional part of  $y$ , and consider the discrepancy  $\Delta(n) := \sup_{0 \leq \alpha < \beta < 1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|$ . We show that  $\Delta(p+1) \rightarrow 0$  for almost all primes  $p$  if and only if  $\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q} = \infty$  for every positive integer  $m$ , where  $\|x\|$  stands for the distance between  $x$  and its nearest integer and where the sum runs over all primes  $q$ .

**Key words:** Sum of divisors function, shifted primes

### 1. Introduction and notation

Let  $\tau(n)$  stand for the number of positive divisors of  $n$ . Given an additive function  $f$  and a real number  $\alpha \in [0, 1)$ , let  $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\} < \alpha}} 1$ , where  $\{y\}$  stands for the fractional part of  $y$ , and consider the discrepancy  $\Delta(n) := \sup_{0 \leq \alpha < \beta < 1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|$ . It is well known that  $h_n(\alpha) \rightarrow \alpha$  as  $n \rightarrow \infty$  uniformly for  $\alpha \in [0, 1)$  if and only if  $\lim_{n \rightarrow \infty} \Delta(n) = 0$ .

Let  $\|x\|$  stand for the distance between  $x$  and its nearest integer and let  $\wp$  stand for the set of all primes. From here on, the letters  $p$  and  $q$  will be used exclusively to denote primes. In 1976, the second author [6] proved that  $\Delta(n) \rightarrow 0$  for almost all  $n$  if and only if  $\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q} = \infty$  for every positive integer  $m$  (see Theorem A below). Observe that there is a small error in the original paper of Kátaı [6]: in relation (5), the number 2 should be removed.

Here, we consider the case of shifted primes  $p+1$  and show that  $\Delta(p+1) \rightarrow 0$  for almost all primes  $p$  if and only if  $\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q} = \infty$  for every positive integer  $m$ .

Finally, we examine an interesting outcome in the particular case  $f(n) = \log n$ .

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**2. Main result**

**Theorem 1** *Let  $f$  be an additive function and  $\alpha \in [0, 1)$ . Set  $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\} < \alpha}} 1$  and  $\Delta(n) := \sup_{0 \leq \alpha < \beta < 1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|$ . Then  $\Delta(p + 1) \rightarrow 0$  for almost all primes  $p$  if and only if  $\sum_{q \in \mathcal{P}} \frac{\|mf(q)\|^2}{q} = \infty$  for every positive integer  $m$ .*

**3. Preliminary results**

Let  $P(n)$  stand for the largest prime factor of  $n$  and  $\pi(x)$  for the number of primes not exceeding  $x$ .

**Lemma 1** *Given  $\delta \in (0, 1/2)$  and a large number  $x$ , set  $\wp_{x,\delta} := \{p \leq x : P(p + 1) \notin [x^\delta, x^{1-\delta}]\}$ . Then, for some absolute constant  $C_1 > 0$ ,*

$$\#\wp_{x,\delta} < C_1 \delta \pi(x).$$

**Proof** The fact that there exists an absolute constant  $c_1 > 0$  such that

$$\#\{p \leq x : P(p + 1) > x^{1-\delta}\} < c_1 \delta \pi(x)$$

is essentially a direct application of Theorem 3.8 in the book of Halberstam and Richert [2]. Therefore, it remains to prove that there exists an absolute constant  $c_2 > 0$  such that

$$\#\{p \leq x : P(p + 1) < x^\delta\} < c_2 \delta \pi(x). \tag{3.1}$$

To do so, we shall first obtain an upper bound for the sum  $T_\delta(x) := \sum_{\substack{p \leq x \\ P(p+1) < x^\delta}} \log(p + 1)$ . Letting as usual

$\pi(x; a, b)$  stand for  $\#\{p \leq x : p \equiv b \pmod{a}\}$ , then, for some absolute constants  $c_3 > 0$ ,  $c_4 > 0$ , and  $c_5 > 0$ , we have that

$$\begin{aligned} T_\delta(x) &= \sum_{\substack{q^k \leq x, k \geq 1 \\ q < x^\delta}} (\log q) \pi(x; q^k, 1) \\ &\leq c_3 \frac{x}{\log x} \sum_{q < x^\delta} (\log q) \left( \frac{1}{q-1} + \frac{1}{q(q-1)} + \dots \right) + O \left( x \sum_{\substack{\sqrt{x} < q^k < x, k \geq 1 \\ q < x^\delta}} \frac{1}{q^k} \right) \\ &\leq c_4 \frac{x}{\log x} \sum_{q < x^\delta} \frac{\log q}{q} \leq c_5 \frac{x}{\log x} \delta \log x = c_5 \delta x. \end{aligned}$$

It follows from this last estimate that, provided  $x > x_0(\delta)$ , we have

$$\#\{p \in [x/2, x] : P(p + 1) < x^\delta\} < \frac{c_5 \delta x}{\log \sqrt{x}} + \sqrt{x} \leq \frac{3c_5 \delta x}{\log x}.$$

Replacing successively in the above the value of  $x$  by  $x/2, x/4, x/8, \dots$ , we obtain that, for some absolute constant  $c_6 > 0$ ,

$$\#\{p \leq x : P(p+1) < x^\delta\} = \sum_{1 \leq j \leq \log x / \log 2} \sum_{\substack{\frac{x}{2^j} < p \leq \frac{x}{2^{j-1}} \\ P(p+1) < x^\delta}} 1 \leq \frac{c_6 \delta x}{\log x},$$

thus proving (3.1) and thereby completing the proof of Lemma 1. □

Now assume that  $0 < \delta < 1/2$  and set

$$\wp_x^* := \{p \in [x/2, x] : x^\delta \leq P(p+1) \leq x^{1-\delta}\}.$$

Given a prime  $p \in \wp_x^*$  with  $P(p+1) = q$ , then

$$p+1 = mq \quad \text{for some positive integer } m. \tag{3.2}$$

Let  $R_m(x)$  be the number of solutions of (3.2) with  $p \in \wp_x^*$ . Then, if we let  $\phi$  stand for the Euler totient function, we have the following result.

**Lemma 2** *There exists an absolute constant  $C_2 > 0$  such that*

$$R_m(x) < C_2 \frac{x}{\log^2(x/m) \phi(m)} < C_2 \frac{x}{\delta^2 (\log x)^2 \phi(m)}.$$

**Proof** For a proof, see Theorem 4.6 in the book of Prachar [7]. □

**Lemma 3** *Given any real number  $\kappa \in (0, 1)$ , there exists an absolute constant  $C_3 > 0$  such that, for all integers  $u \geq 1$ ,*

$$S_u := \sum_{\substack{u \leq m \leq 2u \\ \phi(m)/m < \kappa}} \frac{1}{\phi(m)} < C_3 \kappa. \tag{3.3}$$

Moreover, there exists an absolute constant  $C_4 > 0$  such that

$$\sum_{\substack{m \leq x \\ \phi(m)/m < \kappa}} \frac{1}{\phi(m)} < C_4 \kappa \log x. \tag{3.4}$$

**Proof** Clearly,

$$S_u \leq \sum_{u \leq m \leq 2u} \frac{\kappa m}{\phi(m)} \cdot \frac{m}{u} \cdot \frac{1}{\phi(m)} = \frac{\kappa}{u} \sum_{u \leq m \leq 2u} \left(\frac{m}{\phi(m)}\right)^2. \tag{3.5}$$

Since one can easily establish that there exists a computable constant  $c_7 > 0$  such that

$$\sum_{m \leq x} \left(\frac{m}{\phi(m)}\right)^2 = (1 + o(1))c_7 x \quad (x \rightarrow \infty),$$

it follows from (3.5) that, for some absolute constant  $c_8$ , we have

$$S_u \leq \frac{\kappa}{u} \cdot c_8 u \quad (u \geq 1),$$

thus proving (3.3). Estimate (3.4) is a direct consequence of (3.3). □

Given real numbers  $z_1, \dots, z_M \in [0, 1)$ , let

$$D(z_1, \dots, z_M) := \frac{1}{M} \sup_{0 \leq \alpha < \beta < 1} \left| \sum_{z_\nu \in [\alpha, \beta)} 1 - M(\beta - \alpha) \right|$$

stand for the discrepancy of the sequence of numbers  $z_1, \dots, z_M$ . We have the following result.

**Lemma 4** *Let  $x_1, \dots, x_M \in [0, 1)$  and, for  $\ell = 1, \dots, M$ , let  $x_{M+\ell} = x_\ell + a$ , where  $a \in [0, 1)$ . Then*

$$D(x_1, \dots, x_{2M}) \leq D(x_1, \dots, x_M).$$

**Proof** The proof follows easily from the definition of the discrepancy and will therefore be omitted. □

**Lemma 5** *Let  $x_1, \dots, x_M \in [0, 1)$  and let  $m$  be an arbitrary integer. Then,*

$$\frac{1}{M} \left| \sum_{j=1}^M e(mx_j) \right| \leq 2\pi m D(x_1, \dots, x_M).$$

**Proof** Even though this is a well-known inequality, let us only mention that it can be obtained by the relation

$$\frac{1}{M} \sum_{j=1}^M e(mx_j) = - \int_0^1 \left( \left( \frac{1}{M} \sum_{x_\nu < u} 1 \right) - u \right) 2\pi i m e(mu) du$$

and partial integration. □

**Theorem A** (Kátai [6]) *Let  $f$  be an additive function and  $\alpha \in [0, 1)$ . Further set  $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\} < \alpha}} 1$  and*

$\Delta(n) := \sup_{0 \leq \alpha < \beta < 1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|$ . *Then  $\Delta(n) \rightarrow 0$  for almost all  $n$  if and only if  $\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q} = \infty$*

*for every positive integer  $m$ .*

#### 4. Proof of the main result

Let  $\kappa$ ,  $\delta$ , and  $\varepsilon$  be arbitrarily small positive numbers. We shall find an upper bound for the number of primes  $p \in [x/2, x]$  for which  $\Delta(p+1) \geq \varepsilon$ .

First of all, we know from Lemma 1 that

$$\#\{p \in [x/2, x] : p \notin \wp_x^*\} < C_1 \delta \pi(x). \tag{4.1}$$

On the other hand, it is clear that

$$\#\{p \in [x/2, x] : P^2(p+1) \mid p+1\} < c_9 \delta \pi(x) \tag{4.2}$$

for some constant  $c_9 > 0$ . Hence, we are left to consider the contribution of the other primes.

It follows from Lemma 4 that if (3.2) holds, then  $\Delta(p + 1) > \varepsilon$  only if  $\Delta(m) > \varepsilon$ . Now, according to Lemma 2, we may write that

$$\#\{p \in [x/2, x] : \Delta(p + 1) > \varepsilon\} \leq C_2 \frac{x}{\log^2 x} \sum_{\substack{x/2 \leq m < x^{1-\delta} \\ \Delta(m) > \varepsilon}} \frac{1}{\phi(m)} = C_2 \frac{x}{\log^2 x} S(x). \tag{4.3}$$

Let us write  $S(x) = S_1(x) + S_2(x)$ , where the sum in  $S_1(x)$  runs over those  $m$  for which  $\phi(m)/m \geq \kappa$ , whereas in  $S_2(x)$  it runs over those  $m$  for which  $\phi(m)/m < \kappa$ . As an easy consequence of Theorem A, we have that

$$S_1(x) = o(\log x) \quad (x \rightarrow \infty). \tag{4.4}$$

On the other hand, it follows from inequality (3.4) in Lemma 3 that

$$S_2(x) \leq C_4 \kappa \log x. \tag{4.5}$$

Therefore, gathering (4.1), (4.2), (4.4), and (4.5), it follows from (4.3) that, for some absolute constant  $c_{10} > 0$ ,

$$\#\{p \in [x/2, x] : \Delta(p + 1) > \varepsilon\} \leq c_{10} \delta \pi(x) + C_4 \kappa \pi(x) + o(\pi(x)) \quad (x \rightarrow \infty). \tag{4.6}$$

Applying this very same inequality with  $x$  replaced by  $x/2^j$  as  $j = 0, 1, \dots, \lfloor \log x / \log 2 \rfloor$ , we easily obtain that

$$\frac{1}{\pi(x)} \#\{p \leq x : \Delta(p + 1) > \varepsilon\} \leq c_{10} \delta + C_4 \kappa + o(1) \quad (x \rightarrow \infty),$$

from which it follows that

$$\limsup_{x \rightarrow \infty} \frac{1}{\pi(x)} \#\{p \leq x : \Delta(p + 1) > \varepsilon\} \leq c_{10} \delta + C_4 \kappa.$$

Since  $\kappa$  and  $\delta$  can be chosen arbitrarily small, this completes the proof of the sufficient part of Theorem 1.

We will now show the necessity of the divergence of the series  $\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q}$ . To do so, let us assume the contrary, i.e. that there exists some positive integer  $m$  such that

$$\sum_{q \in \wp} \frac{\|mf(q)\|^2}{q} < \infty. \tag{4.7}$$

Now consider the multiplicative function  $g_m$  defined by

$$g_m(n) = \frac{1}{\tau(n)} \left| \prod_{p^a \parallel n} \left( 1 + e^{2\pi i m f(p)} + e^{2\pi i m f(p^2)} + \dots + e^{2\pi i m f(p^a)} \right) \right|.$$

Observe that  $0 \leq g_m(n) \leq 1$  for all integers  $n \geq 1$  and that, at primes  $p$ ,

$$g_m(p) = \frac{|1 + e^{2\pi i m f(p)}|}{2},$$

so that

$$|g_m(p)|^2 = \frac{2 + 2 \cos 2\pi m f(p)}{4} = \cos^2 \pi m f(p),$$

which implies that

$$g_m(p) = |\cos \pi m f(p)|.$$

From this it follows that there exists an absolute constant  $c_6 > 0$  such that, for all primes  $p$ ,  $0 \leq 1 - g_m(p) \leq 1 - g_m^2(p) = \sin^2 \pi m f(p) \leq c_6 \|mf(p)\|^2$ . Hence, (4.7) implies that

$$\sum_{p \in \wp} \frac{1 - g_m(p)}{p} < \infty. \tag{4.8}$$

On the other hand, recall that the second author [4] proved the analogue of the famous Delange result [1] for shifted primes, namely the following.

**Theorem B** *Let  $g(n)$  be a complex-valued multiplicative function such that  $|g(n)| \leq 1$  for all  $n \in \mathbb{N}$  and such that the series*

$$\sum_{p \in \wp} \frac{g(p) - 1}{p}$$

*converges. Let  $N(g)$  be the product*

$$N(g) = \prod_{p \in \wp} \left( 1 - \frac{1}{p-1} + \sum_{j=1}^{\infty} \frac{g(p^j)}{p^j} \right).$$

*Then,*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} g(p+1) = N(g).$$

In light of (4.8), we may apply Theorem B to the function  $g_m$  and get that

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} g_m(p+1) = N(g_m). \tag{4.9}$$

Now we may assume that  $N(g_m) \neq 0$ , except in the case where  $g_m(2^\ell) = 0$  for  $\ell = 1, 2, \dots$ . However, if  $g_m(2) = 0$ , then it is easily seen that  $g_{2m}(2) \neq 0$ , which implies that  $N(g_{2m}) \neq 0$ . This is why we can make the assumption that  $N(g_m) \neq 0$ .

On the other hand, since  $0 \leq g_m(n) \leq 1$  for all integers  $n \geq 1$ , it follows from (4.9) that for a suitable constant  $\lambda > 0$  there exists a real number  $x_0 > 0$  such that

$$\frac{1}{\pi(x)} \#\{p \leq x : g_m(p+1) > \lambda\} > \lambda$$

for all  $x > x_0$ . Therefore, since  $g_m(n) < c_7 \Delta(n)$  for a suitable constant  $c_7 > 0$  (which follows from Lemma 5), we obtain that there exists a constant  $\lambda_1 > 0$  such that

$$\frac{1}{\pi(x)} \#\{p \leq x : \Delta(p+1) > \lambda_1\} > \lambda \quad (x > x_0),$$

thereby contradicting our assumption that  $\Delta(p+1) \rightarrow 0$  for almost all primes  $p$ , thus completing the proof of Theorem 1.

### 5. The special case $f(n) = \log n$

Consider the functions

$$h_n^*(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{\log d\} < \alpha}} 1 \quad \text{and} \quad \Delta^*(n) := \sup_{0 \leq \alpha < \beta < 1} |h_n^*(\beta) - h_n^*(\alpha) - (\beta - \alpha)|.$$

Hall [3] proved that, given any positive number  $\lambda < 1/2$ ,

$$\Delta^*(n) \leq \frac{1}{\tau(n)^\lambda} \quad \text{for almost all } n. \quad (5.1)$$

The second author [5] improved Hall's result by showing the following.

**Theorem C** *Inequality (5.1) holds for any positive number  $\lambda < \frac{\log \pi}{\log 2} - 1 \approx 0.651$ .*

Interestingly, we can prove that the analogue of Theorem B also holds for shifted primes. Indeed, using Theorem B and Lemma 4, similarly as Theorem 1 was deduced from Theorem A, one can easily show the following.

**Theorem 2** *Given any positive number  $\lambda < \frac{\log \pi}{\log 2} - 1$ ,*

$$\Delta^*(p+1) \leq \frac{1}{\tau(p+1)^\lambda} \quad \text{for almost all primes } p.$$

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