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The numerical range of matrices over \mathbb{F}_4

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Abstract: For any prime power q and any $u = (x_1, \dots, x_n), v = (y_1, \dots, y_n) \in \mathbb{F}_{q^2}^n$ set $\langle u, v \rangle := \sum_{i=1}^n x_i^q y_i$. For any $k \in \mathbb{F}_q$ and any $n \times n$ matrix M over \mathbb{F}_{q^2} , the k -numerical range $\text{Num}_k(M)$ of M is the set of all $\langle u, Mu \rangle$ for $u \in \mathbb{F}_{q^2}^n$ with $\langle u, u \rangle = k$ [5]. Here, we study the case $q = 2$, which is quite different from the case $q \neq 2$.

Key words: numerical range, finite field, binary field

1. Introduction and main results

Let q be a prime power. Let \mathbb{F}_q denote the only field, up to field isomorphisms, with $|\mathbb{F}_q| = q$ ([8, Theorem 2.5]). Let e_1, \dots, e_n be the standard basis of $\mathbb{F}_{q^2}^n$. For all $v, w \in \mathbb{F}_{q^2}^n$, say $v = a_1 e_1 + \dots + a_n e_n$ and $w = b_1 e_1 + \dots + b_n e_n$, set $\langle v, w \rangle = \sum_{i=1}^n a_i^q b_i$. $\langle \cdot, \cdot \rangle$ is the standard Hermitian form of $\mathbb{F}_{q^2}^n$. For any $n \geq 1$ and any $a \in \mathbb{F}_q$ set

$$C_n(a) := \{(x_1, \dots, x_n) \in \mathbb{F}_{q^2}^n \mid x_1^{q+1} + \dots + x_n^{q+1} = a\}.$$

The set $C_n(1)$ is an affine chart of the Hermitian variety of $\mathbb{P}^n(\mathbb{F}_{q^2})$ ([6, Chapter 5], [7, Chapter 23]). Take $M \in M_{n,n}(\mathbb{F}_{q^2})$, i.e. let M be an $n \times n$ matrix with coefficients in \mathbb{F}_{q^2} . For any $k \in \mathbb{F}_q$ set $\text{Num}_k(M) := \{\langle u, Mu \rangle \mid u \in C_n(k)\} \subseteq \mathbb{F}_{q^2}$. Set $\text{Num}(M) := \text{Num}_1(M)$. The set $\text{Num}(M)$ is called the *numerical range* of M . These concepts were introduced in [5] when q is a prime $p \equiv 3 \pmod{4}$ and in [1] in the general case. We always have $0 \in \text{Num}_0(M)$. When $n \geq 2$, we defined in [4] the set $\text{Num}'_0(M)$ as the set of all $\langle u, Mu \rangle$ for some $u \in \mathbb{F}_{q^2}^n \setminus \{0\}$ such that $\langle u, u \rangle = 0$. We have $\text{Num}_0(M) \setminus \{0\} \subseteq \text{Num}'_0(M) \subseteq \text{Num}_0(M)$. For any $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_{q^2})$, set $(M^\dagger)_{ij} = m_{ji}^q$. For any $M \in M_{n,n}(\mathbb{F}_{q^2})$ and any $u \in \mathbb{F}_{q^2}^n$, set $\nu_M(u) := \langle u, Mu \rangle$.

If $q = 2$ then in the definition of $C_n(a)$ we just take $a \in \mathbb{F}_2 = \{0, 1\}$. In particular for $q = 2$ (and for all even q by Remark 2.6), we only have to compute $\text{Num}_1(M)$ and $\text{Num}_0(M)$. The cases “ $q = 2$ ” and “ $q \neq 2$ ” (independently of the parity of q) are quite different, because if $x \in \mathbb{F}_4 \setminus \{0\}$, then $x^3 = 1$ and therefore when $q = 2$, the set $C_n(a)$ is just the set of all $(x_1, \dots, x_n) \in \mathbb{F}_4^n$ such that the number of nonzero entries x_i is $\equiv a \pmod{2}$ (Remark 2.8).

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Obviously all diagonal entries m_{ii} of a matrix M belong to $\text{Num}(M)$. If $q = 2$ and $n = 2$ and $M = (m_{ij})$, then $\text{Num}(M) = \{m_{11}, m_{22}\}$ (Remark 2.10).

We summarize our main results in the following way (here we take $q = 2$ and $M = (m_{ij}) \in M_{n,n}(F_4)$).

Proposition 1.1 *Assume $n \geq 3$. We have $\text{Num}_0(M) = \{0\}$ if and only if $M = c\mathbb{I}_{n \times n}$ for some $c \in \mathbb{F}_4$.*

Theorem 1.2 *Assume $n \geq 2$. We have $\text{Num}_0(M) = \{0, b\}$ with $b \neq 0$ if and only if $(\frac{1}{b}(M - m_{11}\mathbb{I}_{n \times n}))^\dagger = \frac{1}{b}(M - m_{11}\mathbb{I}_{n \times n})$ and $M \neq m_{11}\mathbb{I}_{n \times n}$ and this is the case if and only if $M \neq d\mathbb{I}_{n \times n}$ for any d and there is $c \in \mathbb{F}_4$ such that $(\frac{1}{b}(M - c\mathbb{I}_{n \times n}))^\dagger = \frac{1}{b}(M - c\mathbb{I}_{n \times n})$.*

Theorem 1.3 *Take $N \in M_{n,n}(\mathbb{F}_4)$, $n \geq 3$. We have $\text{Num}_1(N) = \{a, b\}$ with $a \neq b$ if and only if $(\frac{1}{b-a}(N - a\mathbb{I}_{n,n}))^\dagger = \frac{1}{b-a}(N - a\mathbb{I}_{n,n})$ and $N \neq c\mathbb{I}_{n,n}$ for some c .*

Corollary 1.4 *Assume $n \geq 3$. We have $\text{Num}(M) \subseteq \mathbb{F}_2$ if and only if $M^\dagger = M$.*

Proposition 1.5 *We have $|\text{Num}(M)| \leq 1$ if and only if $\text{Num}(M) = \{m_{11}\}$.*

(a) *If $n = 2$ we have $|\text{Num}(M)| = 1$ if and only if $m_{11} = m_{22}$.*

(b) *If $n \geq 3$, then $\text{Num}(M) = \{m_{11}\}$ if and only if $M = m_{11}\mathbb{I}_{n \times n}$.*

We also prove that $\text{Num}(M) = \mathbb{F}_4$ if $n \geq 3$, $M \neq 0\mathbb{I}_{n \times n}$ and M is strictly triangular (Remark 3.5).

2. Preliminaries

For any matrix $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_{q^2})$, let $M^\dagger = (a_{ij})$ be the matrix with $a_{ij} = m_{ji}^q$ for all i, j . M is said to be *Hermitian* if $M^\dagger = M$. Note that the diagonal elements of a Hermitian matrix are contained in \mathbb{F}_q . Let $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ be the standard basis of $\mathbb{F}_{q^2}^n$. Let $\mathbb{I}_{n \times n}$ denote the identity $n \times n$ matrix. For any $a \in \mathbb{F}_q$ and any $n > 0$ we have $C_n(a) \neq \emptyset$ by [1, Remark 3] and hence, $\text{Num}_a(M) \neq \emptyset$ for any a , any n , and any matrix M .

Notation 2.1 *Write $M = (m_{ij})$, $i, j = 1, \dots, n$.*

Remark 2.2 *Take $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_q)$. The vector e_i gives $m_{ii} \in \text{Num}(M)$. Hence, $\text{Num}(M)$ contains all diagonal elements of M .*

Remark 2.3 *For any $a, b \in \mathbb{F}_{q^2}^*$, any $k \in \mathbb{F}_q$, and any $M \in M_{n,n}(\mathbb{F}_{q^2})$, we have $\text{Num}_k(aM) = a\text{Num}_k(M)$ and $\text{Num}_k(M + b\mathbb{I}_{n,n}) = \text{Num}_k(M) + kb^{q+1}$ ([5, Proposition 3.1], [1, Remark 7], [4, Remark 2.4]).*

Remark 2.4 *Take $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_q)$ such that $M^\dagger = M$. For any $u \in \mathbb{F}_q^n$, we have $\langle u, Mu \rangle = \langle M^\dagger u, u \rangle = \langle Mu, u \rangle$. Hence, $\langle u, Mu \rangle \in \mathbb{F}_q$. Thus, $\text{Num}_k(M) \subseteq \mathbb{F}_q$ for every $k \in \mathbb{F}_q$.*

We recall from [5] the following definitions. For any $O, P \in \mathbb{F}_{q^2}$ the *strict affine \mathbb{F}_q -hull* $((O, P))$ of O and P is the set $\{tO + (1 - t)P\}_{t \in \mathbb{F}_q \setminus \{0,1\}}$. If $O = P$, then $((O, P)) = \{O\}$. If $O \neq P$, then $((O, P))$ is the complement of $\{O, P\}$ in the affine \mathbb{F}_q -line of $\mathbb{F}_{q^2} \cong \mathbb{F}_q^2$ spanned by O and P and hence it has cardinality

$q - 2$. For any two nonempty subsets $S, S' \subseteq \mathbb{F}_{q^2}$ set $((S, S')) := \cup_{O \in S, P \in S'} ((O, P))$. With this notation we have the following lemma ([1, Lemma 1]).

Lemma 2.5 *Let M be unitarily equivalent to the direct sums of matrices A and B . Then $\text{Num}(M) = ((\text{Num}(A), \text{Num}(B))) \cup \{\text{Num}_0(A) + \text{Num}(B)\} \cup \{\text{Num}(A) + \text{Num}_0(B)\}$.*

Remark 2.6 *Each element of \mathbb{F}_q^* , q even, is a square. Hence to compute all $\text{Num}_k(M)$ when q is even it is sufficient to compute $\text{Num}_0(M)$ and $\text{Num}_1(M)$.*

Unless otherwise stated, from now on $m_{ij} \in \mathbb{F}_4$ and $q = 2$.

Notation 2.7 *Fix $e \in \mathbb{F}_4 \setminus \mathbb{F}_2$. We have $e^3 = 1$, $e^2 + e = 1$ and $\mathbb{F}_4 = \{0, 1, e, e^2\}$. If $a \in \mathbb{F}_4^*$, then $a^3 = 1$. If a, b, c are 3 different elements of \mathbb{F}_4 , then $\{a, b, c, a + b + c\} = \mathbb{F}_4$. Hence for any $a \in \mathbb{F}_4$ the set $a + \mathbb{F}_4^*$ of all $a + b$, $b \in \mathbb{F}_4^*$ is the set $\mathbb{F}_4 \setminus \{a\}$. We fix some $e \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and write $\mathbb{F}_4 = \{0, 1, e, e^2\}$.*

Remark 2.8 *For each $x \in \mathbb{F}_4^*$ we have $x^3 = 1$. We obviously have $0^3 = 0$. Take $u = (x_1, \dots, x_n) \in \mathbb{F}_4^n$. We have $x_1^3 + \dots + x_n^3 = 1$ (resp. $x_1^3 + \dots + x_n^3 = 0$) if and only if $x_i \neq 0$ for an odd (resp. an even) number of indices i .*

Remark 2.9 *Take $u \in \mathbb{F}_4^n$ and $t \in \mathbb{F}_4^*$. We have $\langle tu, M(tu) \rangle = t^{q+1} \langle u, Mu \rangle = \langle u, Mu \rangle$, because $t^3 = 1$.*

Remark 2.10 *Assume $n = 2$. By Remark 2.8, we have $\text{Num}(M) = \{m_{11}, m_{22}\}$.*

3. Strictly triangular matrices

We first list some cases with $n = 3$ in which we prove that $\text{Num}(M) = \mathbb{F}_4$. All these matrices are triangular matrices with equal entries, m_{11} , on the diagonal. By [5, Lemma 2.7] to compute $\text{Num}(M)$, it is sufficient to compute $\text{Num}(N)$, where N is the strictly triangular matrix $M - m_{11}\mathbb{I}_{3 \times 3}$.

Proposition 3.1 *Fix $a, b, c \in \mathbb{F}_4$ with $ab \neq 0$. Take*

$$M = \begin{pmatrix} c & b & 0 \\ 0 & c & a \\ 0 & 0 & c \end{pmatrix}$$

Then $\text{Num}(M) = \mathbb{F}_4$.

Proof Taking $\frac{1}{a}(M - c\mathbb{I}_{3 \times 3})$ instead of M and applying [5, Lemma 2.7], we reduce to the case $c = 0$ and $a = 1$. Note that even after this reduction step, we have $b \in \mathbb{F}_4 \setminus \{0\}$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume $x_1^3 + x_2^3 + x_3^3 = 1$. We have $\langle u, Mu \rangle = x_2(bx_1^2 + x_2x_3)$. Taking $x_1 = x_3 = 1$ and $x_2 = 0$, we get $0 \in \text{Num}(M)$. From now on we always take $x_2 = 1$ and in particular $x_2^3 = 1$. Thus, we may use any x_1, x_3 with $x_1^3 + x_3^3 = 0$, i.e. any $(x_1, x_3) \in \mathbb{F}_4^2$ with either $x_1 = x_3 = 0$ or $x_1x_3 \neq 0$. Fix $c \in \mathbb{F}_4 \setminus \{b\}$. If $u = (1, 1, c - b)$, then $\langle u, Mu \rangle = 1(b + c - b) = c$. Note that $1 + e \neq 0$ and hence, $b(1 + e) \neq 0$. We take $u = (e, 1, b(1 + e)) = be + b + be = b$. □

Proposition 3.2 Fix $a, b, c, d \in \mathbb{F}_4$ with $abd \neq 0$. Take

$$M = \begin{pmatrix} c & b & d \\ 0 & c & a \\ 0 & 0 & c \end{pmatrix}$$

Then $\text{Num}(M) = \mathbb{F}_4$.

Proof As in the proof of Proposition 3.1, we reduce to the case $c = 0$ and $d = 1$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume $x_1^3 + x_2^3 + x_3^3 = 1$. We have $Mu = (bx_2 + x_3, ax_3, 0)$ and hence, $\langle u, Mu \rangle = bx_2x_1^2 + x_3x_1^2 + ax_3x_2^2$. Taking $u = (1, 0, 0)$, we get $0 \in \text{Num}(M)$. From now on we always take $x_2 = 1$ and in particular $x_2^3 = 1$. Thus, we may use any x_1, x_3 with $x_1^3 + x_3^3 = 0$, i.e. any $(x_1, x_3) \in \mathbb{F}_4^2$ with either $x_1 = x_3 = 0$ or $x_1x_3 \neq 0$. Fix $c \in \mathbb{F}_4 \setminus \{0\}$. It is sufficient to find $x_1, x_3 \in \mathbb{F}_4 \setminus \{0\}$ with $x_1^2(b + x_3) = c + ax_3$. Since $a \neq 0$ and $|\mathbb{F}_4 \setminus \{0\}| = 3$, there is $w \in \mathbb{F}_4 \setminus \{0\}$ such that $b + w \neq 0$ and $c + aw \neq 0$. Since the Frobenius map $t \mapsto t^2$ induces a permutation $\mathbb{F}_4 \setminus \{0\} \rightarrow \mathbb{F}_4 \setminus \{0\}$, there is $z \in \mathbb{F}_4 \setminus \{0\}$ such that $z^2 = (c + aw)/(b + w)$. Take $u = (z, 1, w)$. \square

Proposition 3.3 Fix $a, b, c \in \mathbb{F}_4$ with $(a, b) \neq (0, 0)$. Take

$$M = \begin{pmatrix} c & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$$

Then $\text{Num}(M) = \mathbb{F}_4$.

Proof It is sufficient to do the case $c = 0$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume that 2 or none among x_1, x_2, x_3 are zeroes. We have $Mu = (ax_2 + bx_3, 0, 0)$ and hence, $\langle u, Mu \rangle = x_1^2(ax_2 + bx_3)$. Taking $u = (1, 0, 0)$, we get $0 \in \text{Num}(M)$. Fix $w \in \mathbb{F}_4 \setminus \{0\}$. Since $(a, b) \neq (0, 0)$, there is $(a_2, a_3) \in (\mathbb{F}_4 \setminus \{0\})^2$ such that $aa_2 + ba_3 \neq 0$. Since the Frobenius map $t \mapsto t^2$ induces a permutation $\mathbb{F}_4 \setminus \{0\} \rightarrow \mathbb{F}_4 \setminus \{0\}$, there is $z \in \mathbb{F}_4 \setminus \{0\}$ such that $z^2 = w/(aa_2 + ba_3)$. Take $u = (z, a_2, a_3)$. \square

Proposition 3.4 Take $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_4)$, $n \geq 3$, such that $m_{ij} = 0$ for all $i \geq j$. We have $\text{Num}(M) = \mathbb{F}_4$ if and only if $M \neq 0$.

Proof Since $\text{Num}(0\mathbb{I}_{n \times n}) = \{0\}$, we only need to prove the “if” part. Assume $m_{ij} \neq 0$ for some $i < j$. Take any principal minor of M associated to i, j and some $h \in \{1, \dots, n\} \setminus \{i, j\}$ and apply one of the Propositions above. \square

Remark 3.5 Take M as in one of the Propositions 3.1, 3.2, 3.3, 3.4. A similar proof works for M^t . Thus, we computed $\text{Num}(M)$ for all strictly triangular matrices and proved that $\text{Num}(M) = \mathbb{F}_4$, unless either $n = 2$ or $M = m_{11}\mathbb{I}_{3 \times 3}$.

4. The proofs

Lemma 4.1 Take $n = 2$. We have $\text{Num}_0(M) = \{0, m_{11} + m_{22} + m_{12} + m_{21}, m_{11} + m_{22} + m_{12}e + m_{21}e^2, m_{11} + m_{22} + m_{12}e^2 + m_{21}e\}$.

1. $\text{Num}_0(M) = \{0\}$ if and only if $m_{11} = m_{22}$ and $m_{12} = m_{21} = 0$.
2. If $m_{11} = m_{22}$ and $m_{12} = m_{21} \neq 0$, then $\text{Num}_0(M) = \{0, m_{12}\}$.
3. Assume $m_{11} = m_{22}$. If either $m_{12} = 0$ and $m_{21} \neq 0$ or $m_{21} = 0$ and $m_{12} \neq 0$, then $\text{Num}_0(M) = \mathbb{F}_4$.
4. If $m_{11} = m_{22}$, $m_{12} \neq m_{21}$ and $m_{12}m_{21} \neq 0$, then $\text{Num}_0(M) = \{0, m_{12}, m_{21}\}$.
5. If $m_{11} \neq m_{22}$ and $m_{21} = m_{12} = 0$, then $\text{Num}_0(M) = \{0, m_{11} + m_{22}\}$.
6. If $m_{11} \neq m_{22}$ and $m_{21} = m_{12} \neq 0$, then $\text{Num}_0(M) = \{0, m_{11} + m_{22}, m_{11} + m_{22} + m_{12}\}$; this set has two elements if and only if $m_{22} = m_{11} + m_{21}$.
7. Assume $m_{11} \neq m_{22}$ and either $m_{12} = 0$ and $m_{21} \neq 0$ or $m_{12} \neq 0$ and $m_{21} = 0$, then $|\text{Num}_0(M)| = 3$ and $\text{Num}_0(M) = \mathbb{F}_4 \setminus \{m_{11} + m_{22}\}$.
8. If $m_{11} \neq m_{22}$, $m_{21}m_{12} \neq 0$ and $m_{12} \neq m_{21}$, then $\text{Num}_0(M) = \{0, m_{11} + m_{22} + em_{12} + e^2m_{21}, m_{11} + m_{22} + e^2m_{12} + em_{21}\}$; we have $|\text{Num}_0(M)| = 2$ if and only if either $m_{11} + m_{22} + em_{12} + e^2m_{21} = 0$ or $m_{11} + m_{22} + e^2m_{12} + em_{21} = 0$.

Proof Take $u = (x_1, x_2) \in \mathbb{F}_4^2$ with $x_1^3 + x_2^3 = 0$. The case $u = (0, 0)$ gives $0 \in \text{Num}_0(M)$. By Remark 2.8 to compute the other elements of $\text{Num}_0(M)$, we may assume $x_1 \neq 0$ and $x_2 \neq 0$. Using Remark 2.9 with $t = x_1^{-1}$ we reduce to the case $x_1 = 1$. Taking $x_2 = 1$ (resp. $x_2 = e$, resp. $x_2 = e^2$), we get $m_{11} + m_{12} + m_{21} + m_{22} \in \text{Num}_0(M)$ (resp. $m_{11} + m_{12}e + m_{21}e^2 + m_{22} \in \text{Num}_0(M)$, $m_{11} + m_{22} + m_{12}e^2 + m_{21}e \in \text{Num}_0(M)$). We have $m_{12}e + m_{21}e^2 = m_{12}e^2 + m_{21}e$ if and only if $m_{12}(e + e^2) = m_{21}(e + e^2)$, i.e. if and only if $m_{12} = m_{21}$. We have $m_{12} + m_{21} = m_{12}e + m_{21}e^2$ if and only if $m_{12}(1 + e) = m_{21}(1 + e^2)$, i.e. if and only if $m_{12}e = m_{21}$. We have $m_{12} + m_{21} = m_{12}e^2 + m_{21}e$ if and only if $(m_{12}(1 + e^2) = m_{21}(1 + e))$, i.e. if and only if $m_{12} = em_{21}$, i.e. $m_{12}e^2 = m_{21}$.

(a) Assume $m_{11} = m_{22}$. If $m_{12} = m_{21} = 0$, then $\text{Num}_0(M) = \{0\}$. Now assume $m_{12} = m_{21} \neq 0$. Since $e^2 + e = 1$, we get that $\text{Num}_0(M) = \{0, m_{12}e, m_{12}\}$. If $m_{12} = 0 \neq m_{21}$ (resp. $m_{21} = 0$ and $m_{12} \neq 0$), then $\text{Num}_0(M)$ contains $0, m_{21}, em_{21}, e^2m_{21}$ (resp. $0, m_{12}, em_{12}, e^2m_{12}$); in both cases we get $\text{Num}_0(M) = \mathbb{F}_4$. Now assume $m_{12} \neq m_{21}$ and $m_{12}m_{21} \neq 0$. Set $t := m_{12}/m_{21}$. Either $t = e$ or $t = e^2$. Assume $t = e$ (the case $t = e^2$ being similar). $\text{Num}_0(M)$ is the union of $0, m_{21}(1 + e) = m_{21}e^2 = m_{12}, m_{21}(e + e^2) = m_{21}$ and $m_{21}(e^2 + e^2) = 0$. Hence, $\text{Num}_0(M) = \{0, m_{12}, m_{21}\}$.

(b) Assume $m_{11} \neq m_{22}$. If $m_{21} = m_{12} = 0$, then $\text{Num}_0(M) = \{0, m_{11} + m_{22}\}$. If $m_{21} = m_{12} \neq 0$, then $\text{Num}_0(M)$ is the union of $\{0\}$, $m_{11} + m_{22}$ and $m_{11} + m_{22} + m_{12}$ (recall that $2m_{12} = 0$ and $e + e^2 = 1$). Assume $m_{12} = 0$ and $m_{21} \neq 0$. We get $\text{Num}_0(M) = \{0, m_{11} + m_{22} + m_{21}, m_{11} + m_{22} + m_{21}e, m_{11} + m_{22} + m_{21}e^2\}$. Since $\{m_{21}, m_{21}e, m_{21}e^2\} = \mathbb{F}_4^*$, we get $|\text{Num}_0(M)| = 3$ and $\text{Num}_0(M) = \mathbb{F}_4 \setminus \{m_{11} + m_{22}\}$. The same answer comes if $m_{12} \neq 0$ and $m_{21} = 0$. Now assume $m_{12} \neq 0, m_{21} \neq 0$, and $m_{12} \neq m_{21}$. We first check that $A := \{m_{12} + m_{21}, em_{12} + e^2m_{21}, e^2m_{12} + em_{21}\}$ has cardinality 2. Since $e \neq e^2$ and $m_{12} \neq m_{21}$, we have $em_{12} + e^2m_{21} \neq e^2m_{12} + em_{21}$. We have $m_{12} + m_{21} = em_{12} + e^2m_{21}$ if and only if $(1 + e)m_{12} = (1 + e^2)m_{21}$, i.e. if and only if $e^2m_{12} = em_{21}$, i.e. if and only if $em_{12} = m_{21}$. In the same way, we see that $m_{12} + m_{21} = e^2m_{12} + em_{21}$ if and only if $e^2m_{12} = m_{21}$. Since $m_{21}/m_{12} \notin \{0, 1\}$, we have $m_{21}/m_{12} \in \{e, e^2\}$. Hence $A = \{em_{12} + e^2m_{21}, e^2m_{12} + em_{21}\}$. We get part (8). \square

Lemma 4.2 Take $q = 2$, $n = 3$ and $m_{11} = m_{22} = m_{33}$.

1. If $m_{ij} = 0$ for all $i \neq j$, then $\text{Num}_0(M) = \{0\}$.
2. If $m_{ij} = 0$ and $m_{ji} \neq 0$ for some i, j , then $\text{Num}_0(M) = \mathbb{F}_4$.
3. If $m_{ij} \neq m_{ji}$, then $\text{Num}_0(M) \supseteq \{0, m_{ij}, m_{ji}\}$.
4. If $m_{ij} = m_{ji} \neq 0$, then $\text{Num}_0(M) \supseteq \{0, m_{ij}\}$.

All nonzero m_{ij} are in $\text{Num}_0(M) \setminus \{0\}$ and they are the only elements of $\text{Num}_0(M) \setminus \{0\}$ unless there is i, j with $m_{ij} = 0$ and $m_{ji} \neq 0$.

Proof Take $u = (x_1, x_2, x_3) \in \mathbb{F}_4$ with $x_1^3 + x_2^3 + x_3^3 = 0$. By Remark 2.8 either $u = 0$ or there is $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $x_h \neq 0$ if and only if $h \in \{i, j\}$. Apply Lemma 4.1 to the restriction of M to $\mathbb{F}_4 e_i + \mathbb{F}_4 e_j$. \square

Remark 2.8, Lemma 4.1, and the proof of Lemma 4.2 give the following result.

Lemma 4.3 Take $n = 3$ and $m_{11} = m_{22} \neq m_{33}$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be the 2×2 matrices with $a_{ij} = m_{hk}$, $h = i + 1$, $k = i + 1$, and $b_{ij} = m_{vw}$, $v = i$ if $i = 1$, $v = 3$ if $i = 2$, $w = j$ if $j = 1$, $w = 3$ if $j = 2$. A and B are as in one of the last 4 cases of Lemma 4.1.

1. If $m_{12}m_{21} = 0$ and $m_{12} \neq m_{21}$, then $\text{Num}_0(M) = \mathbb{F}_4$
2. If $m_{12} = m_{21} = 0$, then $\text{Num}_0(M) = \text{Num}_0(A) \cup \text{Num}_0(B)$.
3. If $m_{12}m_{21} \neq 0$ and $m_{12} \neq m_{21}$, then $\text{Num}_0(M) = \text{Num}_0(A) \cup \text{Num}_0(B) \cup \{m_{12}, m_{21}\}$; we have $|\text{Num}_0(M)| \geq 3$; we have $\text{Num}_0(M) \neq \mathbb{F}_4$ if and only if $\text{Num}_0(A) \cup \text{Num}_0(B) \supseteq \{0, m_{12}, m_{21}\}$.

Take the set-up of (3), i.e. assume $m_{12}m_{21} \neq 0$ and $m_{12} \neq m_{21}$. $\text{Num}_0(A) \supseteq \{0, m_{12}, m_{21}\}$ if and only if either $m_{13} = m_{31} = 0$ and $m_{11} + m_{33} \in \{m_{12}, m_{21} \text{ or } m_{31} = m_{13} \neq 0 \text{ and } m_{11} + m_{33}, m_{11} + m_{33} + m_{13}\} \subseteq \{0, m_{12}, m_{21} \text{ or } m_{13}m_{31} = 0, m_{13} \neq m_{31} \text{ and } \mathbb{F}_4 \setminus \{m_{11} + m_{33}\} \text{ or } m_{31}m_{13} \neq 0, m_{13} \neq m_{31} \text{ and } \{m_{11} + m_{33} + em_{13} + e^2m_{31}, m_{11} + m_{33} + e^2m_{13} + em_{31}\} \subseteq \{0, m_{12}, m_{21}\}$. The same list works for B exchanging the indices 1 and 2.

Lemma 4.4 Take $n = 3$ and $|\{m_{11}, m_{22}, m_{33}\}| = 3$. Let $A_h = (a_{ij}^h)$, $i, j = 1, 2$, $h = 1, 2, 3$, be the matrix obtained from M deleting the h -th row and the h -th column; each A_h is as in one of the last 4 cases of Lemma 4.1. We have $\text{Num}_0(M) = \text{Num}_0(A_1) \cup \text{Num}_0(A_2) \cup \text{Num}_0(A_3)$. Hence, $\text{Num}_0(M) = \mathbb{F}_4$ if one of the following conditions is satisfied:

1. $m_{ij} = 0$ for all $i \neq j$.
2. There is $i \in \{1, 2, 3\}$ such that, writing $\{1, 2, 3\} = \{i, j, h\}$, we have $m_{jh}m_{jh} = 0$ and $m_{jh} \neq m_{hj}$.
3. There is $i \in \{1, 2, 3\}$ such that, writing $\{1, 2, 3\} = \{i, j, h\}$, we have $m_{ij} \neq 0$, $m_{ji} \neq 0$, $m_{hi} \neq 0$, $m_{ih} \neq 0$, and $\{m_{ij}, m_{ji}, m_{ih}, m_{hi}\} = \mathbb{F}_4^*$.

Proof Remark 2.8, Lemma 4.1, and the proof of Lemma 4.2 give $\text{Num}_0(M) = \text{Num}_0(A_1) \cup \text{Num}_0(A_2) \cup \text{Num}_0(A_3)$.

(a) Assume $m_{ij} = 0$ for all $i \neq j$. Since $m_{11} + m_{22}$, $m_{11} + m_{33}$, and $m_{22} + m_{33}$ are distinct elements of \mathbb{F}_4^* , we have $\{0, m_{11} + m_{22}, m_{11} + m_{33}, m_{22} + m_{33}\} = \mathbb{F}_4$.

(b) Assume $m_{12}m_{21} = m_{13}m_{31} = 0$ and $m_{12} + m_{21} \neq 0$, $m_{13} + m_{31} \neq 0$. We have $\text{Num}_0(A_3) = \mathbb{F}_4 \setminus \{m_{11} + m_{22}\}$ and $\text{Num}_0(A_2) = \mathbb{F}_4 \setminus \{m_{11} + m_{33}\}$ and thus, $\text{Num}_0(M) = \mathbb{F}_4$. The same proof works if either $m_{23}m_{32} = m_{13}m_{31} = 0$ and $m_{23} + m_{32} \neq 0$, $m_{13} + m_{31} \neq 0$ or $m_{12}m_{21} = m_{23}m_{32} = 0$ and $m_{12} + m_{21} \neq 0$, $m_{23} + m_{32} \neq 0$.

(c) Assume $m_{12} \neq 0$, $m_{21} \neq 0$, $m_{13} \neq 0$, and $m_{31} \neq 0$ and $\{m_{12}, m_{21}, m_{31}, m_{13}\} = \mathbb{F}_4^*$. The last 2 cases in Lemma 4.1 give $\text{Num}_0(A_3) \cup \text{Num}_0(A_2) = \mathbb{F}_4$. The same proof works for $\text{Num}_0(A_1) \cup \text{Num}_0(A_2)$ and $\text{Num}_0(A_3) \cup \text{Num}_0(A_1)$. \square

Lemma 4.5 Assume $n = 3$. $\text{Num}(M)$ is the union of $\{m_{11}, m_{22}, m_{33}\}$, $\sum_{i,j=1}^3 m_{ij}$, $m_{11} + m_{22} + m_{33} + e(m_{12} + m_{23} + m_{31}) + e^2(m_{13} + m_{21} + m_{32})$, $m_{11} + m_{22} + m_{33} + e^2(m_{12} + m_{23} + m_{31}) + e(m_{13} + m_{21} + m_{32})$ and B_h , $h = 1, 2, 3$, where (writing $\{i, j, h\} = \{1, 2, 3\}$) $B_h = \{m_{ii} + m_{ij} + m_{ji} + m_{jj} + m_{hh} + e(m_{ih} + m_{jh}) + e^2(m_{hi} + m_{hj}), m_{ii} + m_{ij} + m_{ji} + m_{jj} + m_{hh} + e^2(m_{ih} + m_{jh}) + e(m_{hi} + m_{hj})\}$.

Proof Take $u = (x_1, x_2, x_3) \in \mathbb{F}_4^3$ with $x_1^3 + x_2^3 + x_3^3 = 1$. By Remark 2.8, there is an odd number of indices i with $x_i \neq 0$. Taking u with exactly one nonzero coordinate, we get $\{m_{11}, m_{22}, m_{33}\} \subseteq \text{Num}(M)$. Thus, it is sufficient to test all u with $x_i \neq 0$ for every i . Taking $u = (t, t, t)$ for some $t \neq 0$, we get $\sum_{i,j=1}^3 m_{ij} \in \text{Num}(M)$. Now assume that u has exactly two different entries, say $x_i = x_j$ with $i \neq j$ and $x_h \neq x_i$, $\{i, j, h\} = \{1, 2, 3\}$. By Remark 2.9, we may assume that $x_i = 1$ and hence, either $x_h = e$ or $x_h = e^2$. In the first case, $\langle u, Mu \rangle = m_{ii} + m_{ij} + m_{ji} + m_{jj} + m_{hh} + e(m_{ih} + m_{jh}) + e^2(m_{hi} + m_{hj})$. In the second case, we have $\langle u, Mu \rangle = m_{ii} + m_{ij} + m_{ji} + m_{jj} + m_{hh} + e^2(m_{ih} + m_{jh}) + e(m_{hi} + m_{hj})$. Now assume that all entries of u are different. By Remark 2.9, we may assume that $x_1 = 1$. Hence, either $(x_2, x_3) = (e, e^2)$ or $(x_2, x_3) = (e^2, e)$. In the first case, we have $\langle u, Mu \rangle = m_{11} + m_{22} + m_{33} + e(m_{12} + m_{23} + m_{31}) + e^2(m_{13} + m_{21} + m_{32})$. In the second case, we have $\langle u, Mu \rangle = m_{11} + m_{22} + m_{33} + e^2(m_{12} + m_{23} + m_{31}) + e(m_{13} + m_{21} + m_{32})$. \square

Corollary 4.6 Assume $n = 3$ and $|\{m_{11}, m_{22}, m_{33}\}| = 3$. We have $\text{Num}(M) = \mathbb{F}_4$ if one of the following conditions is satisfied

1. $m_{12} + m_{23} + m_{31} = t(m_{13} + m_{21} + m_{32})$ with $t \in \{e, e^2\}$;
2. $m_{ij} = m_{12}$ for all $i \neq j$.

Proof Since $|\{m_{11}, m_{22}, m_{33}\}| = 3$, we have $\mathbb{F}_4 = \{m_{11}, m_{22}, m_{33}, m_{11} + m_{22} + m_{33}\}$ (Remark 2.7). Hence, it is sufficient to check if $m_{11} + m_{22} + m_{33} \in \text{Num}(M)$. To get (1), use the third and fourth sum in Lemma 4.5 and that $e^3 + e^3 = 0$. To get (2), use B_1 in Lemma 4.5. \square

Corollary 4.7 Assume $n = 3$ and $|\{m_{11}, m_{22}, m_{33}\}| = 3$. Fix 5 of the elements m_{ij} with $i \neq j$, say all except m_{hk} . There is a choice of m_{hk} with $\text{Num}(M) = \mathbb{F}_4$.

Proof Since $|\{m_{11}, m_{22}, m_{33}\}| = 3$, we have $\mathbb{F}_4 = \{m_{11}, m_{22}, m_{33}, m_{11} + m_{22} + m_{33}\}$ (Remark 2.7). Hence, it is sufficient to check that $m_{11} + m_{22} + m_{33} \in \text{Num}(M)$ for some choice of m_{hk} . We do the case $h = 1, k = 2$, because the other cases are similar. We take $m_{12} := e(m_{13} + m_{21} + m_{32}) + m_{23} + m_{31}$ and apply part (1) of Corollary 4.6. The proof shows that if $m_{13} + m_{21} + m_{32} \neq 0$, we have at least 2 choices for m_{12} . \square

Lemma 4.8 Take $n = 3$. We have $|\text{Num}(M)| = 1$ if and only if $M = m_{11}\mathbb{I}_{3 \times 3}$.

Proof Since the “if” part is trivial, we assume $|\text{Num}(M)| = 1$. Since we have $\{m_{11}, m_{22}, m_{33}\} \subseteq \text{Num}(M)$ (Remark 2.2), we have $m_{11} = m_{22} = m_{33}$. Since $e^3 = 1$, Lemma 4.5 gives $m_{12} + m_{23} + m_{31} = e(m_{13} + m_{21} + m_{32})$ and $m_{12} + m_{23} + m_{31} = e^2(m_{13} + m_{21} + m_{32})$. Hence, $m_{12} + m_{23} + m_{31} = m_{13} + m_{21} + m_{32} = 0$, i.e. $m_{31} = m_{12} + m_{23}$ and $m_{32} = m_{13} + m_{21}$. Lemma 4.5 implies that $m_{ij} + m_{ji} + e(m_{ih} + m_{jh}) + e^2(m_{hi} + m_{hj}) = 0$ and $m_{ij} + m_{ji} + e^2(m_{ih} + m_{jh}) + e(m_{hi} + m_{hj}) = 0$ for all $\{i, j, h\} = \{1, 2, 3\}$. Since $e^2 - e = 1$, subtracting these equalities, we get $m_{ih} + m_{jh} = m_{hi} + m_{hj}$ for all i, h, j . In particular, we have $m_{13} + m_{23} = m_{31} + m_{32} = m_{12} + m_{23} + m_{13} + m_{21}$, i.e. $m_{12} = m_{21}$. In the same way, we get $m_{ij} = m_{ji}$ for all $i \neq j$. Then the set B_h in the statement of Lemma 4.6 gives $e(m_{ih} + m_{jh}) + e^2(m_{ih} + m_{ji}) = 0$. Since $e \neq e^2$, we get $m_{ih} + m_{jh} = 0$, i.e. $m_{ih} = m_{jh}$, for all $\{i, j, h\} = \{1, 2, 3\}$. Since M is symmetric, we get $m_{ij} = m_{12}$ for all $i \neq j$. Since $m_{12} = 3m_{12} = m_{12} + m_{23} + m_{31} = 0$, we get $m_{ij} = 0$ for all $i \neq j$. \square

Proof [Proof of Proposition 1.1:] The “if” part is obvious, while the “only if” part follows from part (1) of Lemma 4.1. \square

Proof [Proof of Proposition 1.5:] We always have $\{m_{11}, \dots, m_{nn}\} \subseteq \text{Num}(M)$ (Remark 2.2) and this inclusion is an equality if $n = 2$ (Remark 2.10), proving the case $n = 2$. The case $n = 3$ is true by Lemma 4.8. Now assume $n \geq 4$ and $|\text{Num}(M)| = 1$. Hence $m_{ii} = m_{11}$ for all i . By Lemma 4.8 applied to all $\mathbb{F}_4 e_i + \mathbb{F}_4 e_j + \mathbb{F}_4 e_h$ we have $m_{ij} = 0$ for all $i \neq j$. \square

Proof [Proof of Theorem 1.2:] Taking $\frac{1}{b}M$ instead of M , we reduce to the case $b = 1$ (Remark 2.3). Note that $\text{Num}_0(M) = \text{Num}_0(M - c\mathbb{I}_{n \times n})$ for any $c \in \mathbb{F}_4$ (Remark 2.3). Hence, Remark 2.4 and Proposition 4.4 give the “if” part. Note that $M - m_{11}\mathbb{I}_{n \times n}$ is Hermitian if and only if $m_{ij}^2 = m_{ji}$ for all $i \neq j$ and $m_{ii} - m_{11} \in \mathbb{F}_2$ for all i . Hence, $M - m_{11}\mathbb{I}_{n \times n}$ is Hermitian if and only if $M - m_{ii}\mathbb{I}_{n \times n}$ is Hermitian for some $i \in \{1, \dots, n\}$ and hence, that $M - m_{11}\mathbb{I}_{n \times n}$ is Hermitian if and only if there is $c \in \mathbb{F}_4$ with $M - c\mathbb{I}_{n \times n}$ Hermitian.

Now, assume that $\text{Num}_0(M) \subseteq \{0, 1\}$. Taking $A := M|_{\mathbb{F}_4 e_i + \mathbb{F}_4 e_j}$ with $i \neq j$ we reduce to the case $n = 2$; we write $A = (a_{hk}), h, k = 1, 2$. Then taking $M - a_{11}\mathbb{I}_{2 \times 2}$, we reduce to the case $a_{11} = 0$. After this reduction, we need to prove that $M^\dagger = M$. If $n = 2$ we assume $\text{Num}_0(M) = \{0, 1\}$, but if $n \geq 3$ we only assume that $\text{Num}_0(A) \subseteq \{0, 1\}$.

First, assume that $a_{22} = 0$. Set $\alpha := a_{12} + a_{21}$, $\beta := ea_{12} + e^2a_{21}$ and $\gamma := e^2a_{12} + a_{21}$. Since $2a = 0$ for all $a \in \mathbb{F}_4$, $e^2 + 1 = e$, $e + 1 = e^2$ and $e^2 + 1 = \alpha$, we have $\alpha + \beta = \gamma$, $\beta + \gamma = \alpha$ and $\alpha + \gamma = \beta$. Lemma 4.1 gives $\alpha, \beta, \gamma \in \mathbb{F}_2$. First, assume that $\alpha = 0$, i.e. $a_{12} = a_{21}$. We get $\beta = (e^2 + e)a_{12}$ and so $a_{12} \in \mathbb{F}_2$. Thus, A is Hermitian in this case.

Now, assume that $a_{22} \neq 0 = a_{11}$. If $a_{12} = a_{21} = 0$, then part (5) of Lemma 4.1 implies that $a_{22} \in \mathbb{F}_2$ and hence, A is Hermitian. Case (7) of Lemma 4.1 excludes the case where $a_{12}a_{21} = 0$ and

$(a_{12}, a_{21}) \neq (0, 0)$. If $a_{12} = a_{21} \neq 0$, then case (6) of Lemma 4.1 gives $a_{22} = a_{11} + a_{21}$, i.e. $a_{22} = a_{12} = a_{21}$; since $a_{11} + a_{22} + a_{12} + a_{21} \in \text{Num}_0(M) \subseteq \{0, 1\}$ and $a_{11} = 0$, we get $a_{ij} \in \mathbb{F}_2$ for all $i, j = 1, 2$. Since $a_{21} = a_{12}$, this 2×2 matrix is Hermitian. Now, assume that $a_{12} \neq a_{21}$ and $a_{12}a_{21} \neq 0$. Part (8) of Lemma 4.1 gives that either $\delta := a_{22} + ea_{12} + e^2a_{21} = 0$ or $\eta := a_{22} + e^2a_{12} + ea_{21} = 0$ and that (since $a_{12} \neq a_{21}$) either $\delta = 0$ and $\eta = 1$ or $\delta = 1$ and $\eta = 0$. Since $e^2 + e = 1$, we have $1 = \delta + \eta = (e^2 + e)a_{12} + (e^2 + e)a_{21} = a_{21} + a_{12}$. Since $a_{12} \neq 0$ and $a_{21} \neq 0$, we get $(a_{12}, a_{21}) \in \{(e, e^2), (e^2, e)\}$ and hence, $a_{21} = a_{12}^2$. Since $e^3 = 1$, if $(a_{12}, a_{21}) = (e^2, e)$ (resp. $(a_{12}, a_{21}) = (e, e^2)$), then $\delta \in \mathbb{F}_2$ (resp. $\eta \in \mathbb{F}_2$) gives $a_{22} \in \mathbb{F}_2$. Hence, this 2×2 matrix is Hermitian. \square

Proof [Proofs of Theorem 1.3 and Corollary 1.4]: Take $N \in M_{n,n}(\mathbb{F}_4)$ such that $\text{Num}(N) = \{a, b\}$ with $a \neq b$ and set $M := \frac{1}{b-a}(N - a\mathbb{I}_{n,n})$. By Remark 2.3, we have $\text{Num}(M) = \{0, 1\}$. By Theorem 1.2, the matrices N and M are not a multiple of $\mathbb{I}_{n,n}$. Hence, to prove Theorem 1.3, it is sufficient to prove Corollary 1.4. Also note that in the last assertion of Theorem 1.3, it is sufficient to assume that $N \neq c\mathbb{I}_{n \times n}$ with $c \in \{a, b\}$.

If $M^\dagger = M$, then for any $u \in \mathbb{F}_4^n$, we have $\langle u, Mu \rangle = \langle Mu, u \rangle = (\langle u, Mu \rangle)^2$ and hence, $\text{Num}(M) \subseteq \mathbb{F}_2$, proving the “if” part.

Now, assume that $\text{Num}(M) \subseteq \mathbb{F}_2$. Since $\langle e_i, Me_i \rangle = m_{ii}$, we have $m_{ii} \in \mathbb{F}_2$ for all i . Taking the restriction to $\mathbb{F}_4e_i + \mathbb{F}_4e_j + \mathbb{F}_4e_h$ for all i, j, h with $1 \leq i < j < h \leq n$, we reduce to the case $n = 3$.

(a) First, assume that $m_{ii} = m_{11}$ for all i . By Remark 2.3, taking $\mathbb{I}_{2 \times 2} + M$ instead of M if $m_{11} = 1$, we reduce to the case $m_{11} = m_{22} = m_{33} = 0$. Set $\alpha := m_{12} + m_{23} + m_{31}$ and $\beta := m_{13} + m_{21} + m_{32}$. By Lemma 4.5 $\alpha + \beta \in \mathbb{F}_2$, $e\alpha + e^2\beta \in \mathbb{F}_2$ and $e^2\alpha + e\beta \in \mathbb{F}_2$. Since $e^2 + e = 1$ we get $\alpha + \beta \in \mathbb{F}_2$. For any $\{i, j, h\} = \{1, 2, 3\}$ set $\delta_i := m_{jh} + m_{hj}$. By Lemma 4.5, each element of B_h , $h = 1, 2, 3$, is contained in \mathbb{F}_2 . Since $e^2 + e = 1$, the sum of the two elements of B_h gives $\delta_j + \delta_i \in \mathbb{F}_2$.

Since $\sum_{ij} m_{ij} \in \text{Num}(M)$ (Lemma 4.5), we have $\delta_1 + \delta_2 + \delta_3 \in \mathbb{F}_2$. Hence, $\delta_i \in \mathbb{F}_2$ for all i .

Let $B = (b_{ij})$, $i, j = 1, 2, 3$, be the 3×3 -matrix with $b_{ii} = 0$ for all i , $b_{ij} = m_{ij}$ if $j < i$ and $b_{ji} = b_{ij}^2$ if $i < j$. Thus, B is a Hermitian matrix and therefore, $\text{Num}(D) \subseteq \mathbb{F}_2$ if $D := M + B$. Note that $D = (d_{ij})$ $d_{ij} = 0$ if either $i \geq j$ or $i < j$ and $m_{ji} = m_{ij}^2$ and that $d_{ij} = 1$ if $i < j$ and $m_{ji} \neq m_{ij}^2$, because $x^2 + x = 1$ if $x \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and $x^2 + x = 0$ if $x \in \mathbb{F}_2$.

(a1) Assume that D has exactly one nonzero entry. First, assume that $d_{12} = 1$; take $u = (1, e, 1)$; we have $Du = (e, 0, 0)$ and $\langle u, Du \rangle = e \notin \mathbb{F}_2$, a contradiction. If $d_{13} = 1$ take $u = (1, 1, e)$. If $d_{23} = 1$, take $u = (1, 1, e)$.

(a2) Now, assume that D has 2 nonzero entries. First, assume that $d_{12} = d_{13} = 1$ and $d_{23} = 0$; take $u = (1, e, 1)$; we have $Du = (e + 1, 0, 0)$ and $\langle u, Du \rangle = e + 1 \notin \mathbb{F}_2$, a contradiction. Now, assume that $d_{12} = 0$ and $d_{13} = d_{23} = 1$; take $u = (1, e, e)$; we have $Du = (e, e, 0)$ and $\langle u, Du \rangle = e + e^3 = e + 1 \notin \mathbb{F}_2$, a contradiction. Now, assume that $d_{12} = d_{23} = 1$ and $d_{13} = 0$; take $u = (1, e, e)$; we have $Du = (e, e, 0)$ and $\langle u, Du \rangle = e + e^3 = e + 1 \notin \mathbb{F}_2$, a contradiction.

(a3) Now, assume that $d_{12} = d_{13} = d_{23} = 1$; take $u = (1, e, 1)$; since $e + 1 = e^2$, we have $Du = (e^2, e, 0)$ and $\langle u, Du \rangle = e^2 + e^3 \notin \mathbb{F}_2$, a contradiction.

(b) Now, assume that $m_{ii} \neq m_{jj}$ for some $i \neq j$. Let $E = (e_{ij}) \in M_{3,3}(\mathbb{F}_2)$ be the diagonal matrix with $e_{ii} = m_{ii}$ for all i . Since $m_{ii} \in \mathbb{F}_2$ for all i , we have $E^\dagger = E$ and hence, $G := M + E$ is Hermitian if and only

if G is Hermitian. Since E is Hermitian, $\langle u, Eu \rangle \in \mathbb{F}_2$ for all $u \in \mathbb{F}_4^3$. Hence, $\langle u, Gu \rangle = \langle u, Mu \rangle + \langle u, Eu \rangle \in \mathbb{F}_2$ for all $u \in C_3(1)$. Since all diagonal elements of G are zero, step (a) gives $G^\dagger = G$ and so $M^\dagger = M$. \square

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