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SABINA HRUSTIC

MUSTAFA KULENOVIC

SAMRA MORANJKIC

ZEHRA NURKANOVIC

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Global dynamics of perturbation of certain rational difference equation

Sabina HRUSTIĆ[1], **Mustafa R. S. KULENOVIĆ**[2], **Samra MORANJKIĆ**[1], **Zehra NURKANOVIĆ**[1,*]

[1]Department of Mathematics, Faculty of Natural Sciences and Mathematics, University of Tuzla, Tuzla, Bosnia and Herzegovina

[2]Department of Mathematics, University of Rhode Island, Kingston, RI, USA

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Abstract: We investigate the global asymptotic stability of the difference equation of the form

$$x_{n+1} = \frac{Ax_n^2 + F}{ax_n^2 + ex_{n-1}}, \quad n = 0, 1, \dots,$$

with positive parameters and nonnegative initial conditions such that $x_0 + x_{-1} > 0$. The map associated to this equation is always decreasing in the second variable and can be either increasing or decreasing in the first variable depending on the parametric space. In some cases, we prove that local asymptotic stability of the unique equilibrium point implies global asymptotic stability.

Key words: Difference equation, attractivity, invariant, period doubling bifurcation, periodic solutions

1. Introduction

In this paper, we investigate the global dynamics of the following difference equation

$$x_{n+1} = \frac{Ax_n^2 + F}{ax_n^2 + ex_{n-1}}, \quad n = 0, 1, \dots, \tag{1.1}$$

where $A, F, a, e \in (0, \infty)$ and the initial conditions x_{-1} and x_0 are arbitrary nonnegative real numbers such that $x_0 + x_{-1} > 0$. The special case of Equation (1.1), where $a = 0$,

$$x_{n+1} = \frac{Ax_n^2 + F}{ex_{n-1}}, \quad n = 0, 1, \dots, \tag{1.2}$$

which exhibits nonconservative chaos was studied in detail in [8].

Equation (1.1) is the special case of a general second order quadratic fractional difference equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots, \tag{1.3}$$

*Correspondence: zehra.nurkanovic@untz.ba

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with nonnegative parameters and initial conditions such that $A+B+C>0$, $a+b+c+d+e+f>0$ and $x_0+x_{-1}+f>0$. Several global asymptotic results for some special cases of Equation (1.3) were obtained in [3, 4, 6, 9–11, 16, 17, 21, 22]. Two interesting special cases of Equation (1.3) are the following difference equations:

$$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{Bx_n+Dx_nx_{n-1}+x_{n-1}}, \quad n=0,1,\ldots, \tag{1.4}$$

studied in [12], and

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}+F}{ax_n^2+ex_{n-1}+f}, \quad n=0,1,\ldots, \tag{1.5}$$

studied in [7]. In [7], we performed the extensive local stability analysis of all equilibrium solutions of Equation (1.5) and we concluded that Naimark–Sacker bifurcation is not possible while the period-doubling bifurcation is possible, which was explored in full detail for the special case of (1.5) $A=F=0$ in [6]. The global asymptotic stability results were obtained in [7] for several special cases of Equation (1.5), where the right-hand side does not change its monotonicity, such as the special case $A=E=0$. No global dynamic results on the special cases of Equation (1.5) with mixed monotonicity was given in [7]. In both equations, (1.4) and (1.5), the associated map changes its monotonicity with respect to its variable. In this paper, in some cases when the associated map changes its monotonicity with respect to the first variable, in invariant interval, we will use results first obtained in [2, 13]. Those results were extended to the case of higher order difference equations and systems in [14, 19].

Note that the problem of determining invariant intervals in the case when the associated map changes its monotonicity with respect to its variable, has been considered in [18–20, 23].

In order to obtain the convergence results, we will also use the following theorems.

Theorem 1.1 *(See [1], Theorem 1.4) Let f be the function from*

$$x_{n+1}=f(x_n,x_{n-1}), \quad n=0,1,\ldots, \tag{1.6}$$

with

1. $f\in C\left[(0,\infty)\times(0,\infty),(0,\infty)\right]$;
2. $f(u,v)$ *is nonincreasing in* u *and* v *respectively;*
3. $xf(x,x)$ *is nondecreasing in* x;
4. *Equation (1.6) has a unique positive equilibrium* $\overline{x}$.

Then every positive solution $\{x_n\}_{n=-1}^{\infty}$ *of Equation (1.6) which is bounded from above and from below by positive constants converges to* $\overline{x}$.

The following result which provides the existence of full solutions of general difference equation is from [5], Theorem 1.8.

Theorem 1.2 *Consider the difference equation*

$$x_{n+1}=f(x_n,x_{n-1},\ldots,x_{n-k}), \tag{1.7}$$

where $f \in C[J^{k+1}, J]$ *for some interval* J *of real numbers and some nonnegative integer* k*. Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.7)*. Set* $I = \liminf_{n\to\infty} x_n$ *and* $S = \limsup_{n\to\infty} x_n$*, and suppose that* $I, S \in J$*. Let* $\mathcal{L}_0$ *be a limit point of the sequence* $\{x_n\}_{n=-k}^{\infty}$*. Then, the following statements are true.*

1. *There exists a solution* $\{L_n\}_{n=-\infty}^{\infty}$ *of* (1.7)*, called a full limiting sequence of* $\{x_n\}_{n=-k}^{\infty}$*, such that* $L_0 = \mathcal{L}_0$*, and such that for every* $N \in \mathbb{Z}$*,* L_N *is a limit point of* $\{x_n\}_{n=-k}^{\infty}$*. In particular,*

$$I \leq L_n \leq S \quad \text{for all} \quad N \in \mathbb{Z}.$$

2. *For every* $i_0 \in \mathbb{Z}$*, there exists a subsequence* $\{x_{r_i}\}_{i=0}^{\infty}$ *of the solution* $\{x_n\}_{n=-k}^{\infty}$ *such that*

$$L_N = \lim_{i\to\infty} x_{r_i+N} \text{ for all } N \geq i_0.$$

The rest of this paper is organized as follows. The second section presents the local stability of the unique positive equilibrium solution. The third section gives conditions for existence of the minimal period-two solution and its local stability. The fourth section presents global dynamics in certain regions of the parametric space. The results and techniques depend on monotonic character of the transition function $f(x, y)$ which is either decreasing in both arguments or increasing in first and decreasing in second argument. The results of this paper show that Equation (1.1) is an example of difference equation where the addition of terms (in this case term ax_n^2 in denominator) simplifies and stabilizes global dynamics in the sense that the unique equilibrium solution of the resulting equation is in many cases globally asymptotically stable while it is never asymptotically stable for Equation (1.2). In fact, we conjecture that the unique equilibrium solution of Equation (1.1) is globally asymptotically stable whenever it is locally asymptotically stable, see Conjecture 4.12. In addition, while Equation (1.2) cannot have period-two solutions, the perturbed Equation (1.1) has, in a parametric region, locally stable period-two solution which was conjectured to be global attractor as well, see Conjecture 4.14.

2. Linearized stability analysis

In this section, we present the local stability of the unique positive equilibrium of Equation (2.1). Notice first that we can easily eliminate one parameter, for example parameter a, so we will in the rest of the paper consider equation of the form

$$x_{n+1} = f(x_n, x_{n-1}) = \frac{Ax_n^2 + F}{x_n^2 + ex_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.1}$$

The equilibrium points of Equation (2.1) are the positive solutions of the equation

$$\overline{x} = \frac{A\overline{x}^2 + F}{\overline{x}^2 + e\overline{x}}, \tag{2.2}$$

or equivalently

$$\overline{x}^3 + (e - A)\overline{x}^2 - F = 0. \tag{2.3}$$

Denote by

$$\varphi(x) = x^3 + (e - A)x^2 - F.$$

Now, we have

$$\varphi'(x) = 3x^2 + 2(e-A)x \Rightarrow \varphi'(x) = 0 \Leftrightarrow \left(x = 0 \vee x = \frac{2(A-e)}{3}\right).$$

Notice that

$$\varphi\left(\frac{2(A-e)}{3}\right) = -\frac{4(A-e)^3}{27} - F < 0 \text{ for } A \geq e,$$

and

$$\varphi'(x) > 0 \text{ for } A < e \text{ and } x > 0,$$

and since $\varphi(-\infty) = -\infty$, $\varphi(\infty) = \infty$, $\varphi(0) = -F$, we conclude that there is a unique positive equilibrium point of Equation (2.1). Now, we investigate the stability of the positive equilibrium of Equation (2.1). Set

$$f(u,v) = \frac{Au^2 + F}{u^2 + ev}.$$

Then Equation (2.1) has a linearized equation $z_{n+1} = pz_n + qz_{n-1}$, where

$$p = \frac{\partial f}{\partial u}(\overline{x},\overline{x}) = \frac{2(Ae\overline{x} - F)}{\overline{x}(\overline{x}+e)^2} = \frac{-2(\overline{x}-A)}{\overline{x}+e}, \quad q = \frac{\partial f}{\partial v}(\overline{x},\overline{x}) = -\frac{e(A\overline{x}^2 + F)}{\overline{x}^2(\overline{x}+e)^2} = -\frac{e}{\overline{x}+e}. \tag{2.4}$$

Notice that $q \in (-1, 0)$.

In next result, we use standard local stability analysis, see [13, 15].

Theorem 2.1 *Let $F_0 = 4(A+3e)(A+e)^2$. The unique equilibrium point $\overline{x}$ of Equation (2.1) is:*

i) locally asymptotically stable if $F < F_0$,

ii) a saddle point if $F > F_0$,

iii) a nonhyperbolic point if $F = F_0$.

Proof For equilibrium point to be locally asymptotically stable, the well-known condition $|p| < 1 - q < 2$ must hold. Since $q \in (-1, 0)$, the second condition is already satisfied, so we need to prove the following

$$|p| < 1 - q \Leftrightarrow \quad -\frac{2e+\overline{x}}{e+\overline{x}} < \frac{-2(\overline{x}-A)}{e+\overline{x}} < \frac{2e+\overline{x}}{e+\overline{x}}.$$

Solving the inequality on the left-hand side, we get

$$\frac{2e+\overline{x}}{e+\overline{x}} - \frac{2(\overline{x}-A)}{e+\overline{x}} > 0,$$

and it leads to

$$\overline{x} < 2(A+e),$$

which is true only if $\varphi(2(A+e)) > 0$. Since,

$$\varphi(2(A+e)) = -F + 12e^3 + 28Ae^2 + 20A^2e + 4A^3,$$

the condition is satisfied for

$$F < 4\,(A+3e)\,(A+e)^2 = F_0.$$

Analogously, solving the inequality on the right-hand side, we get

$$\frac{-2(\overline{x}-A)}{e+\overline{x}} - \frac{2e+\overline{x}}{e+\overline{x}} < 0 \Leftrightarrow \overline{x} > \frac{2\,(A-e)}{3},$$

which is obviously true if $A \leq e$, and in the case $A > e$, it demands $\varphi\left(\frac{2(A-e)}{3}\right) < 0$. Since

$$\varphi\left(\frac{2\,(A-e)}{3}\right) = -\frac{4}{27}(A-e)^3 - F,$$

the previous condition is satisfied if $F > -\frac{4}{27}(A-e)^3$, which is always true. Because $q \in (-1,0)$, equilibrium point $\overline{x}$ cannot be a repeller; thus, for the value of parameter $F = F_0$ equilibrium point $\overline{x}$ is obviously nonhyperbolic. Moreover, in that case, equilibrium point $\overline{x}$ is of the form $\overline{x} = 2(A+e)$, and expressions in relation (2.4) are given with

$$p = -\frac{2\,(A+2e)}{2A+3e} \quad \text{and} \quad q = -\frac{e}{2A+3e}.$$

Now, corresponding characteristic equation is

$$\lambda^2 + \frac{2\,(A+2e)}{2A+3e}\lambda + \frac{e}{2A+3e} = 0$$

for which the solutions are $\lambda_1 = -1$ and $\lambda_2 = -\frac{e}{2A+3e} \in (-1,0)$. □

3. Period-two solutions

Now, we present the results about the existence and local stability of minimal period-two solutions of Equation (2.1).

Theorem 3.1 *Suppose $F > F_0$. Then, Equation (2.1) has a minimal period-two solution $\ldots,\phi,\psi,\phi,\psi,\ldots$ where*

$$\begin{aligned}\phi &= \frac{\left(e^2-A^2+\sqrt{(A-e)^2(A+e)^2+4Fe}\right)-\sqrt{2}\sqrt{2Fe+(A+e)(A+3e)\left(A^2-e^2-\sqrt{(A-e)^2(A+e)^2+4Fe}\right)}}{4e},\\ \psi &= \frac{\left(e^2-A^2+\sqrt{(A-e)^2(A+e)^2+4Fe}\right)+\sqrt{2}\sqrt{2Fe+(A+e)(A+3e)\left(A^2-e^2-\sqrt{(A-e)^2(A+e)^2+4Fe}\right)}}{4e}.\end{aligned}$$

Proof Assume that there exists a minimal period-two solution (ϕ,ψ) of Equation (2.1), where ϕ and ψ are distinct nonnegative real numbers. Then, (ϕ,ψ) satisfies

$$\begin{cases} \phi = \dfrac{A\psi^2+F}{\psi^2+e\phi}, \\[2ex] \psi = \dfrac{A\phi^2+F}{\phi^2+e\psi}, \end{cases} \tag{3.1}$$

or, equivalently

$$\begin{cases} \phi\left(\psi^2+e\phi\right)=A\psi^2+F, \\ \psi\left(\phi^2+e\psi\right)=A\phi^2+F. \end{cases} \tag{3.2}$$

By subtracting and adding those equations, we get

$$(e+A)(\phi+\psi)-\phi\psi=0, \quad \phi\psi(\phi+\psi)+(e-A)(\phi+\psi)^2-2(e-A)\phi\psi-2F=0. \tag{3.3}$$

Let

$$\begin{cases} \phi+\psi=u, \\ \quad \phi\psi=v. \end{cases} \tag{3.4}$$

Now, (3.3) takes the form

$$\begin{aligned} u\left(e+A\right)-v &= 0, \\ uv+u^2\left(e-A\right)-2v\left(e-A\right)-2F &= 0. \end{aligned}$$

If we substitute v from the first equation and replace it in the second one, we get

$$eu^2-(e^2-A^2)u-F=0,$$

from where it follows

$$u_{\pm}=\frac{(e^2-A^2)\pm\sqrt{(e^2-A^2)^2+4eF}}{2e}.$$

It is obviously $u_-<0$, so there is only one positive solution u_+. By using (3.4), we have

$$\begin{aligned} \phi+\psi &= \frac{1}{2e}\left(e^2-A^2+\sqrt{(e^2-A^2)^2+4eF}\right), \\ \phi\psi &= \frac{1}{2e}\left(e+A\right)\left(e^2-A^2+\sqrt{(e^2-A^2)^2+4eF}\right), \end{aligned}$$

i.e.

$$\phi=\frac{1}{2}u_+-\frac{1}{2}\sqrt{\left(u_+-4\left(A+e\right)\right)u_+}, \quad \psi=\frac{1}{2}u_++\frac{1}{2}\sqrt{\left(u_+-4\left(A+e\right)\right)u_+}.$$

The periodic solution is real if

$$u_+>4\left(A+e\right)\Leftrightarrow F>4\left(A+3e\right)\left(A+e\right)^2=F_0.$$

□

By substitution $x_{n-1}=u_n$, $x_n=v_n$, Equation (2.1) becomes the system of equations

$$\begin{cases} u_{n+1}=v_n, \\ v_{n+1}=\dfrac{Av^2+F}{v^2+eu}. \end{cases} \tag{3.5}$$

The map T corresponding to (3.5) is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} v \\ h(u,v) \end{pmatrix}, \tag{3.6}$$

where $h(u,v) = \frac{Av^2+F}{eu+v^2}$. The second iteration of the map T is

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} v \\ h(u,v) \end{pmatrix} = \begin{pmatrix} h(u,v) \\ H(u,v) \end{pmatrix}, \tag{3.7}$$

where

$$H(u,v) = \frac{Ah(u,v)+F}{h^2(u,v)+ev}.$$

Theorem 3.2 *Assume $F > F_0$. Then the minimal period-two solution (ϕ,ψ) of Equation (2.1) is locally asymptotically stable.*

Proof The Jacobian matrix of the map T^2 is of the form

$$J_{T^2} = \begin{pmatrix} \frac{\partial h(u,v)}{\partial u} & \frac{\partial h(u,v)}{\partial v} \\ \frac{\partial H(u,v)}{\partial u} & \frac{\partial H(u,v)}{\partial v} \end{pmatrix},$$

where

$$\frac{\partial h(u,v)}{\partial u} = -\frac{e\left(Av^2+F\right)}{(eu+v^2)^2},$$

$$\frac{\partial h(u,v)}{\partial v} = -\frac{2v(F-Aeu)}{(eu+v^2)^2},$$

$$\frac{\partial H(u,v)}{\partial u} = \frac{2e\left(Av^2+F\right)^2\left(eu+v^2\right)(F-Aev)}{(v^3(A^2v+e(2eu+v^2))+2AFv^2+e^3u^2v+F^2)^2},$$

$$\frac{\partial H(u,v)}{\partial v} = -\frac{\left(eu+v^2\right)\left(A^3ev^4\left(v^2-3eu\right)+2A^2eFv^2(v(2u+3v)-eu)+AF^2\left(e^2u+ev(4u+5v)-4v^3\right)+F\left(e\left(eu+v^2\right)^3-4F^2v\right)\right)}{(v^3(A^2v+e(2eu+v^2))+2AFv^2+e^3u^2v+F^2)^2}.$$

By using (3.1), the following holds

$$\frac{\partial h(u,v)}{\partial u}|_{(\phi,\psi)} = -\frac{e\phi}{e\phi+\psi^2}, \qquad \frac{\partial h(u,v)}{\partial v}|_{(\phi,\psi)} = \frac{2\psi(A-\phi)}{e\phi+\psi^2},$$

$$\frac{\partial H(u,v)}{\partial u}|_{(\phi,\psi)} = \frac{2e\phi^2(\psi-A)}{(e\psi+\phi^2)\,(e\phi+\psi^2)}, \quad \frac{\partial H(u,v)}{\partial v}|_{(\phi,\psi)} = \frac{4\phi\psi(A-\phi)(A-\psi)}{(e\psi+\phi^2)(e\phi+\psi^2)} - \frac{e\psi}{e\psi+\phi^2}.$$

Hence,

$$J_{T^2}((\phi,\psi)) = \begin{pmatrix} -\frac{e\phi}{e\phi+\psi^2} & \frac{2\psi(A-\phi)}{e\phi+\psi^2} \\ -\frac{2e\phi^2(A-\psi)}{(e\psi+\phi^2)(e\phi+\psi^2)} & \frac{4\phi\psi(A-\phi)(A-\psi)}{(e\psi+\phi^2)(e\phi+\psi^2)} - \frac{e\psi}{e\psi+\phi^2} \end{pmatrix}.$$

Furthermore,

$$p = TrJ_{T^2}((\phi,\psi)) = \frac{4\phi\psi(A-\phi)(A-\psi) - e(\phi^3+\psi^3) - 2e^2\phi\psi}{(e\psi+\phi^2)\,(e\phi+\psi^2)},$$

$$q = DetJ_{T^2}((\phi,\psi)) = \frac{e^2\phi\psi}{(e\psi+\phi^2)\,(e\phi+\psi^2)}.$$

Notice $0 < q < 1$. Let us see when the following inequalities hold $|p| < 1 + q < 2$, i.e. $|p| < 1 + q$. We have

$$|p| < 1 + q \Leftrightarrow -(1+q) < p < 1 + q$$

Solving the inequality on the left-hand side, we get

$$\begin{aligned} p > -(1+q) \Leftrightarrow p + 1 + q &> 0, \\ \frac{\phi\psi\left(4A^2 - 4A(\phi+\psi) + 5\phi\psi\right)}{\left(e\psi + \phi^2\right)\left(e\phi + \psi^2\right)} &> 0. \end{aligned}$$

The straightforward calculation gives

$$\begin{aligned} 4A^2 - 4A(\phi+\psi) + 5\phi\psi &= 4A^2 - 4A(\phi+\psi) + 5(A+e)(\phi+\psi) \\ &= (\phi+\psi)(A+5e) + 4A^2 > 0, \end{aligned}$$

hence, the inequality on the left-hand side is always satisfied. Similarly, we need to prove

$$\begin{aligned} p - 1 - q &< 0, \\ \frac{\phi\psi\left(4A^2 - 4A(\phi+\psi) + 3\phi\psi\right) - 4e^2\phi\psi - 2e\left(\phi^3 + \psi^3\right)}{\left(e\psi + \phi^2\right)\left(e\phi + \psi^2\right)} &< 0. \end{aligned}$$

In order to determine the sign of the numerator, we will transform it in the equivalent form:

$$\begin{aligned} &-2e\phi^3 - 2e\psi^3 + \phi\psi\left(-4A\phi - 4A\psi + 3\phi\psi + 4A^2 - 4e^2\right) \\ &= -2e\left((\phi+\psi)^3 - 3\phi\psi(\phi+\psi)\right) + (A+e)(\phi+\psi)\left(-4A(\phi+\psi) + 3(A+e)(\phi+\psi) + 4A^2 - 4e^2\right) \\ &= (\phi+\psi)\left(4(A+e) - (\phi+\psi)\right)\left(2e(\phi+\psi) + A^2 - e^2\right). \end{aligned}$$

Notice the following: first factor is obviously positive, the third one also, since the following holds

$$\begin{aligned} 2e(\phi+\psi) + A^2 - e^2 &= 2e\frac{\left(e^2 - A^2 + \sqrt{(A-e)^2(A+e)^2 + 4Fe}\right)}{2e} + A^2 - e^2 \\ &= \sqrt{(A-e)^2(A+e)^2 + 4Fe} > 0. \end{aligned}$$

The second factor can be written as

$$\begin{aligned} 4(A+e) - (\phi+\psi) &= 4(A+e) - \frac{\left(e^2 - A^2 + \sqrt{(A-e)^2(A+e)^2 + 4Fe}\right)}{2e} \\ &= \frac{1}{2e}\left((A+e)(A+7e) - \sqrt{(A-e)^2(A+e)^2 + 4Fe}\right), \end{aligned}$$

and it is negative if

$$\begin{aligned} (A+e)(A+7e) < \sqrt{(A-e)^2(A+e)^2 + 4Fe} &\Leftrightarrow (A+e)^2(A+7e)^2 < (A-e)^2(A+e)^2 + 4Fe \\ &\Leftrightarrow F > 4(A+3e)(A+e)^2 = F_0, \end{aligned}$$

which completes the proof of the theorem. □

4. Global asymptotic stability

Notice that the function $f(u,v)$ is always decreasing with respect to the second variable, and it could be increasing or decreasing with respect to the first variable. The critical point of the function $f(u,v)$ in the first variable is

$$\frac{\partial f\left(u,v\right)}{\partial u}=0\Leftrightarrow 2u\left(Ave-F\right)=0\Leftrightarrow u=0\vee v=\frac{F}{Ae}.$$

Thus, if $v\geq\frac{F}{Ae}$, function $f(u,v)$ is increasing, and if $v\leq\frac{F}{Ae}$ function $f(u,v)$ is decreasing. Since

$$f\left(\frac{F}{Ae},\frac{F}{Ae}\right)=\frac{A\left(\frac{F}{Ae}\right)^2+F}{\left(\frac{F}{Ae}\right)^2+e\frac{F}{Ae}}=A,$$

we distinguish the following three cases:

(1) $\frac{F}{Ae}<A\Leftrightarrow F<A^2e$,

(2) $\frac{F}{Ae}>A\Leftrightarrow F>A^2e$,

(3) $\frac{F}{Ae}=A\Leftrightarrow F=A^2e$.

Denote $F_g=A^2e$, and notice that $F_g<F_0$, which means that the function $f(u,v)$ changes its monotonicity inside the interval where the equilibrium point $\overline{x}$ is locally asymptotically stable.

Case (1) $F<F_g<F_0$

In this case, the function $f(u,v)$ is increasing with respect to the first variable and decreasing with respect to the second variable on the invariant interval. The invariant interval is of the form

$$[L,U]=\left[\frac{F}{Ae},A\right].$$

Since

$$\min_{(x,y)\in[L,U]^2}f(x,y)=f(L,U)\ \text{ and }\ \max_{(x,y)\in[L,U]^2}f(x,y)=f(U,L),$$

it has to be $f(L,U)\geq L$ and $f(U,L)\leq U$. A straightforward calculation shows that

$$\begin{aligned}f(L,U)&=f\left(\frac{F}{Ae},A\right)=\frac{AF(Ae^2+F)}{A^3e^3+F^2}\geq L=\frac{F}{Ae}\\&\Leftrightarrow A^2Fe(Ae^2+F)\geq F(A^3e^3+F^2)\Leftrightarrow F\leq A^2e,\end{aligned}$$

which is true. Furthermore,

$$f(U,L)=f\left(A,\frac{F}{Ae}\right)=A\leq A,$$

which implies that $[L,U]=\left[\frac{F}{Ae},A\right]$ is an invariant interval. We need to show that the equilibrium point belongs to the invariant interval, i.e., we will show that $\varphi\left(\frac{F}{Ae}\right)\varphi(A)<0$. Indeed,

$$\varphi\left(\frac{F}{Ae}\right)\varphi(A)=-\frac{F(A^2e-F)^2(Ae^2+F)}{A^3e^3}<0.$$

Lemma 4.1 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (2.1). The following statements are true for $n = 0, 1, \ldots$:*

(a) If $x_{n-1} \leq A$, then $x_{n+1} \geq \dfrac{F}{Ae}$;

(b) If $x_{n-1} \geq \dfrac{F}{Ae}$, then $x_{n+1} \leq A$.

In other words, $\left[\dfrac{F}{Ae}, A\right]$ is an invariant interval.

Proof

(a) Suppose that $x_{n-1} \leq A$. Then, we have

$$x_{n+1} - \frac{F}{Ae} = \frac{(A^2e - F)x_n^2 + Fe(A - x_{n-1})}{Ae(x_n^2 + ex_{n-1})} > 0,$$

since $F < F_g = A^2 e$, i.e., $A^2 e - F > 0$.

(b) Also, if $x_{n-1} \geq \dfrac{F}{Ae}$, we have

$$x_{n+1} - A = \frac{F - Aex_{n-1}}{x_n^2 + ex_{n-1}} \leq 0.$$

□

Lemma 4.2 *If $F < F_g$, then $\left[\dfrac{F}{Ae}, A\right]$ is an attracting interval. In other words, there exists $N \in \mathbb{Z}$ such that $x_n \in \left[\dfrac{F}{Ae}, A\right]$ for all $n \geq N$.*

Proof Let $I = \liminf\limits_{n\to\infty} x_n$ and $S = \limsup\limits_{n\to\infty} x_n$. Then, if either $I \in \left[\dfrac{F}{Ae}, A\right]$ or $S \in \left[\dfrac{F}{Ae}, A\right]$, the proof is done, since by Lemma 4.1, if $I \in \left[\dfrac{F}{Ae}, A\right]$, then $S \in \left[\dfrac{F}{Ae}, A\right]$ and vice versa. Assume now that $I \notin \left[\dfrac{F}{Ae}, A\right]$ and $S \notin \left[\dfrac{F}{Ae}, A\right]$. It follows from Lemma 4.1 that $I < \dfrac{F}{Ae}$ and $S > A$. Hence, there is an open neighborhood O containing S such that $O \cap \left[\dfrac{F}{Ae}, A\right] = \emptyset$. By Theorem 1.2, let S_{n+1} be a full-limiting sequence such that $\lim\limits_{n\to\infty} S_{n+1} = S$. Thus, there exists a positive integer N, such that $S_{n-1} \in O$ for $n \geq N$. According to Lemma 4.1, if $S_{n-1} > A > \dfrac{F}{Ae}$, then $S_{n+1} < \dfrac{F}{Ae}$, which is a contradiction. Thus, it must be the case that both I and S are in the interval $\left[\dfrac{F}{Ae}, A\right]$. □

In this case, depending on the corresponding monotonicity of the map associated to Equation (2.1), we will consider the system of equations

$$f(m, M) = m \quad \text{and} \quad f(M, m) = M, \tag{4.1}$$

(see [13], Theorem 1.4.5). Number of the solutions of System (4.1) is analyzed in the following Lemma.

Lemma 4.3 *System* (4.1) *has:*

(A1) only one solution - equilibrium solution, if any of the following conditions are satisfied:

(i) $F_+ < F < F_g < F_0\,;$

(ii) $F = F_+ < F_g < F_0$ *and* $A \leq \frac{5}{3}e\,;$

(iii) $0 < F_c \leq F < F_+ < F_g < F_0$ *and* $A < \frac{5}{3}e\,;$

(A2) three positive solutions if any of the the following conditions hold:

(i) $F = F_+ < F_g < F_0$ *and* $A > \frac{5}{3}e\,;$

(ii) $F < F_c < F_+ < F_g < F_0\,;$

(A3) five positive solutions if $\max\{0, F_c\} < F < F_+ < F_g < F_0$ *and* $A > \frac{5}{3}e\,;$

where $F_+ = \frac{1}{27}\left(2\sqrt{(A^2+Ae+e^2)^3} - (A-e)(2A+e)(A+2e)\right)$ *and* $F_c = -\frac{4}{27}A^2(A-3e)\,.$

See Figure 1 for visual interpretation of different regions in Lemma 4.3 in the parametric (A, F) plane.

Proof System (4.1) is of the form

$$\begin{cases} \frac{Am^2+F}{m^2+eM} = m, \\ \frac{AM^2+F}{M^2+em} = M, \end{cases} \tag{4.2}$$

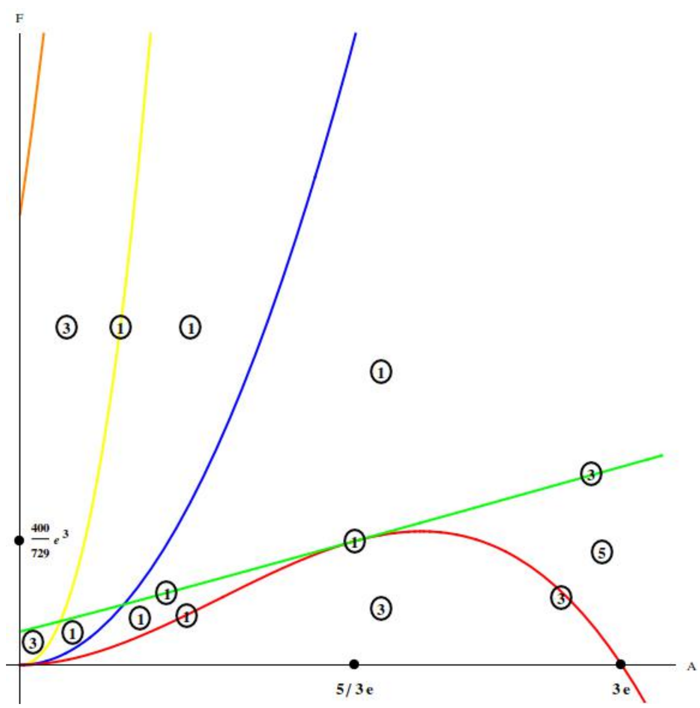


Figure 1. Visual representation of Lemma 4.3 in (A, F) plane.

or

$$\begin{cases} Am^2 + F = m^3 + emM, \\ AM^2 + F = M^3 + emM. \end{cases} \tag{4.3}$$

By subtracting equations, we get

$$(m - M)[(m + M)^2 - A(m + M) - mM] = 0,$$

and by adding them,

$$A(m^2 + M^2) + 2F = m^3 + M^3 + 2emM.$$

If we put

$$m + M = y, \quad mM = x, \tag{4.4}$$

we have

$$\begin{cases} y^2 - Ay - x = 0, \\ y^3 - Ay^2 + (2A - 3y + 2e)x - 2F = 0. \end{cases} \tag{4.5}$$

Since $x > 0$, $y > 0$, we obtain

$$x > 0 \Leftrightarrow y^2 - Ay > 0 \Leftrightarrow y > A. \tag{4.6}$$

System (4.5) is equivalent to the next equation

$$-2y^3 + 2(2A + e)y^2 - 2A(A + e)y - 2F = 0. \tag{4.7}$$

Let $H(y) = -2y^3+2(2A+e)y^2-2A(A+e)y-2F$ and notice that $H(0) = -2F$, $H(+\infty) = -\infty$, $H(-\infty) = +\infty$. Furthermore, we have

$$H'(y) = -6y^2 + 4(2A + e)y - 2A(A + e),$$

and

$$H'(y) = 0 \Leftrightarrow y_\pm = \frac{2A + e \pm \sqrt{A^2 + Ae + e^2}}{3} > 0.$$

Observe the following

$$H(y_-)H(y_+) = 4F^2 + \frac{8}{27}(A - e)(2A + e)(A + 2e)F - \frac{4}{27}A^2e^2(A + e)^2,$$

i.e. we can consider $H(y_-)H(y_+)$ as quadratic function by variable F. Now, we distinguish three cases:

(a) If $H(y_-)H(y_+) > 0$, there is no positive solution of Equation (4.7), as it could be seen at Figure 2a.

$$H(y_-)H(y_+) = 0 \Leftrightarrow F_\pm = \frac{1}{27}\left(-(A - e)(2A + e)(A + 2e) \pm 2\sqrt{(A^2 + Ae + e^2)^3}\right).$$

Now we have

$$F_- = -\frac{1}{27}(A - e)(2A + e)(A + 2e) - \frac{2}{27}\sqrt{(A^2 + Ae + e^2)^3} < 0$$

and

$$F_+ = -\frac{1}{27}(A - e)(2A + e)(A + 2e) + \frac{2}{27}\sqrt{(A^2 + Ae + e^2)^3} > 0.$$

Thus, $H(y_-)H(y_+) > 0 \Leftrightarrow F > F_+$, i.e., if $F \in (F_+, F_g)$, and that completes the proof of the statement (A1) (i).

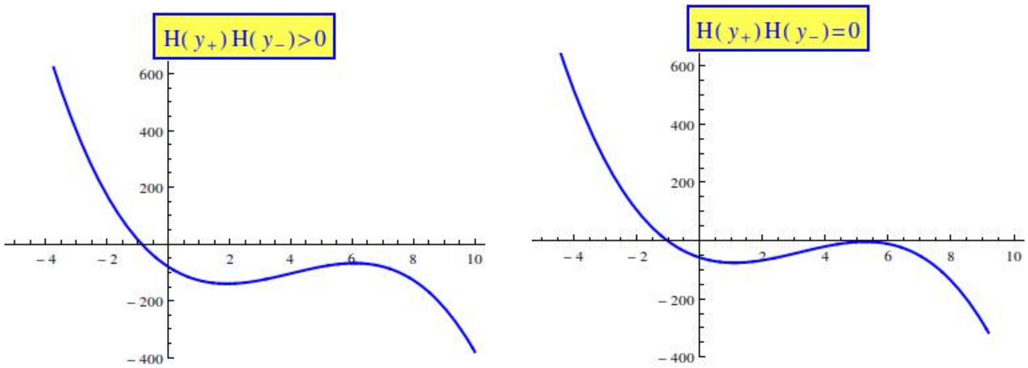


Figure 2. Some positions of the function $H(y_-)H(y_+)$.

(b) If $H(y_-)H(y_+) = 0$, i.e. $F = F_+$, then Equation (4.7) has only one positive solution $y_+ = \frac{2A+e+\sqrt{A^2+Ae+e^2}}{3}$. See Figure 2b. Now, from (4.4) and (4.5), replacing y with y_+, we get:

$$\begin{cases} m + M = y_+, \\ mM = y_+^2 - Ay_+. \end{cases} \tag{4.8}$$

The solutions of the above system are

$$(m_1, M_1) = \left(\frac{2A+e+\gamma+\sqrt{9A^2-3Ae-6e(\gamma+e)}}{6}, \frac{2A+e+\gamma-\sqrt{9A^2-3Ae-6e(\gamma+e)}}{6} \right)$$

and

$$(m_2, M_2) = \left(\frac{2A+e+\gamma-\sqrt{9A^2-3Ae-6e(\gamma+e)}}{6}, \frac{2A+e+\gamma+\sqrt{9A^2-3Ae-6e(\gamma+e)}}{6} \right),$$

where

$$\gamma = \sqrt{A^2+Ae+e^2} > 0.$$

If $9A^2 - 3Ae - 6e(\gamma + e) < 0 \Leftrightarrow A < \frac{5e}{3}$, there is no other real solutions of the system except equilibrium solution. If $9A^2 - 3Ae - 6e(\gamma + e) = 0 \Leftrightarrow A = \frac{5e}{3}$, then $y_+ = \frac{4A}{3}$, and System (4.8) has solution

$$m = \frac{2A+e-\gamma}{6} = e = M = \overline{x},$$

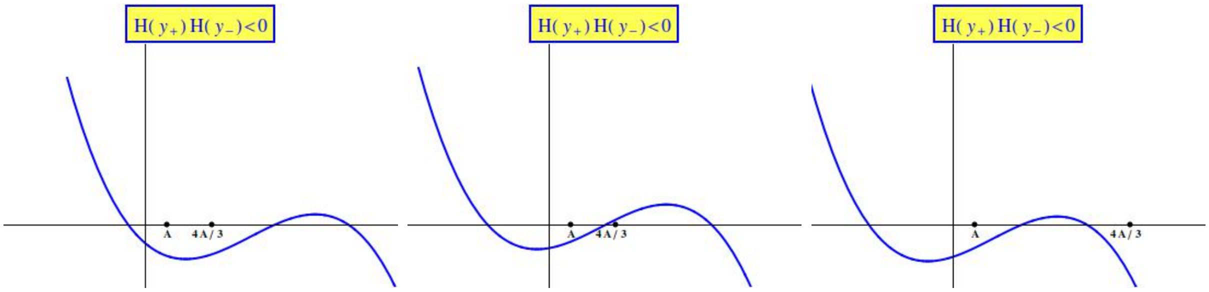
and that completes the proof of the statement (A1) (ii).
If

$$9A^2 - 3Ae - 6e\left(\sqrt{A^2+Ae+e^2} + e\right) > 0$$

and

$$2A + e + \sqrt{A^2+Ae+e^2} - \sqrt{9A^2 - 3Ae - 6e\left(\sqrt{A^2+Ae+e^2}+e\right)} > 0,$$

which is equivalent to $A > \frac{5e}{3}$, m_1 and M_1 are positive and real, so System (4.8) has three solution and that completes the proof of the statement (A2) (i).



(a) (b) (c)

Figure 3. Some positions of the function $H(y_-)H(y_+)$.

(c) If $H(y_-)H(y_+) < 0$, then Equation (4.7) has two real and positive solutions $y_{1,2}$. See Figure 3. Condition $H(y_-)H(y_+) < 0$ is equivalent to $F \in (0, F_+)$, where $F_+ < F_g$. Then system

$$\begin{cases} m + M = y_i, \\ mM = y_i^2 - Ay_i, \end{cases} \quad i = 1, 2$$

is equivalent to the Equation

$$M^2 - y_i M + y_i^2 - Ay_i = 0,$$

whose solutions are given as

$$M_{1,2} = \frac{y_i \pm \sqrt{y_i^2 - 4(y_i^2 - Ay_i)}}{2} = \frac{y_i \pm \sqrt{-3y_i^2 + 4Ay_i}}{2}.$$

i) If $-3y_i^2 + 4Ay_i \leq 0 \Leftrightarrow y_i \in \left[\frac{4}{3}A, +\infty\right)$, which is true if $H\left(\frac{4}{3}A\right) \leq 0$ and $\frac{4}{3}A < y_+$, there is no real solution of System (4.3) except the equilibrium solution.

$$H\left(\frac{4}{3}A\right) \leq 0 \quad \Leftrightarrow \quad -\frac{8}{27}A^2(A - 3e) - 2F \leq 0$$
$$\Leftrightarrow \quad F \geq -\frac{4}{27}A^2(A - 3e) = F_c.$$

It is obviously $F_c < F_+$. The following holds

$$y_+ > \frac{4}{3}A \quad \Leftrightarrow \quad \frac{1}{3}(2A + e + \sqrt{(2A+e)^2 - 3A(A+e)}) > \frac{4}{3}A$$
$$\Leftrightarrow \quad \sqrt{A^2 + Ae + e^2} > 2A - e$$
$$\Leftrightarrow \quad A < \frac{5}{3}e.$$

Notice that

$$\begin{aligned} y_+ > A \quad &\Leftrightarrow \quad \frac{1}{3}(2A+e+\sqrt{(2A+e)^2-3A(A+e)}) > A \\ &\Leftrightarrow \quad \sqrt{A^2+Ae+e^2} > A-e \\ &\Leftrightarrow \quad 3Ae > 0, \ \text{which is always satisfied.} \end{aligned}$$

So if $F \in [F_c, F_+)$ and $A < \dfrac{5}{3}e$, System (4.3) has only one solution which proves $(A1)$ (iii).

$ii)$ If $-3y_i^2 + 4Ay_i > 0$, then $y_i \in (0, \frac{4}{3}A)$. By using (4.6), it has to be $y_i > A$, so $y_i \in (A, \frac{4}{3}A)$. Now, we need to analyze when zeros of Equation (4.7) y_i, $i = 1, 2$, are in the interval $(A, \frac{4}{3}A)$.

If just one zero of Equation (4.7) belongs to the interval $(A, \frac{4}{3}A)$, which is true if $H(A)H\left(\frac{4}{3}A\right) < 0$, and since $H(A) = -2F < 0$, the previous condition is reduced to $H(\frac{4A}{3}) > 0$, i.e. $F < F_c$, then System (4.3) has three solutions (case $(A2)$ (ii)). See Figure 3b.

If both of zeros y_i belong to the interval $(A, \frac{4}{3}A)$, which is satisfied if $H\left(\frac{4}{3}A\right) < 0 \Leftrightarrow F > F_c$ and $A < y_+ < \frac{4}{3}A \Leftrightarrow A > \dfrac{5e}{3}$, System (4.3) has five solutions which completes the proof of the case $(A3)$. See Figure 3c.

□

Theorem 4.4 *If any of conditions $(A1)$ $(i), (ii)$, or (iii) of Lemma 4.3 holds, then the equilibrium solution $\overline{x}$ of Equation (2.1) is globally asymptotically stable.*

Proof Proof of the statement follows from Theorem 1.4.5 in [13], Theorem 2.1 and Lemmas 4.2 and 4.3. □

Case (2) $F_g < F < F_0$

In this case, in view of the fact that $f(x, y)$ is decreasing in both arguments, we will consider the system of equations

$$f(m, m) = M \quad \text{and} \quad f(M, M) = m. \tag{4.9}$$

Number of the solutions of System (4.9) is analyzed in the following Lemma.

Lemma 4.5 *(a) If $F_g < F \le F_d = 4A^2(A+e) < F_0$, then the system of the algebraic equations (4.9) has the unique solution $(m, M) = (\overline{x}, \overline{x})$.*

(b) If $F_g < F_d < F < F_0$, then the system of the algebraic equations (4.9) has three solutions: (m_1, M_1), (M_1, m_1), and the equilibrium point $(\overline{x}, \overline{x})$.

Proof

(a) System (4.9) of the form

$$\begin{cases} \frac{AM^2+F}{M^2+eM} = m, \\ \frac{Am^2+F}{m^2+em} = M, \end{cases} \tag{4.10}$$

or

$$\begin{cases} AM^2 + F = mM\,(M+e)\,, \\ Am^2 + F = mM\,(m+e)\,. \end{cases} \tag{4.11}$$

By subtracting the equations, it leads to

$$(M-m)\,(AM + Am - Mm) = 0. \tag{4.12}$$

Thus, $M = m$ or $m = \frac{AM}{M-A}$, $M > A$. By substituting m in the first equation in (4.11), we get the following

$$(A+e)\,AM^2 - FM + AF = 0. \tag{4.13}$$

Solutions of Equation (4.13) are

$$M_{1,2} = \frac{F \pm \sqrt{F(F - 4A^2\,(A+e))}}{2A\,(A+e)}.$$

If the discriminant $F(F - 4A^2\,(A+e)) < 0$, i.e. $F < 4A^2\,(A+e) = F_d$, then Equation (4.13) does not have real solutions, and the equilibrium point $\overline{x}$ is a unique solution of (4.10). If $F - 4A^2\,(A+e) = 0$, i.e. $F = F_d$, then

$$M = \frac{F}{2\,(A+e)\,A} = \frac{4A^2\,(A+e)}{2\,(A+e)\,A} = 2A \text{ and } m = \frac{AM}{M-A} = \frac{2A^2}{A} = 2A,$$

i.e. $m = M$, so the conclusion is the same as in the previous case.

(b) If $F - 4A\,(A+e)\,A > 0$, i.e., $F > F_d$, the solutions of Equation (4.13) are real. The following holds

$$0 < M_1 = \frac{F - \sqrt{F(F - 4A^2\,(A+e))}}{2A\,(A+e)} < M_2 = \frac{F + \sqrt{F(F - 4A^2\,(A+e))}}{2A\,(A+e)},$$

and $M_{1,2} \in \left[A, \frac{F}{Ae}\right]$. Next, we have

$$m_1 = \frac{AM_1}{M_1 - A} = \frac{A\left(\frac{F - \sqrt{F(F-4A^2(A+e))}}{2A(A+e)}\right)}{\frac{F - \sqrt{F(F-4A^2(A+e))}}{2A(A+e)} - A} = \frac{F + \sqrt{F(F - 4A^2\,(A+e))}}{2A\,(A+e)} = M_2,$$

$$m_2 = \frac{AM_2}{M_2 - A} = \frac{A\left(\frac{F + \sqrt{F(F-4A^2(A+e))}}{2A(A+e)}\right)}{\frac{F + \sqrt{F(F-4A^2(A+e))}}{2A(A+e)} - A} = \frac{F - \sqrt{F(F - 4A^2\,(A+e))}}{2A\,(A+e)} = M_1,$$

so the conclusion follows. □

In this case, the function $f\,(u,v)$ is nonincreasing in both variables on invariant interval which is of the form

$$[L,U] = \left[A, \frac{F}{Ae}\right],$$

with the property that $f:[L,U]^2 \to [L,U]$. Indeed, since

$$\max_{(x,y)\in[L,U]^2} f(x,y) = f(L,L) \quad \text{and} \quad \min_{(x,y)\in[L,U]^2} f(x,y) = f(U,U),$$

we need to show that $f(U,U) \geq L$ and $f(L,L) \leq U$. By straightforward calculation, we get

$$f(U,U) = f\left(\frac{F}{Ae}, \frac{F}{Ae}\right) = A,$$

$$f(L,L) = f(A,A) = \frac{A^3+F}{A^2+Ae} \leq \frac{F}{Ae} \Longleftrightarrow \frac{A^3+F}{A^2+Ae} - \frac{F}{Ae} \leq 0 \Longleftrightarrow F \geq A^2 e,$$

which is true. Furthermore, since

$$\varphi\left(\frac{F}{Ae}\right)\varphi(A) = -\frac{F(A^2e-F)^2(Ae^2+F)}{A^3e^3} < 0,$$

the equilibrium point is inside the invariant interval $[L,U]$.

Lemma 4.6 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (2.1). The following statements are true.*

(a) If $x_{n-1} \leq \frac{F}{Ae}$, then $x_{n+1} \geq A$;

(b) If $x_{n-1} \geq A$, then $x_{n+1} \leq \frac{F}{Ae}$.

In other words, $\left[A, \frac{F}{Ae}\right]$ is an invariant interval.

Proof The proof is similar to that of Lemma 4.1. □

Lemma 4.7 *If $F > F_g$, then $\left[A, \frac{F}{Ae}\right]$ is an attracting interval. In other words, there exists $N \in \mathbb{Z}$ such that $x_n \in \left[A, \frac{F}{Ae}\right]$ for all $n \geq N$.*

Proof The proof is similar to that of Lemma 4.2. □

Theorem 4.8 *If $F_g < F \leq F_d < F_0$, where $F_d = 4A^2(A+e)$, then the equilibrium $\overline{x}$ is globally asymptotically stable.*

Proof By using Lemmas 4.5 (a) and 4.7, Theorem 1.4.7 in [13] and Theorem 2.1, we get the conclusion that the equilibrium $\overline{x}$ is globally asymptotically stable. □

Remark 4.9 Also, to prove Theorem 4.8, we can apply Theorem 1.1. In this case, for $F_g < F \leq F_d$, and $\left[A, \frac{F}{Ae}\right]$, there exists $N \in \mathbb{N}$ for which $A \leq x_n \leq \frac{F}{Ae}$ holds, for all $n \geq N$ which implies that every solution

$\{x_n\}_{n=-1}^{\infty}$ of Equation (2.1) is bounded from above and from below by positive constants. Since

$$xf(x,x) = x\frac{Ax^2+F}{x^2+ex} = \frac{Ax^2+F}{x+e},$$

and

$$\frac{d}{dx}(xf(x,x)) = \frac{(2A-x)x}{x+e} > 0,$$

if

$$\overline{x} < 2A \Leftrightarrow \varphi(2A) > \varphi(\overline{x}) = 0 \Leftrightarrow F < 4A^2(A+e) \Leftrightarrow F < F_d,$$

and f clearly satisfies the conditions 1 and 2 of Theorem 1.1. By Theorem 1.1, every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (2.1) converges to $\overline{x}$.

Case (3) $F = F_g < F_0$

If we replace F with $F_g = A^2 e$, then $\overline{x} = A$, and Equation (2.1) is of the form

$$x_{n+1} = \frac{\overline{x}x_n^2 + \overline{x}^2 e}{x_n^2 + ex_{n-1}}. \tag{4.14}$$

Lemma 4.10 *Assume that $F = F_g < F_0$. Then Equation (4.14) does not possess a minimal period-four solution.*

Proof Suppose the opposite, that Equation (4.14) has a minimal period-four solution $\dots x, y, z, t, x, y, z, t, \dots$, i.e.

$$\begin{cases} z = \dfrac{\overline{x}y^2 + \overline{x}^2 e}{y^2 + ex}, \\ t = \dfrac{\overline{x}z^2 + \overline{x}^2 e}{z^2 + ey}, \\ x = \dfrac{\overline{x}t^2 + \overline{x}^2 e}{t^2 + ez}, \\ y = \dfrac{\overline{x}x^2 + \overline{x}^2 e}{x^2 + et}. \end{cases} \tag{4.15}$$

By eliminating z and t, we obtain

$$\begin{cases} x\left(\dfrac{\overline{x}\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + \overline{x}^2 e}{\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + ey}\right)^2 + ex\left(\dfrac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right) - \overline{x}\left(\dfrac{\overline{x}\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + \overline{x}^2 e}{\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + ey}\right)^2 - \overline{x}^2 e = 0, \\ yx^2 + ey\left(\dfrac{\overline{x}\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + \overline{x}^2 e}{\left(\frac{\overline{x}y^2+\overline{x}^2e}{y^2+ex}\right)^2 + ey}\right) - \overline{x}x^2 - \overline{x}^2 e = 0, \end{cases} \tag{4.16}$$

or, after straightforward calculation,

$$\frac{\overline{x}\left(\overline{x}-x\right)\left(\Lambda+\Gamma\right)}{\left(ex+y^{2}\right)\left(ey^{5}+\overline{x}^{4}e^{2}+\overline{x}^{2}y^{4}+2\overline{x}^{3}ey^{2}+2e^{2}xy^{3}+e^{3}x^{2}y\right)^{2}}=0,$$

$$\frac{\left(\overline{x}-y\right)\left(\overline{x}^{5}e^{3}+\overline{x}^{4}e^{2}x^{2}+2\overline{x}^{4}e^{2}y^{2}+\overline{x}^{2}x^{2}y^{4}+2e^{2}x^{3}y^{3}+\overline{x}^{3}ey^{4}+ex^{2}y^{5}+e^{3}x^{4}y+2\overline{x}^{3}ex^{2}y^{2}\right)}{ey^{5}+\overline{x}^{4}e^{2}+\overline{x}^{2}y^{4}+2\overline{x}^{3}ey^{2}+2e^{2}xy^{3}+e^{3}x^{2}y}=0,$$

where

$$\begin{aligned}\Lambda &= \overline{x}^{5}y^{10}+e^{3}y^{12}+\overline{x}^{3}e^{7}x^{5}+2\overline{x}^{6}e^{6}x^{3}+2\overline{x}^{2}e^{2}y^{11}+\overline{x}^{3}e^{2}y^{10}+4\overline{x}^{3}e^{3}y^{9}+2\overline{x}^{4}e^{4}y^{7}+8\overline{x}^{5}e^{2}y^{8}\\ &+8\overline{x}^{6}e^{3}y^{6}+6\overline{x}^{7}e^{2}y^{6}+4\overline{x}^{7}e^{4}y^{4}+4\overline{x}^{8}e^{3}y^{4}+\overline{x}^{8}e^{5}y^{2}+\overline{x}^{9}e^{4}y^{2}+6e^{5}x^{2}y^{8}+4e^{6}x^{3}y^{6}+e^{7}x^{4}y^{4}\\ &+\overline{x}^{9}e^{5}x+3\overline{x}^{4}ey^{10}+4\overline{x}^{6}ey^{8}+4e^{4}xy^{10}+4\overline{x}^{2}e^{3}xy^{9}+5\overline{x}^{3}e^{3}xy^{8}+8\overline{x}^{3}e^{4}xy^{7}+6\overline{x}^{4}e^{2}xy^{8}>0,\end{aligned}$$

$$\begin{aligned}\Gamma &= 4\overline{x}^{4}e^{5}xy^{5}+12\overline{x}^{5}e^{3}xy^{6}+4\overline{x}^{6}e^{2}xy^{6}+6\overline{x}^{6}e^{4}xy^{4}+6\overline{x}^{7}e^{3}xy^{4}+4\overline{x}^{8}e^{4}xy^{2}+2\overline{x}^{2}e^{4}x^{2}y^{7}\\ &+10\overline{x}^{3}e^{4}x^{2}y^{6}+4\overline{x}^{3}e^{5}x^{2}y^{5}+10\overline{x}^{3}e^{5}x^{3}y^{4}+5\overline{x}^{3}e^{6}x^{4}y^{2}+6\overline{x}^{4}e^{3}x^{2}y^{6}+2\overline{x}^{4}e^{4}x^{3}y^{4}\\ &+2\overline{x}^{4}e^{6}x^{2}y^{3}+12\overline{x}^{5}e^{4}x^{2}y^{4}+4\overline{x}^{5}e^{5}x^{3}y^{2}+6\overline{x}^{6}e^{5}x^{2}y^{2}+\overline{x}^{5}exy^{8}>0.\end{aligned}$$

Hence, $x=\overline{x}$, $y=\overline{x}$, $z=\frac{\overline{x}y^{2}+\overline{x}^{2}e}{y^{2}+ex}=\overline{x}$ and $t=\frac{\overline{x}z^{2}+\overline{x}^{2}e}{z^{2}+ey}=\overline{x}$ is only solution of System (4.15). Thus, Equation (4.14) does not posses a minimal period-four solution. □

Theorem 4.11 *Assume that* $F=F_{g}=A^{2}e<F_{0}$*. Then, the unique equilibrium point* $\overline{x}=A$ *of Equation* (4.14) *is globally asymptotically stable. Also, every solution of Equation* (4.14) *oscillates about the equilibrium point* $\overline{x}$ *with semicycles of length two.*

Proof Notice that

$$x_{n+1}-\overline{x}=\frac{e\overline{x}\left(\overline{x}-x_{n-1}\right)}{x_{n}^{2}+ex_{n-1}},$$

i.e. x_{n+1} and x_{n-1} are from the different sides of the equilibrium point. Also, it means that x_{n+1} and x_{n+5} are always from the same side of the equilibrium point $\overline{x}$. Since

$$x_{n}-x_{n+4} = \left(x_{n}-\overline{x}\right)\frac{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}}{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}^{2}e^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}},$$

the following holds

$$\frac{x_{n}-x_{n+4}}{x_{n}-\overline{x}}=\frac{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}}{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}^{2}e^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}}>0.$$

Furthermore,

$$\frac{x_{n}-x_{n+4}}{x_{n}-\overline{x}}<1\Leftrightarrow\frac{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}}{x_{n+1}^{2}x_{n+3}^{2}+\overline{x}^{2}e^{2}+\overline{x}ex_{n+1}^{2}+ex_{n}x_{n+3}^{2}}<1\Leftrightarrow\overline{x}^{2}e^{2}>0,$$

which is always true. Also,

$$x_{n}>\overline{x}\Rightarrow x_{n}>x_{n+4}>\overline{x},\quad n\in\mathbb{N},$$

$$x_{n}<\overline{x}\Rightarrow x_{n}<x_{n+4}<\overline{x},\quad n\in\mathbb{N},$$

which means that every sequence $\{x_{4k}\}_{k=1}^{\infty}$, $\{x_{4k+1}\}_{k=0}^{\infty}$, $\{x_{4k+2}\}_{k=0}^{\infty}$, $\{x_{4k+3}\}_{k=0}^{\infty}$ is monotone and bounded. That implies that each of the sequences is convergent. Since, by Theorem 3.1 and Lemma 4.10, Equation (4.14) has neither minimal period-two nor period-four solutions, the following holds

$$\lim_{k\to\infty} x_{4k} = \lim_{k\to\infty} x_{4k+1} = \lim_{k\to\infty} x_{4k+2} = \lim_{k\to\infty} x_{4k+3} = \overline{x},$$

which implies that the equilibrium $\overline{x}$ is an attractor and by using Theorem 2.1, that completes the proof of the theorem. □

Case (4) $F = F_0$

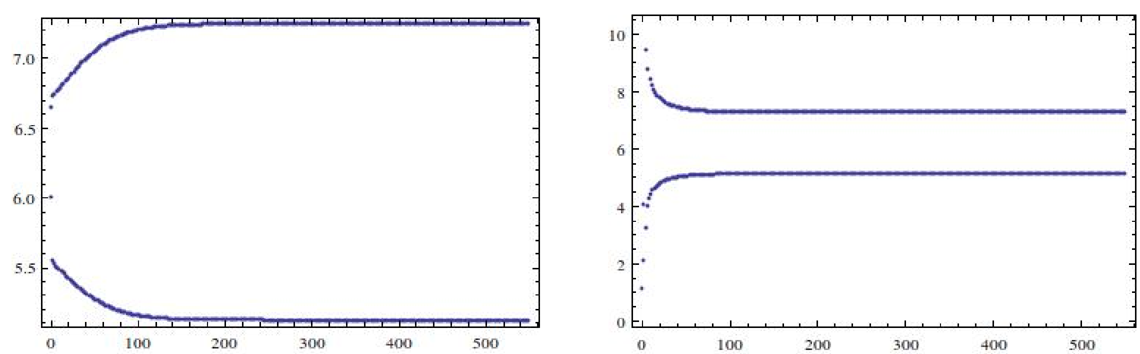
In this case, equilibrium point is a nonhyperbolic point and by Theorem 3.1, there is no period-two solutions. We give simulations for some numerical values of parameters.
Based on many numerical simulations, we believe that the following conjectures are true.

Conjecture 4.12 *If* $F < F_0$, *then the equilibrium point* $\overline{x}$ *of Equation* (2.1) *is globally asymptotically stable.*

Conjecture 4.13 *If* $F = F_0$, *then every solution of Equation* (2.1) *converges to the equilibrium point* $\overline{x}$.

Conjecture 4.14 *If* $F > F_0$, *then every solution of Equation* (2.1) *converges to either the equilibrium point* $\overline{x}$ *or to unique period-two solution* (ϕ, ψ). *More precisely, every solution which starts off the global stable manifold of the equilibrium* $E(\bar{x}, \bar{x})$ *converges to the period-two solution* (ϕ, ψ).

For some numerical values of parameters, we give a visual evidence for Conjecture 4.14. See Figures 4 and 5.



(a) $(x_0, x_{-1}) = (3.4, 6)$ (b) $(x_0, x_{-1}) = (3.1, 1.1)$

Figure 4. The orbits for values of parameters $A = 2$, $e = 1$ and $F = 190 > F_0 = 180$.

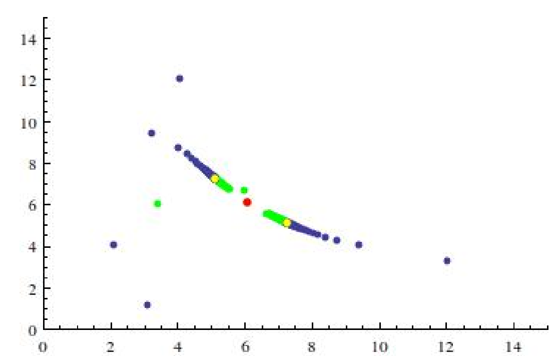


Figure 5. The phase portrait for values of parameters $A = 2$, $e = 1$, and $F = 190 > F_0 = 180$, and initial conditions $(x_0, x_{-1}) = (3.4, 6)$-green, $(x_0, x_{-1}) = (3.1, 1.1)$-blue.

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