

1-1-2019

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Recommended Citation

KAMBLE, PRAKASH; SHRIGAN, MALLIKARJUN; and ALTINKAYA, ŞAHSENE (2019) "On λ -pseudo q -bi-starlike functions," *Turkish Journal of Mathematics*: Vol. 43: No. 2, Article 13. <https://doi.org/10.3906/mat-1810-80>

Available at: <https://journals.tubitak.gov.tr/math/vol43/iss2/13>

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On λ -pseudo q -bi-starlike functions

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Received: 18.10.2018

Accepted/Published Online: 28.01.2019

Final Version: 27.03.2019

Abstract: Making use of the λ -pseudo- q -differential operator, we aim to investigate a new, interesting class of bi-starlike functions in the conic domain. Furthermore, we obtain certain sharp bounds of the Fekete–Szegő functional for functions belonging to this class.

Key words: Fekete–Szegő inequality, bi-starlike functions, q -differential operator

1. Introduction

Let \mathcal{A} denote the family of functions analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and given by the following Taylor–Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad (1.1)$$

We denote by \mathcal{S} the class of starlike functions $f \in \mathcal{A}$, which are univalent in \mathbb{U} (e.g., see [1, 4, 5, 9, 11]).

Let $\mathcal{S}^*(\beta)$ be the usual subclass of starlike functions \mathcal{S} of order β , $0 \leq \beta < 1$, so that $f \in \mathcal{S}^*(\beta)$ if and only if, for $z \in \mathbb{U}$,

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \beta.$$

For $\alpha > 0$, let $\mathcal{B}(\alpha)$ denote the class of Bazilevič functions defined in the open unit disk \mathbb{U} , normalized by the condition $f(0) = f'(0) - 1 = 0$, and such that, for $z \in \mathbb{U}$,

$$\operatorname{Re} \left(f'(z) \left(\frac{zf(z)}{z} \right)^{\alpha-1} \right) > 0.$$

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2010 AMS Mathematics Subject Classification: 30C45

The class $\mathcal{B}(\alpha)$ reduces to the starlike function and bounded turning function whenever $\alpha = 0$ and $\alpha = 1$, respectively. This class is extended to $\mathcal{B}(\alpha, \beta)$, which satisfies the geometric condition

$$\operatorname{Re} \left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \right) > \beta,$$

where α is a nonnegative real number and $0 \leq \beta < 1$. This class of functions was intensively studied by Singh [18] and considered subsequently by London and Thomas [14]. Recently, Babalola [3] introduced a new subclass $\mathcal{L}_\lambda(\beta)$ of λ -pseudo-starlike functions of order β satisfying the geometric condition

$$\operatorname{Re} \left(\frac{z(f'(z))^\lambda}{f(z)} \right) > \beta, \quad (z \in \mathbb{U}, 0 \leq \beta < 1, \lambda \geq 1).$$

We note that, if $\lambda = 1$, we have the class of starlike functions of order β , which in this context is 1-pseudo-starlike functions of order β . If $\beta = 0$, we simply write \mathcal{L}_λ instead of $\mathcal{L}_\lambda(0)$. For $\lambda = 2$, we note that functions in $\mathcal{L}_2(\beta)$ are defined by

$$\operatorname{Re} \left(f'(z) \frac{z f'(z)}{f(z)} \right) > \beta, \quad (z \in \mathbb{U}),$$

which is a product combination of geometric expression for bounded turning and starlike functions, an interesting analytic presentation on univalent functions in the open unit disk \mathbb{U} . Joshi et al. [8] defined the subclasses $S_\Sigma^\lambda(k, \alpha)$ and $S_\Sigma^\lambda(k, \beta)$ of bi-univalent functions associated with λ -bi-pseudo-starlike functions in the unit disk \mathbb{U} . Recently, Altinkaya and Özkan [2] introduced the subclasses $\mathcal{L}_\lambda(\beta)$ and $\mathcal{L}_\lambda(\beta, \phi)$ of Sălăgean type λ -pseudo-starlike functions. For these function classes, they found upper bounds for the initial coefficients as well as Fekete–Szegő inequalities.

Definition 1.1 Let \mathcal{P} be analytic and normalized Carathéodory functions with positive real part in \mathbb{U} . Let $\mathcal{P}(p_k)(0 \leq k < \infty)$ denote the family of functions p , such that $p \in \mathcal{P}$ and $p \prec \mathcal{P}$ in \mathbb{U} , where p_k maps the unit disk conformally onto the domain Ω_k such that $1 \in \Omega_k$ and $\partial\Omega_k$ is defined by

$$\partial\Omega_k = \{u + iv : u^2 = k^2(u - 1)^2 + k^2v^2\}.$$

Moreover, Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, and parabolic for $k = 1$ and it covers the right half plane when $k = 0$. The extremal functions of class $\mathcal{P}(p_k)(0 \leq k < \infty)$ were presented and investigated by Kanas et al. in [12] and [13]. Obviously, for $k = 0$, we have

$$p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + \dots,$$

for $k = 1$, we have

$$p_1(z) = 1 + \frac{2}{\pi^2} \log^2 \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right),$$

and for $0 < k < 1$ and $A = A(k) = (2/\pi) \arccos k$, we have

$$p_k(z) = 1 + \frac{2}{1 - k^2} \sinh^2 (A(k) \operatorname{arctanh} \sqrt{z}).$$

By virtue of

$$p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z)$$

or

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z),$$

and the properties of domains, we have

$$Re(p(z)) > Re(p_k(z)) > \frac{k}{k+1}.$$

The q -differential operator plays a vital role in the theory of geometric function theory. The various subclasses of the normalized analytic function class \mathcal{A} have been studied from different view points. Both q -calculus and fractional calculus provide important tools that have been used in order to investigate various subclasses of \mathcal{A} . Historically speaking, the firm footing of the usage of q -calculus in the context of geometric function theory was provided and q -hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [19, p. 347 et seq.]). Ismail et al. [6] introduced the class of generalized complex functions via q -calculus on some subclasses of analytic functions. Recently, Purohit and Raina [16] investigated applications of the fractional q -calculus operator to define new classes of functions that are analytic in unit disk \mathbb{U} (see, for details, [7], [10], and [20]–[23]).

For $0 < q < 1$, the q -derivative of a function $f \in \mathcal{A}$ given by (1.1) is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0), \tag{1.2}$$

and $D_q f(0) = f'(0)$, $D_q^2 f(z) = D_q(D_q f(z))$. From (1.1), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}. \tag{1.4}$$

As $q \rightarrow 1^-$, $[k]_q \rightarrow k$. For a function $g(z) = z^k$, we observe that

$$D_q(g(z)) = D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = k z^{k-1},$$

$$\lim_{q \rightarrow 1^-} (D_q(g(z))) = k z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

We define the Sălăgean q -differential operator (also refer to [10]) using the q -differential operator as follows:

$$\begin{aligned} \mathcal{D}_q^0 f(z) &= f(z), \\ \mathcal{D}_q^1 f(z) &= z\mathcal{D}_q f(z), \\ \mathcal{D}_q^n f(z) &= z\mathcal{D}_q(\mathcal{D}_q^{n-1} f(z)), \\ \mathcal{D}_q^n f(z) &= z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}). \end{aligned} \tag{1.5}$$

We note that $\lim_{q \rightarrow 1^-}$

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}). \tag{1.6}$$

Definition 1.2 Let $0 \leq k < 1, \lambda \geq 1, n \in \mathbb{N}_0, 0 < q < 1$. For $p_k(z)$ as defined in Definition 1.1, the function f given by (1.1) belongs to $\mathcal{S}_{\lambda, k}^q(p_k)$ if

$$\left(\frac{z [((\mathcal{D}_q^n f)z)']^\lambda}{(\mathcal{D}_q^n f)z} \right) \prec p_k(z) \quad (z \in \mathbb{U}). \tag{1.7}$$

Let $\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots (c_1 > 0)$ be an analytic function with positive real part on \mathbb{U} .

Definition 1.3 For $\lambda \geq 1, 0 < q < 1$, we say a function f given by (1.1) belongs to the class $\mathcal{S}_{\lambda, \phi}^q$ if it satisfies the quasi-subordination condition

$$\left(\frac{z [((\mathcal{D}_q^n f)z)']^\lambda}{(\mathcal{D}_q^n f)z} \right) \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}). \tag{1.8}$$

In order to derive our main results, we use the following lemma.

Lemma 1.4 [15] Let $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots \in \mathcal{U}$ such that $|w(z)| < 1$ in \mathbb{U} . If t is a complex number, then

$$|w_2 + t w_1^2| \leq \max\{1, |t|\}.$$

The inequality is sharp for the function $w(z) = z$ or $w(z) = z^2$.

In this paper, motivated by the earlier work of Babalola [3] and Altinkaya and Özkan [2], we introduce a new approach for studying a subclass of λ -pseudo bi-starlike functions using the q -differential operator and estimate the Fekete–Szegő body of the coefficient using subordination [17].

2. Main results

We investigate $|a_3 - \sigma a_2^2|$ for the function $f \in \mathcal{A}$ for the class $\mathcal{S}_{\lambda,k}^q(p_k)$ associated with conical domains.

Theorem 2.1 *Let $0 \leq k < 1, \lambda \geq 1, 0 < q < 1$ and $p_k(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$ defined in Definition 1.1. If the function f given by (1.1) belongs to $\mathcal{S}_{\lambda,k}^q(p_k)$, then for any complex σ we have*

$$|a_3 - \sigma a_2^2| \leq \frac{p_1}{(3\lambda - 1)[3]_q^n} \max \left\{ 1, \left| \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda - 1)[3]_q^n}{\{(2\lambda - 1)[2]_q^n\}^2} \right| \right\}. \tag{2.1}$$

Proof By (1.7), we have

$$\frac{z[(\mathcal{D}_q^n f)z]^\lambda}{(\mathcal{D}_q^n f)z} = p_k(z) \quad (z \in \mathbb{U}). \tag{2.2}$$

We note that

$$z[(\mathcal{D}_q^n f)z]^\lambda = z + 2\lambda[2]_q^n a_2 z^2 + (3\lambda[3]_q^n a_3 + 2\lambda(\lambda - 1)a_2^n [2]_q^{2n}) z^3 + \dots \tag{2.3}$$

and

$$p_k(w(z))(\mathcal{D}_q^n f)(z) = z + (p_1 w_1 + [2]_q^n a_2) z^2 + (p_1 w_2 + p_2 w_1^2 + [2]_q^n a_2 p_1 w_1 + [3]_q^n a_3) z^3 + \dots \tag{2.4}$$

Comparing coefficients of (2.2), (2.3), and (2.4), we obtain

$$a_2 = \frac{p_1 w_1}{(2\lambda - 1)[2]_q^n} \tag{2.5}$$

and

$$a_3 = \frac{p_1 w_2}{(3\lambda - 1)[3]_q^n} + \frac{p_2 w_1^2}{(3\lambda - 1)[3]_q^n} + \frac{(4\lambda - 1 - 2\lambda^2) p_1 w_1^2}{(3\lambda - 1)(2\lambda - 1)^2 [3]_q^n}. \tag{2.6}$$

Hence, by (2.5) and (2.6), we get the following:

$$a_3 - \sigma a_2^2 = \frac{p_1}{(3\lambda - 1)[3]_q^n} (w_2 + \vartheta w_1^2),$$

where

$$\vartheta = \left| \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda - 1)[3]_q^n}{\{(2\lambda - 1)[2]_q^n\}^2} \right|. \tag{2.7}$$

Using Lemma 1.4 and equation (2.7), we yield (2.1). This completes the proof. □

Corollary 2.2 *Let $f \in \mathcal{S}_{\lambda,k}^q(p_k)$, then*

$$|a_2| = \frac{p_1}{(2\lambda - 1)[2]_q^n} \tag{2.8}$$

and

$$|a_3| \leq \frac{p_1}{(3\lambda - 1)[3]_q^n} \max \left\{ 1, \left| \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)}{(2\lambda - 1)^2 [2]_q^{2n}} \right| \right\}, \tag{2.9}$$

where $0 \leq k < 1, \lambda \geq 1, 0 < q < 1$.

For the class of functions $f \in \mathcal{S}_{\lambda, \varphi}^q(\phi)$, we can prove the following:

Theorem 2.3 *Let $\lambda \geq 1, 0 < q < 1$. If the function f given by (1.1) belongs to $\mathcal{S}_{\lambda, \varphi}^q(\phi)$, then for any complex σ we have*

$$|a_3 - \sigma a_2^2| \leq \frac{1}{3\lambda[3]_q^n} \left(c_1 + \max \left\{ c_1, \left| \frac{2\lambda(2-\lambda)[2]_q^{2n} - 3\sigma\lambda[3]_q^n}{4\lambda^2[2]_q^{2n}} \right| c_1^2 + |c_2| \right\} \right). \tag{2.10}$$

Proof If $f \in \mathcal{S}_{\lambda, \varphi}^q(\phi)$, then

$$\frac{z[(\mathcal{D}_q^n f)z]^\lambda}{(\mathcal{D}_q^n f)z} = \varphi(z) (\phi(z) - 1) \quad (z \in \mathbb{U}). \tag{2.11}$$

We have

$$z[(\mathcal{D}_q^n f)z]^\lambda = z + 2\lambda[2]_q^n a_2 z^2 + (3\lambda[3]_q^n a_3 + 2\lambda(\lambda - 1)[2]_q^{2n} a_2^2) z^3 + \dots$$

and

$$\varphi(z) (\phi(z) - 1) (\mathcal{D}_q^n f)(z) = c_1 A_0 w_1 z^2 + (c_1 A_1 w_1 + A_0 (c_1 w_2 + c_2 w_1^2 + [2]_q^n c_1 A_0 w_1 a_2)) z^3 + \dots \tag{2.12}$$

From (2.11) and (2.12), it is easily seen that

$$a_2 = \frac{c_1 A_0 w_1}{2\lambda[2]_q^n}, \tag{2.13}$$

$$a_3 = \frac{c_1 A_1 w_1}{3\lambda[3]_q^n} + \frac{c_1 A_0 w_2}{3\lambda[3]_q^n} + \frac{A_0}{3\lambda[3]_q^n} \left(c_2 - \frac{(2-\lambda)c_1^2 A_0}{2\lambda} \right) w_1^2, \tag{2.14}$$

and

$$|a_3 - \sigma a_2^2| \leq \frac{1}{3\lambda[3]_q^n} [|c_1 A_1 w_1| + |c_1 A_0 \Psi|], \tag{2.15}$$

where

$$\Psi = \left\{ w_2 - \left(\frac{(2-\lambda)c_1 A_0}{2\lambda} + \frac{3\lambda c_1 A_0 w_1^2 \sigma [3]_q^n}{4\lambda^2 [2]_q^{2n}} - \frac{c_2}{c_1} \right) w_1^2 \right\}. \tag{2.16}$$

Since φ is analytic in \mathbb{U} , using the inequalities $|A_n| \leq 1$ and $|w_1| \leq 1$, we get

$$|a_3 - \sigma a_2^2| \leq \frac{c_1}{3\lambda[3]_q^n} [1 + |\Phi|], \tag{2.17}$$

where

$$\Phi = \left| w_2 - \left(-\frac{c_2}{c_1} - \left[\frac{(2-\lambda)c_1}{2\lambda} + \frac{3\sigma\lambda[3]_q^n c_1}{4\lambda^2 [2]_q^{2n}} \right] c_1 \right) w_1^2 \right|. \tag{2.18}$$

Applying Lemma 1.4 and equation (2.18) yields result (2.10). □

Corollary 2.4 Let $f \in \mathcal{S}_{\lambda, \varphi}^q(\phi)$, then

$$|a_2| \leq \frac{c_1 A_0}{2\lambda [2]_q^n} \quad (2.19)$$

and

$$|a_3| \leq \frac{1}{3\lambda [3]_q^n} \left(c_1 + \max \left\{ c_1, \left| \frac{(2-\lambda)c_1^2}{2\lambda} \right| + |c_2| \right\} \right), \quad (2.20)$$

where $\lambda \geq 1, 0 < q < 1$.

References

- [1] Abdel-Gawad HR, Thomas DK. The Fekete-Szegő coefficient problems for strongly close-to-convex functions. P Am Math Soc 1992; 114: 345-349.
- [2] Altinkaya Ş, Özkan SY. On Sălăgean type pseudo-starlike functions. Acta et Commentationes Universitatis Tartuensis de Mathematica 2017; 21: 275-285.
- [3] Babalola KO. On λ -pseudo starlike functions. J Classical Anal 2013; 2: 137-147.
- [4] Darus M, Hussain S, Raza M, Sokół J. On a subclass of starlike functions. Result Math 2018; 73: 22.
- [5] Frasin BA, Auof MK. New subclasses of bi-univalent functions. Appl Math Lett 2011; 24: 1569-1673.
- [6] Ismail MEH, Merkes E, Styer D. A generalization of starlike functions. Complex Variable Theory Appl 1990; 14: 77-84.
- [7] Jahangiri JM, Hamidi SG. Coefficient estimates for certain classes of bi-univalent functions. Int J Math Math Sci 2013; 2013: 190560.
- [8] Joshi SB, Altinkaya Ş, Yalçın S. Coefficient estimates for Sălăgean type bi-pseudo-starlike functions. Kyungpook Mathematical Journal 2017; 57: 613-621.
- [9] Keogh FR, Merkes EP. A Coefficient inequality for certain classes of analytic functions. P Am Math Soc 1969; 20: 8-12.
- [10] Kamble PN, Shrigan MG. Initial coefficient estimates for bi-univalent functions. Far East J Math 2018; 102: 271-282.
- [11] Kamble PN, Shrigan MG, Srivastava HM. A novel subclass of univalent functions involving operators of fractional calculus. Int J Appl Math 2017; 30: 501-514.
- [12] Kanas S, Răducanu D. Conic regions and k-uniform convexity. J Comput Appl Math 1999; 105: 327-336.
- [13] Kanas S, Răducanu D. Some subclass of analytic functions related to conic domains. Math Slovaca 2014; 64: 1183-1196.
- [14] London RR, Thomas DK. The derivative of Bazilevič functions. P Am Math Soc 1988; 104: 85-89.
- [15] Ma W, Minda D. A Unified Treatment of Some Special Classes of Univalent Functions. Cambridge, MA, USA: MIT Press, 1994.
- [16] Purohit SD, Raina RK. Certain subclasses of analytic functions associated with fractional q-calculus operators. Math Scand 2011; 109: 55-70.
- [17] Robertson MS. Quasi-subordination and coefficient conjecture. B Am Math Soc 1970; 76: 1-9.
- [18] Singh R. On Bazilevič functions. B Am Math Soc 1973; 38: 261-271.
- [19] Srivastava HM. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In: Srivastava HM, Owa S, editors. Univalent Functions, Fractional Calculus, and Their Applications. New York, NY, USA: John Wiley and Sons, 1989, pp. 329-354.

- [20] Srivastava HM, Altinkaya Ş, Yalçın S. Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q -derivative operator. *Filomat* 2018; 32: 503-516.
- [21] Srivastava HM, Bulut S, Çağlar M, Yağmur N. Coefficient estimates for a general subclass of analytic and bi-univalent functions. *Filomat* 2013; 27: 831-842.
- [22] Srivastava HM, Gaboury S, Ghanim F. Coefficient estimates for some general subclasses of analytic and bi-univalent functions. *Afrika Math* 2017; 28: 693-706.
- [23] Srivastava HM, Sümer S, Hamidi SG, Jahangiri JM. Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator. *Bull Iranian Math Soc* 2017; 44: 149-157.