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A blow-up result for nonlocal thin-film equation with positive initial energy

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Abstract: In this note, we consider a thin-film equation including a diffusion term, a fourth order term and a nonlocal source term under the periodic boundary conditions. In particular, a finite time blow-up result is established for the case of positive initial energy provided that

$$\frac{\pi^2}{a^2} \leq \frac{2}{p-1},$$

where a is the length of the interval and $p > 1$ is the power of nonlinear force term. Also upper and lower blow-up times are estimated.

Key words: Nonlinear thin film equation, positive initial energy, blow up, periodic boundary condition, Non-local source term

1. Introduction

In this note we consider the following initial and periodic boundary value problem :

$$u_t - u_{xx} + u_{xxxx} = |u|^{p-1}u - \frac{1}{a}\int_0^a |u|^{p-1}u\,dx, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,t) = u(x+a,t), \quad \text{for all } x \in \mathbb{R}, \text{ and } t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}, \tag{1.3}$$

where $p > 1, u_0 \in \dot{H}^2_{per}(\Omega)$, $\Omega = (0,a)$ and $\displaystyle\int_0^a u_0(x)dx = 0$ with $u_0 \not\equiv 0$. The novelty in the problem above is the existence of the diffusion term and the periodic boundary conditions which are natural boundary conditions for this type models [9].

The following general fourth-order reaction diffusion equation

$$u_t + A_1\Delta u + A_2\Delta^2 u + A_3\nabla\cdot(|\nabla u|^2\nabla u) + A_4\Delta|\nabla u|^2 = g(x,t) + \eta(x,t), \tag{1.4}$$

arises in theories such as the thin film theory, lubrication theory, phase transitions etc. (see [12]). In (1.4), $u(x,t)$ and $A_1\Delta u$ denote the height of a film in epitaxial growth and the diffusion due to evaporation condensation, respectively. The terms $A_2\Delta^2 u$ and $A_3\nabla\cdot(|\nabla u|^2\nabla u)$ are the capilarity-driven surface diffusion and the hopping

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of atoms, respectively. The term $A_4\Delta|\nabla u|^2$ describes motion of an atom to a neighbouring kink. The functions $g(x,t)$ and $\eta(x,t)$ represent the mean deposition flux, and some Gaussian noise, respectively. For a detailed description of this model we refer the readers to [9].

In [12], Qu and Zhou considered $1D$ form of the equation in (1.4) and derived a threshold result of global existence and nonexistence of solutions when $A_1 = A_3 = A_4 = 0$. In this work, the flux term is the nonlocal-source term

$$g(x,t) = |u|^{p-1}v - \frac{1}{a}\int_0^a |u|^{p-1}u$$

and boundary conditions are

$$u_x(0,t) = u_x(a,t) = 0, \quad u_{xxx}(0,t) = u_{xxx}(a,t) = 0.$$

In [7], using potential well theory, Zhou established a blow-up result for the same problem in [12] assuming that the initial energy is positive. Also, he derived an upper bound for the blow-up time. Existence of blow up solutions is a long standing topic in the study of nonlinear models of partial differential equations. Interested readers may refer to some or all of the references [1, 3–6, 8, 9, 13, 14]. In this work, using the potential well method the existence of finite time blow up solutions will be studied under the assumption $0 < J(u_0) < E_m$ and $0 < I(u_0)$ where

$$J(u) = \frac{1}{2}\|u_x\|^2 + \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{1.5}$$

$$I(u) = \|u_x\|^2 + \|u_{xx}\|^2 - \|u\|_{p+1}^{p+1}, \tag{1.6}$$

and E_m is the potential well depth given below in (1.8). The existence of blow-up solutions and lower bounds for their blow-up times will be estimated. By the zero average of initial function from (1.1), we obtain that $\frac{d}{dt}\int_0^a u\,dx = 0$.

Now we present some notations and mathematical tools which we shall need:

Let us denote the $L^2(\Omega)$-inner product and the $L^2(\Omega)$-norm by $(u,v) = \int_0^a u(x)v(x)\ dx$ by $\|\cdot\|$, respectively. Let $\dot{H}^2_{per}(\Omega) := \left\{ u \in H^2_{per}(\Omega) : \int_\Omega u\,dx = 0 \right\}$. The pair $\left(\dot{H}^2_{per}(\Omega), \|\cdot\|_{\dot{H}^2_{per}(\Omega)}\right)$ is a Hilbert space with the inner product and the norm $(u,v)_{\dot{H}^2_{per}(\Omega)} = \int_0^a u_x v_x\,dx + \int_0^a u_{xx}v_{xx}\ dx$, $\|u\|^2_{\dot{H}^2_{per}(\Omega)} := \|u_x\|^2 + \|u_{xx}\|^2$, respectively.

By the Sobolev embedding theorem, the inclusion $\dot{H}^1_{per}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is continuous, so there exists an optimal embedding constant B such that:

$$\|u\|_{p+1} \le B\|u_x\|. \tag{1.7}$$

In the rest of this text, we shall use B as the optimal embedding constant. Define the function

$$g(\alpha) := \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\alpha^2 - \frac{1}{p+1}(B\alpha)^{p+1}.$$

It is obvious that $g(\alpha)$ has a critical point at

$$\alpha_1 = \left[\frac{a^2+\pi^2}{a^2}\right]^{\frac{1}{p-1}} B^{-\frac{p+1}{p-1}},$$

and attains its maximum value at this point as

$$E_m := \frac{p-1}{2(p+1)}\left[\frac{a^2+\pi^2}{a^2}\right]^{\frac{p+1}{p-1}} B^{\frac{-2(p+1)}{p-1}} = \left(\frac{a^2+\pi^2}{a^2}\right)\frac{p-1}{2(p+1)}\alpha_1^2, \tag{1.8}$$

because $g(\alpha)$ is increasing on $(0,\alpha_1)$ and is decreasing on (α_1,∞) with $\lim\limits_{\alpha\to\infty} g(\alpha) = -\infty$.

The rest of this note is organized as follows: Section 2 is devoted to a local existence result and a regularity theorem. In Section 3 a blow-up result is established. In Section 4 a lower blow-up time is estimated.

2. Local Existence

Definition 2.1 *A function $u(x,t)$ is called a weak solution of (1.1) if*

$$u \in L^\infty(0,T,\dot{H}^2_{per}(\Omega)) \text{ and } u_t \in L^2(0,T,L^2(\Omega))$$

and satisfies

$$\int_0^t \int_\Omega \left[u_t\phi + u_x\phi_x + u_{xx}\phi_{xx} - \left(|u|^{p-1}u - \int_\Omega |u|^{p-1}u\right)\phi\right] dxds = 0, \tag{2.1}$$

for all $\phi \in \dot{H}^2_{per}(\Omega)$.

Now we give the following existence result for weak solutions and its proof:

Theorem 2.2 *Assume that $p > 1$, $u_0 \in \dot{H}^2_{per}(\Omega)$, and $I(u_0) > 0$ then the problems (1.1)-(1.3) has a unique local solution $u(x,t)$ with $u \in L^\infty([0,T];\dot{H}^2_{per}(\Omega))$ and $u' \in L^\infty([0,T];L^2(\Omega))$.*

Proof Let $\{\omega_n\}_{n\in\mathbb{N}}$ be the set of eigenfunctions of the problem

$$-u_{xx} = \lambda u, \quad u(x+a) = u(x).$$

The eigenvalues of this has the property $\lambda_n \le \lambda_{n+1}$, for all $n\in\mathbb{N}$, and $\lim_{\to\infty}\lambda_n = \infty$. The eigenfunctions are orthogonal in the spaces $\dot{H}^2_{per}(\Omega)$, $\dot{H}^1_{per}(\Omega)$, and $L^2(\Omega)$. We normalize the eigenfunctions in $L^2(\Omega)$: $(\omega_i,\omega_j) = \delta_{ij}$.

We proceed by constructing approximate solutions $u_m := \sum\limits_{i=1}^m g_{im}(t)\omega_i(x)$

satisfying

$$(\dot{u}_m,\omega_j) + (u_{mxx},\omega_{jxx}) + (u_{mx},\omega_{jx}) = (f(u_m),\omega_j), \tag{2.2}$$

and

$$u_m(x,0) = u_{0m}(x) = \sum_{i=1}^m (u_0,\omega_i)\omega_i, \tag{2.3}$$

where $f(u_m) = |u_m|^{p-1}u_m - \dfrac{1}{a}\displaystyle\int_0^a |u_m|^{p-1}u_m\,dx$. The problem (2.2)–(2.3) is equivalent to the following initial value problem of a system of first order ordinary differential equations for $\{g_{jm}(t)\}_{j=1}^m$:

$$g'_{jm}(t) = (\lambda_j^2 + \lambda_j)g_{jm}(t) + f_{jm}(t), \quad g_{jm}(0) = (u_0, w_j). \tag{2.4}$$

For $p > 1$, the function $f_{jm}(t)$ is a continuously differentiable function of g_{jm}. Thus, the problem (2.4) has a unique local solution $g_{jm}(t)$ on $[0, T_1]$ for $j = 1, 2, \cdots, m$.

Now we multiply (2.2) by $g'_{jm}(t)$ and sum from 1 to m

$$\|u'_m(t)\|^2 + \frac{d}{dt}\left[\frac{1}{2}\|u_{mxx}\|^2 + \frac{1}{2}\|u_{mx}\|^2 - \frac{1}{p+1}|u_m|_{p+1}^{p+1}\right] = 0. \tag{2.5}$$

Integrating from 0 to t we obtain

$$\int_0^t \|u'_m(\tau)\|^2\,d\tau + J(u_m) = J(u_{0m}). \tag{2.6}$$

By the convergence of $u_m(x,0) \to u_0(x)$ in $\dot{H}^2_{per}(\Omega)$, we get $J(u_m(x,0)) \to J(u_0) < d$. Then for sufficiently large m, we have

$$\int_0^t \|u'_m(\tau)\|^2\,d\tau + J(u_m) < d$$

for $0 \le t \le T_1$. By the assumption $I(u_0) > 0$, $I(u_m(t))$ is positive on some interval $[0, T_2]$. Let T be the minimum of T_1 and T_2. By

$$J(u_m) = \frac{p-1}{2(p+1)}\left(\|u_{mxx}\|^2 + \|m_{mx}\|^2\right) + \frac{1}{p+1}I(u_m),$$

for sufficiently large m and any $t \in [0, T]$, we obtain

$$\int_0^t \|u'_m(\tau)\|^2\,d\tau + \frac{p-1}{2(p+1)}\|u_{mxx}\|^2 < d.$$

Hence, we obtain the following a priori estimates

$$\begin{cases} \int_0^t \|u'_m(\tau)\|^2\,d\tau < d, \quad \text{for} \quad t \in [0,T], \\ \sup\limits_{[0,T]}\|u'_m(t)\|^2 < d, \\ \|u_{mxx}\|^2 < (\frac{2(p+1)}{p-1}d)^{\frac{1}{2}}, \quad \text{for} \quad t \in [0,T], \\ \|u_m\|_{p+1}^p \le B^p\|u_{mxx}\|^p < B^p(\frac{2(p+1)}{p-1}d)^{\frac{p}{2}}, \quad \text{for} \quad t \in [0,T]. \end{cases}$$

Therefore, the sequence $\{u_m\}$ has a subsequence, which is denoted by itself has the following convergence properties:

$$(*)\begin{cases} u'_m \overset{w}{\to} u', & \text{in } L^2\left([0,T]; L^2(\Omega)\right), \\ u'_m \overset{w*}{\to} u', & \text{in } L^\infty([0,T]; L^2(\Omega)), \\ u_m \overset{w*}{\to} u, & \text{in } L^\infty([0,T]; \dot{H}^2_{per}(\Omega)), \\ u_m \overset{st}{\to} u, & \text{in } C([0,T]; \dot{H}^1_{per}(\Omega)), \\ |u_m|^{p-1}u_m \overset{w*}{\to} |u|^{p-1}u, & \text{in } L^\infty\left([0,T]; L^2(\Omega)\right), \end{cases}$$

Hence, the problem admits a unique local weak solution on $[0, T]$. □

Now we adapt the following regularity theorem for the smoothness of weak solutions from [2](Chapter 6.3, Theorem 4):

Theorem 2.3 *Suppose* $f \in L^2(\Omega)$ *and and the boundary* $\partial\Omega$ *is* C^2 *and* $u \in \dot{H}^1_{per}(\Omega)$ *is a weak solution of the elliptic boundary value problem*

$$\begin{cases} -u_{xx} = f \quad in \ (0,a) \\ u(x) = u(x+a) \end{cases}$$

Then $u \in \dot{H}^2_{per}(\Omega)$ *and* $\|u\|^2_{\dot{H}^2_{per}(\Omega)} \leq C(\|f\|^2 + \|u\|^2_{\dot{H}^1_{per}(\Omega)})$*, where* C *depends on* Ω*.*

Theorem 2.4 *Let* $u_0 \in \dot{H}^2_{per}(\Omega)$*,* $f \in L^2([0,T]; L^2(\Omega))$ *and* $u \in L^\infty([0,T]; \dot{H}^2_{per}(\Omega))$ *be a weak solution of* (1.1)-(1.3)*. Then* $u \in \dot{H}^4_{per}(\Omega)$*.*

Proof For a.e t we have the identity

$$(u', v) + (u_{xxxx}, v) - (u_{xx}, v) = (f, v) \quad \text{for each } v \in \dot{H}^2_{per}(\Omega).$$

We rewrite $(u_{xxxx}, v) = (h, v)$ for $h = f + u_{xx} - u'$ for a.e. t in $[0,T]$. By $(*)$ $h \in L^2([0,T]; L^2(\Omega))$ and hence $u \in \dot{H}^4_{per}(\Omega)$ follows from the previous theorem. □

3. main result

For the establishment of blow-up solution and an upper bound for the blow-up time we have the following result:

Theorem 3.1 *Assume that* $0 < J(u_0) < E_m$ *and* $\|u_{0x}\| > \alpha_1$*, then the solution* $u(x,t)$ *of* (1.1)-(1.3) *blows up at a finite time*

$$T_* \leq T_{max} = \frac{2\left(\|u_0\|^2 + \|u_{0x}\|^2\right)^{-\frac{p-1}{2}}}{C(p+1)},$$

where $C = C_1/C_2$ *with*

$$C_1 = \frac{p-1}{p+1}[1 - (\frac{\alpha_1}{\alpha_2})^{p+1}] \ and \ C_2 = (2^{-\frac{p+1}{2}})(a^{\frac{p-1}{2}})$$

and T_{max} *is the an upper bound for the blow up time.*

First we introduce the following lemmata which are analogous to the ones in [7] and are necessary for the proof of this theorem.

Lemma 3.2 *The potential energy functional functional* $J(u)(t)$ *given in* (1.5) *is nonincreasing in* t *because of* $J'(u(t)) = -\|u_t\|^2 \leq 0$ *and*

$$J(u) = J(u_0) - \int_0^t \|u_s\|^2 \, ds.$$

This lemma is a corollary of Theorem 2.2. However, we shall include the proof of the following lemma because its use differs slightly in our case:

Lemma 3.3 *Assume that the axioms of Theorem 3.1 hold. Then there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|u_x(.,t)\| \geq \alpha_2, \quad \textit{for all}\, t \geq 0, \tag{3.1}$$

and

$$\|u_x(.,t)\|_{p+1} \geq B\alpha_2, \quad \textit{for all}\, t \geq 0. \tag{3.2}$$

Proof Let $\alpha = \|u_x\|$. Using the Wirtinger's inequality and Sobolev imbedding theorem we deduce that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u_x\|^2 + \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\|u_x\|^2 - \frac{1}{p+1}\left(B\|u\|\right)_{p+1}^{p+1} \\ &= \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\alpha^2 - \frac{1}{p+1}\left(B\alpha\right)^{p+1} \\ &=: g(\alpha). \end{aligned} \tag{3.3}$$

Since $J(u_0) < E_m$, there exists $\alpha_2 > \alpha_1 > 0$ such that $J(u_0) = g(\alpha_2)$. Let $\alpha_0 = \|u_{0x}\| > \alpha_1$. By (3.3), we have $g(\alpha_0) \leq J(u_0) = g(\alpha_2)$. Since $\alpha_0, \alpha_2 \geq \alpha_1$, we obtain $\alpha_0 \geq \alpha_2$. Hence, (3.1) is true for $t = 0$.

To prove that (3.1) is true for $t > 0$ we assume that (3.1) is not true for some t_0. Using the continuity of $\|u_x(.,t)\|$, which follows from $(*)$, and $\alpha_1 < \alpha_2$ we may choose t_0 so that $\alpha_1 < \|u_x(.,t_0)\| < \alpha_2$. Then from (3.3) it follows that

$$J(u_0) = g(\alpha_2) < g(\|u_x(.,t_0)\|) \leq J(u)(t_0)$$

which contradicts the fact that $J(u)(t)$ is nonincreasing.

From Lemma 3.2 it follows that $J(u_0) \geq J(u)$. When this is combined with (3.3) we find

$$\begin{aligned} \frac{1}{p+1}\|u\|_{p+1}^{p+1} &\geq \frac{1}{2}\|u_x\|^2 + \frac{1}{2}\|u_{xx}\|^2 - J(u_0) \\ &\geq \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\alpha_2^2 - J(u_0) \\ &= \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\alpha_2^2 - g(\alpha_2) \\ &= \frac{1}{p+1}(B\alpha_2)^{p+1}. \end{aligned} \tag{3.4}$$

Hence, (3.2) follows. □

Lemma 3.4 *Under the assumptions of Theorem 3.4 we have*

$$\frac{\alpha_2}{\alpha_1} \geq \left[(p+1)\left(\frac{a^2+\pi^2}{2a^2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{\frac{1}{p-1}} > 1 + \frac{\pi^2}{a^2}. \tag{3.5}$$

Proof Let $\beta = \frac{\alpha_2}{\alpha_1} > 1$. Now we have

$$J(u_0) = g(\alpha_2) = g(\alpha_1\beta) = (\alpha_1\beta)^2\left[\frac{a^2+\pi^2}{a^2} - \frac{1}{p+1}B^{p+1}(\beta\alpha_1)^{p-1}\right]$$
$$= (\alpha_1\beta)^2\left(\frac{a^2+\pi^2}{2a^2} - \frac{1}{p+1}\beta^{p-1}\right). \tag{3.6}$$

Dividing both sides the previous equality by $(\alpha_1\beta)^2$, we obtain

$$\left(\frac{a^2+\pi^2}{2a^2} - \frac{1}{p+1}\beta^{p-1}\right) = \frac{J(u_0)}{(\beta\alpha_1)}^2 < \frac{J(u_0)}{\alpha_1^2}.$$

By this inequality, we have

$$(p+1)^{\frac{1}{p-1}}\left[\frac{a^2+\pi^2}{2a^2} - \frac{J(u_0)}{\alpha_1^2}\right]^{\frac{1}{p-1}} \leq \beta = \frac{\alpha_2}{\alpha_1}.$$

Since $J(u_0) < E_m = \left(\frac{a^2+\pi^2}{a^2}\right)\frac{p-1}{2(p+1)}\alpha_1^2$,

$$\frac{J(u_0)}{\alpha_1^2} \leq \frac{a^2+\pi^2}{2a^2}\frac{p-1}{p+1}.$$

So

$$(p+1)\left[\frac{a^2+\pi^2}{2a^2}\right]\left(1-\frac{p-1}{p+1}\right) = \frac{a^2+\pi^2}{a^2}.$$

□

Lemma 3.5 *Let $H(u) = E_m - J(u)$. Under the assumptions of Theorem 3.1 the functions $H(u)$ enjoys the property*

$$0 < H(u_0) \leq H(u) \leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{3.7}$$

provided that $\dfrac{\pi^2}{a^2} \leq \dfrac{2}{p-1}$.

Proof Since $J(u)$ is nonincreasing in t, $H(u)(t)$ is nondecreasing in t. By the assumption $J(u_0) < E_m$, we have

$$0 < E_m - J(u_0) = H(u_0) \leq H(u). \tag{3.8}$$

Now, for $\alpha_2 > \alpha_1$ and by the help of (3.1), we derive

$$\begin{aligned} H(u) &= E_m - \frac{1}{2}\|u_x\|^2 - \frac{1}{2}\|u_{xx}\|^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq E_m - \frac{1}{2}\left(\frac{a^2+\pi^2}{a^2}\right)\|u_x\|^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq E_m - \frac{1}{2}\alpha_1^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \left(\frac{a^2+\pi^2}{a^2}\right)\frac{p-1}{2(p+1)}\alpha_1^2 - \frac{1}{2}\alpha_1^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \left(\left(\frac{a^2+\pi^2}{a^2}\right)\frac{p-1}{2(p+1)} - \frac{1}{2}\right)\alpha_1^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{3.9}$$

Since $\dfrac{\pi^2}{a^2} \leq \dfrac{2}{p-1}$, the inequality (3.7) follows. □

Now we can prove our main result:

Proof Define $\phi(t) = \frac{1}{2}\int_0^a u^2\,dx$. Then

$$\begin{aligned} \phi'(t) &= -\|u_x\|^2 - \|u_{xx}\|^2 + \|u\|_{p+1}^{p+1} \\ &= -2J(u) - \frac{2}{p+1}\|u\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1} \\ &= 2H(u) - 2E_m + \frac{p-1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{3.10}$$

Now, using

$$E_m := \frac{p-1}{2(p+1)}\left[\frac{a^2+\pi^2}{a^2}\right]^{\frac{p+1}{p-1}} B^{\frac{-2(p+1)}{p-1}}$$

and (3.2) we have

$$\begin{aligned} 2E_m &= \frac{p-1}{p+1}\left[\frac{a^2+\pi^2}{a^2}\right]^{\frac{p+1}{p-1}} B^{-2\frac{p+1}{p-1}} = \frac{p-1}{p+1}\left[\frac{a^2+\pi^2}{a^2}\right]^{\frac{p+1}{p-1}} (BB^{-\frac{p+1}{p-1}})^{p+1} \\ &= \frac{p-1}{p+1}(B\alpha_1)^{p+1} = \frac{p-1}{p+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}(B\alpha_2)^{p+1} \\ &\leq \frac{p-1}{p+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{3.11}$$

Hence, we obtain

$$\phi'(t) \geq C_1\|u\|_{p+1}^{p+1} + 2H(u), \tag{3.12}$$

where

$$C_1 = \frac{p-1}{p+1}\left[1-(\frac{\alpha_1}{\alpha_2})^{p+1}\right],$$

is a positive number. On the other hand, by Hölder's inequality, we have

$$\phi^{\frac{p+1}{2}}(t) \geq C_2\|u\|_{p+1}^{p+1}, \tag{3.13}$$

where $C_2 = (2^{-\frac{p+1}{2}})(a^{\frac{p-1}{2}})$. Combining (3.12) and (3.13), we obtain

$$\phi'(t) \geq C\phi^{\frac{p+1}{2}}(t),$$

where $C = C_1/C_2$, and

$$\phi(t) \geq \left(\phi^{-\frac{p-1}{2}}(0) - \frac{p-1}{2}Ct\right)^{-\frac{2}{p-1}}, \tag{3.14}$$

with $\phi(0) = \frac{1}{2}\|u_0\|^2$. Let

$$T_{max} := \frac{2^{\frac{p+1}{2}}}{C(p-1)}\|u_0\|^{-(p-1)}. \tag{3.15}$$

Hence, $\phi(t)$ blows up at some finite time $T_* \leq T_{max}$. By (3.15) and (3.5), we easily estimate T_* as

$$T_* \leq T_{max} = \frac{2^{\frac{p+1}{2}}\|u_0\|^{-(p-1)}}{C(p-1)} = \frac{a^{\frac{p-1}{2}}\|u_0\|^{-(p-1)}(p+1)}{(p-1)^2\left(1-(\frac{\alpha_1}{\alpha_2})^{p+1}\right)}. \tag{3.16}$$

□

4. A lower blow-up time

In this section by adapting a result of Phillipin[10] we will obtain a lower blow-up time estimate. Our goal is to show the existence of a time interval $(0, T_0)$ in which $\|u\|^2_{\dot{H}^2_{per}(\Omega)}$ remains bounded. Here is our result:

Theorem 4.1 *Let $u(x,t)$ be a solution of the problem (1.1)–(1.3). Assume that the constant $p > 1$. Then*

$$\phi(t) = \int_0^a (u_{xx})^2\,dx,$$

remains bounded for $t \in (0, T_{min})$ such that

$$T_{min} = \frac{1}{\phi^{p-1}(0)(p-1)\gamma}, \tag{4.1}$$

where γ is the best optimal constant of the Kondrachov inequality.

In the proof of this theorem we will use $u_{xxx}, u_{xxxx} \in^2 (0,a)$ due to Theorem 2.4.

Proof Differentiating $\phi(t)$, we obtain

$$\phi'(t) = 2\int_0^a u_{xx}u_{xxt}\,dx = \int_0^a u_t u_{xxxx}\,dx.$$

Plugging $u_t = u_{xx} - u_{xxxx} + |u|^{p-1}u - \frac{1}{a}\int_0^a |u|^{p-1}u\,dx$ into above equality and using integration by parts, we obtain

$$\phi'(t) = -\|u_{xxx}\|^2 - \|u_{xxxx}\|^2 + \int_0^a u|u|^{p-1}u_{xxxx}\,dx. \tag{4.2}$$

Applying the arithmetic-geometric mean inequality to the last term above, we obtain

$$\int_0^a u|u|^{p-1}u_{xxxx}\,dx \le \frac{1}{4}\int_0^a |u|^{2p}\,dx + \int_0^a (u_{xx})^2\,dx. \tag{4.3}$$

Thus, we have

$$\phi'(t) \le \frac{1}{4}\int_0^a |u|^{2p}\,dx.$$

Thanks to Kondrachov inequality $\int_0^a |u|^{2p}\,dx \le \gamma\|u_{xx}\|^{2p}, \quad \text{for} \quad p > 1$. Thus,

$$\phi'(t) \le \gamma(\phi(t))^p, \quad p > 1.$$

Solving the previous inequality we obtain:

$$\phi^{1-p}(t) \ge \phi^{1-p}(0) - (p-1)\gamma t. \tag{4.4}$$

Hence, (4.1) follows from (4.4). □

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