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A new general subclass of m -fold symmetric bi-univalent functions given by subordination

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Abstract: In a recent work, Orhan et al. (Afrika Matematika, 2016) defined a subclass of analytic bi-univalent one-fold symmetric functions. The main purpose of this paper is to generalize and improve the results of Orhan et al.

Key words: Analytic functions, m -fold symmetric bi-univalent functions, coefficient bounds, subordination

1. Introduction

Let A denote the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

in $U = \{z : |z| < 1\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Theorem 1 [8] *The range of every function of class S contains a disk of radius $\frac{1}{4}$.*

By Theorem 1, every function $f \in A$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (2)$$

A function $f \in A$ is called bi-univalent if both f and f^{-1} are univalent in U . We say that f is in the class Σ for such functions. The function $f \in A$ is said to be subordinate to another analytic function g , shown as

$$f(z) \prec g(z), \quad (3)$$

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provided that there is an analytic function w defined on U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1$$

fulfilling the following condition:

$$f(z) = g(w(z)).$$

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in U, \quad m \in \mathbb{N}) \tag{4}$$

is univalent and maps the unit disk U into a region with m -fold symmetry. A function is called m -fold symmetric (see [18]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in U, \quad m \in \mathbb{N}). \tag{5}$$

We say that f is in the class S_m for such functions, which are normalized by series expansion (5). The series expansion for f^{-1} is given as follows:

$$g(w) = w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots, \tag{6}$$

where $f^{-1} = g$. This form was recently given in [18]. Σ_m denotes the class of such functions. Recently, many authors investigated bounds for various subclasses of m -fold symmetric bi-univalent functions (see [1, 2, 4–6, 15, 16, 19]).

In the present work, we also use the symbol \mathcal{P} denoting the class of analytic functions of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$$

such that

$$R(p(z)) > 0 \quad (z \in U).$$

According to the study in [14], the m -fold symmetric function p in the class \mathcal{P} has the following form:

$$p(z) = 1 + p_m z + p_{2m} z^{2m} + p_{3m} z^{3m} + \dots. \tag{7}$$

Assume that φ is an analytic function together with positive real part in U such that

$$\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) > 0,$$

and $\varphi(U)$ is symmetric with respect to the real axis. The function φ has a series expansion of the following form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (B_1 > 0). \tag{8}$$

Let $u(z)$ and $v(z)$ be two analytic functions in U with

$$u(0) = v(0) = 0 \quad \text{and} \quad \max\{|u(z)|, |v(w)|\} < 1.$$

Observe that

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \dots \tag{9}$$

and

$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \dots \tag{10}$$

Also, we notice that

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2, |c_m| \leq 1, |c_{2m}| \leq 1 - |c_m|^2. \tag{11}$$

By some simple calculations we can state that

$$\varphi(u(z)) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \dots (|z| < 1) \tag{12}$$

and

$$\varphi(v(w)) = 1 + B_1 c_m w^m + (B_1 c_{2m} + B_2 c_m^2) w^{2m} + \dots (|w| < 1). \tag{13}$$

The aim of this paper is to introduce a new subclasses $\mathcal{S}_{\Sigma_m}^\varphi(\lambda, \mu)$ of Σ_m and derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in $\mathcal{S}_{\Sigma_m}^\varphi(\lambda, \mu)$, motivated essentially by the work of Ma and Minda [11].

2. Coefficient estimates for $\mathcal{S}_{\Sigma_m}^\varphi(\lambda, \mu)$

Definition 2 For a function $f \in \Sigma_m$, we say that $f \in \mathcal{S}_{\Sigma_m}^\varphi(\lambda, \mu)$ if the following conditions are satisfied:

$$\left((1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) \prec \varphi(z) \quad (\lambda \geq 1, \mu \geq 0, z \in U)$$

and

$$\left((1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) \prec \varphi(w) \quad (\lambda \geq 1, \mu \geq 0, w \in U),$$

where the function $g = f^{-1}$.

Remark 3

1. Letting $\mu = 1$, we have the class

$$\mathcal{S}_{\Sigma_m}^\varphi(\lambda, 1) = B_{\Sigma_m}(\lambda, \varphi)$$

(see Def. 4 in [19]).

2. For $\mu = 1$ and $\lambda = 1$ we have the class

$$\mathcal{S}_{\Sigma_m}^{\varphi}(1, 1) = \mathcal{H}_{\Sigma, m}(\varphi)$$

(see Def. 4 in [19]).

3. For $\mu = 0$ and $\lambda = 1$, we have the class of m -fold symmetric bi-starlike functions (see [10]).

In the case of one-fold symmetric functions, we have the following classes:

4. For $m = 1$, we have the class

$$\mathcal{S}_{\Sigma_1}^{\varphi}(\lambda, \mu) = N_{\Sigma}^{\mu, \lambda}(\varphi)$$

(see [12]).

5. For $m = 1, \lambda = 1$, and $\mu = 1$, we have the following class:

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1, 1) = H_{\Sigma}^{\varphi}$$

(see [3]).

For $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(1, 1) = \mathcal{H}_{\Sigma}^{\beta}(0 < \beta \leq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(1, 1) = \mathcal{H}_{\Sigma}^{\alpha}(0 \leq \alpha < 1),$$

respectively (see [17]).

6. For $m = 1$ and $\mu = 1$, we have the following class:

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1, \lambda) = R_{\Sigma}(\lambda, \varphi), \quad \lambda \geq 0$$

(see [3]).

For $m = 1, \mu = 1$, and $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(\lambda, 1) = \mathcal{B}_{\Sigma}(\beta, \lambda), \quad (0 < \beta \leq 1, \lambda \geq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda, 1) = \mathcal{B}_{\Sigma}(\alpha, \lambda), \quad (0 \leq \alpha < 1, \lambda \geq 1),$$

respectively (see [17]).

7. For $m = 1$ and $\lambda = 1$, we have the class

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1, \mu) = \mathcal{F}_{\Sigma}^{\mu}(\varphi) \quad (\mu \geq 0)$$

(see [19]).

8. For $m = 1$, $\mu = 0$, $\lambda = 1$, and $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(1, 0) = \mathcal{S}_{\Sigma, \beta}^*, \quad (0 < \beta \leq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(1, 0) = \mathcal{S}_{\Sigma}^*(\alpha), \quad (0 \leq \alpha < 1),$$

respectively (see [12]).

9. For $m = 1$, $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(\lambda, \mu) = N_{\Sigma}^{\mu, \lambda}(\beta), \quad (0 < \beta \leq 1, \lambda \geq 1, \mu \geq 0)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda, \mu) = N_{\Sigma}^{\mu, \lambda}(\alpha), \quad (0 \leq \alpha < 1, \lambda \geq 1, \mu \geq 0),$$

respectively (see [7]).

Theorem 4 Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, \mu)$. Then

$$|a_{m+1}| \leq \frac{B_1 \sqrt{2B_1}}{\sqrt{|B_1^2(\mu + 2m\lambda)(\mu + m) - 2B_2(\mu + m\lambda)^2| + 2B_1(\mu + m\lambda)^2}} \tag{14}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(\frac{m+1}{2} - \frac{(\mu+m\lambda)^2}{(\mu+2m\lambda)B_1}\right) \frac{2B_1^3}{|B_1^2(\mu+2m\lambda)(\mu+m) - 2B_2(\mu+m\lambda)^2| + 2B_1(\mu+m\lambda)^2} + \frac{B_1}{\mu+2m\lambda} & , \quad B_1 \geq \frac{2(\mu+m\lambda)^2}{(m+1)(\mu+2m\lambda)} \\ \frac{B_1}{\mu+2m\lambda} & , \quad B_1 < \frac{2(\mu+m\lambda)^2}{(m+1)(\mu+2m\lambda)} \end{cases} \tag{15}$$

where $\lambda \geq 1$, $\mu \geq 0$.

Proof Let $f \in \mathcal{S}_{\Sigma_m}^\varphi(\lambda, \mu)$. Then there are two analytic functions $u : U \rightarrow U$ and $v : U \rightarrow U$, with

$$u(0) = v(0) = 0,$$

satisfying the following conditions:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \varphi(u(z)) \quad \text{and} \quad (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = \varphi(v(w)). \tag{16}$$

Using the equalities (12) and (13) in (16) and by equating the coefficients in equation (16), we have

$$(\mu + m\lambda) a_{m+1} = B_1 b_m, \tag{17}$$

$$(\mu + 2m\lambda) \left(\frac{\mu - 1}{2} a_{m+1}^2 + a_{2m+1} \right) = B_1 b_{2m} + B_2 b_m^2, \tag{18}$$

and

$$-(\mu + m\lambda) a_{m+1} = B_1 c_m, \tag{19}$$

$$(\mu + 2m\lambda) \left[\left(m + \frac{\mu + 1}{2} \right) a_{m+1}^2 - a_{2m+1} \right] = B_1 c_{2m} + B_2 c_m^2. \tag{20}$$

From (17) and (19) we obtain

$$b_m = -c_m \tag{21}$$

and

$$(\mu + m\lambda)^2 a_{m+1}^2 = B_1^2 (b_m^2 + c_m^2). \tag{22}$$

Also, from (19), (20), and (22), we have

$$(\mu + 2m\lambda) (\mu + m) a_{m+1}^2 = B_1 (b_{2m} + c_{2m}) + \frac{2B_2 (\mu + m\lambda)^2}{B_1^2} a_{m+1}^2.$$

Therefore, we have

$$\left[(\mu + 2m\lambda) (\mu + m) B_1^2 - 2B_2 (\mu + m\lambda)^2 \right] a_{m+1}^2 = B_1^3 (b_{2m} + c_{2m}). \tag{23}$$

By using the inequalities in (11) for the coefficients b_{2m} and c_{2m} , we obtain

$$\left| (\mu + 2m\lambda) (\mu + m) B_1^2 - 2B_2 (\mu + m\lambda)^2 \right| |a_{m+1}|^2 \leq 2B_1^3 (1 - |b_m^2|), \tag{24}$$

and by using (17) in (24) we have

$$|a_{m+1}|^2 \leq \frac{2B_1^3}{\left| B_1^2 (\mu + 2m\lambda) (\mu + m) - 2B_2 (\mu + m\lambda)^2 \right| + 2B_1 (\mu + m\lambda)^2}, \tag{25}$$

which implies assertion (14). Similarly, in order to find the bound on $|a_{2m+1}|$, we subtract (20) from (18). Then we obtain

$$2(\mu + 2m\lambda) a_{2m+1} = (\mu + 2m\lambda)(m + 1)a_{m+1}^2 + B_1(b_{2m} - c_{2m}). \tag{26}$$

Thus, in view of (17), (21), and (26), and applying (11) for the coefficients b_{2m}, b_m and c_{2m}, c_m , we have

$$\begin{aligned} 2(\mu + 2m\lambda) |a_{2m+1}| &\leq (\mu + 2m\lambda)(m + 1) |a_{m+1}|^2 + 2B_1(1 - |b_m^2|), \\ |a_{2m+1}| &\leq \left(\frac{m + 1}{2} - \frac{(\mu + m\lambda)^2}{(\mu + 2m\lambda) B_1} \right) |a_{m+1}|^2 + \frac{B_1}{\mu + 2m\lambda}. \end{aligned} \tag{27}$$

Upon substituting the value of $|a_{m+1}|^2$ from inequality (25) and putting it in (27), we have the desired result. \square

3. Corollaries of the main theorem

For the case of m -fold symmetric functions, we have following:

Remark 5 For $\mu = 1$, Theorem 4 reduces to the corresponding results of Tang et al. (Theorem 7, p. 11086 in [19]), which we recall here as Corollary 6 below.

Corollary 6 (See [19]) Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\sigma}(\lambda, 1)$. Then

$$|a_{m+1}| \leq \frac{B_1 \sqrt{2B_1}}{\sqrt{|B_1^2(1 + 2m\lambda)(1 + m) - 2B_2(1 + m\lambda)^2| + 2B_1(1 + m\lambda)^2}} \tag{28}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(\frac{m+1}{2} - \frac{(1+m\lambda)^2}{(1+2m\lambda)B_1} \right) \frac{2B_1^3}{|B_1^2(1+2m\lambda)(1+m) - 2B_2(1+m\lambda)^2| + 2B_1(1+m\lambda)^2} + \frac{B_1}{1+2m\lambda}, & B_1 \geq \frac{2(1+m\lambda)^2}{(m+1)(1+2m\lambda)} \\ \frac{B_1}{1+2m\lambda}, & B_1 < \frac{2(1+m\lambda)^2}{(m+1)(1+2m\lambda)} \end{cases}, \tag{29}$$

where $0 \leq \lambda < 1$.

Remark 7 For $\lambda = 1$ and $\mu = 1$, Theorem 4 reduces to the corresponding results of Tang et al. (Theorem 1 in [19]), which we recall here as Corollary 8 below.

Corollary 8 (See [19]) Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\sigma}(1, 1)$. Then

$$|a_{m+1}| \leq \frac{B_1 \sqrt{2B_1}}{\sqrt{(1 + m) [|B_1^2(1 + 2m) - 2B_2(1 + m)| + 2B_1(1 + m)]}} \tag{30}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(1 - \frac{2(1+m)}{(1+2m)B_1} \right) \frac{2B_1^3}{(1+m) [|B_1^2(1+2m) - 2B_2(1+m)| + 2B_1(1+m)]} + \frac{B_1}{1+2m}, & B_1 \geq \frac{2(1+m)}{1+2m} \\ \frac{B_1}{1+2m}, & B_1 < \frac{2(1+m)}{1+2m} \end{cases}. \tag{31}$$

Remark 9 For $\lambda = 1$ and $\mu = 0$, Theorem 4 reduces to the following corollary:

Corollary 10 Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\varphi}(1, 0)$. Then

$$|a_{m+1}| \leq \frac{B_1 \sqrt{B_1}}{m \sqrt{|B_1^2 - B_2| + B_1}} \tag{32}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(\frac{m+1}{2} - \frac{m}{2B_1} \right) \frac{B_1^3}{m^2 [|B_1^2 - B_2| + B_1]} + \frac{B_1}{2m} & , \quad B_1 \geq \frac{m}{1+m} \\ \frac{B_1}{2m} & , \quad B_1 < \frac{m}{1+m} \end{cases} \tag{33}$$

For the case of one-fold symmetric functions, we have following:

Remark 11 Theorem 4 reduces to Corollary 12:

Corollary 12 Let f given by (5) be in the class $\mathcal{S}_{\Sigma_1}^{\varphi}(\lambda, \mu)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{2B_1}}{\sqrt{|B_1^2 (\mu + 2\lambda) (\mu + 1) - 2B_2 (\mu + \lambda)^2| + 2B_1 (\mu + \lambda)^2}} \tag{34}$$

and

$$|a_3| \leq \begin{cases} \left(1 - \frac{(\mu + \lambda)^2}{(\mu + 2\lambda) B_1} \right) \frac{2B_1^3}{|B_1^2 (\mu + 2\lambda) (\mu + 1) - 2B_2 (\mu + \lambda)^2| + 2B_1 (\mu + \lambda)^2} & , \quad B_1 \geq \frac{(\mu + \lambda)^2}{\mu + 2\lambda} \\ \frac{B_1}{\mu + 2\lambda} & , \quad B_1 < \frac{(\mu + \lambda)^2}{\mu + 2\lambda} \end{cases} \tag{35}$$

where $0 \leq \lambda < 1, \mu \geq 0$.

From among the many choices of λ, μ and the function φ that would provide the following corollaries:

i) In the case of $\varphi = \left(\frac{1+z}{1-z} \right)^{\beta}$, we have

$$|a_2| \leq \frac{2\beta}{\sqrt{|(\mu + 2\lambda) (\mu + 1)\beta - (\mu + \lambda)^2 \beta| + (\mu + \lambda)^2}} \tag{36}$$

and

$$|a_3| \leq \begin{cases} 2\beta \left[\left(1 - \frac{(\mu + \lambda)^2}{2\beta(\mu + 2\lambda)} \right) \frac{2\beta}{|(\mu + 2\lambda)(\mu + 1)\beta - (\mu + \lambda)^2 \beta| + (\mu + \lambda)^2} + \frac{1}{\mu + 2\lambda} \right] & , \quad \beta \geq \frac{(\mu + \lambda)^2}{2(\mu + 2\lambda)} \\ \frac{2\beta}{\mu + 2\lambda} & , \quad \beta < \frac{(\mu + \lambda)^2}{2(\mu + 2\lambda)} \end{cases} \tag{37}$$

and

ii) for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{\left|(\mu+2\lambda)(\mu+1)(1-\alpha) - (\mu+\lambda)^2\right| + (\mu+\lambda)^2}} \tag{38}$$

and

$$|a_3| \leq \begin{cases} 2(1-\alpha) \left[\left(1 - \frac{(\mu+\lambda)^2}{2(1-\alpha)(\mu+2\lambda)}\right) \frac{2(1-\alpha)}{\left|(\mu+2\lambda)(\mu+1)(1-\alpha) - (\mu+\lambda)^2\right| + (\mu+\lambda)^2} + \frac{1}{\mu+2\lambda} \right] & , \quad 1-\alpha \geq \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \\ \frac{2(1-\alpha)}{\mu+2\lambda} & , \quad 1-\alpha < \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \end{cases} \tag{39}$$

Remark 13 For $\mu = 1$, Theorem 4 reduces the corresponding results of Peng and Han [13], which we recall here as Corollary 14.

Corollary 14 [13] Let f given by (5) be in the class $S_{\Sigma_1}^\varphi(\lambda, 1)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2(1+2\lambda) - B_2(1+\lambda)^2\right| + B_1(1+\lambda)^2}} \tag{40}$$

and

$$|a_3| \leq \begin{cases} \left(1 - \frac{(1+\lambda)^2}{(1+2\lambda)B_1}\right) \frac{B_1^3}{\left|B_1^2(1+2\lambda) - B_2(1+\lambda)^2\right| + B_1(1+\lambda)^2} + \frac{B_1}{1+2\lambda} & , \quad B_1 \geq \frac{(1+\lambda)^2}{1+2\lambda} \\ \frac{B_1}{1+2\lambda} & , \quad B_1 < \frac{(1+\lambda)^2}{1+2\lambda} \end{cases} \tag{41}$$

From among the many choices of λ and the function φ that would provide the following corollaries:

For the class of strongly starlike functions, the function φ is given by $\varphi = \left(\frac{1+z}{1-z}\right)^\beta$. Theorem 4 reduces the corresponding corollary of Tang et al. [19] (see Corollary 27, p. 1087 in [19]), which we recall here as Corollary 15:

Corollary 15 [19] Let f given by (5) be in the class $S_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^\beta}(\lambda, 1)$. Then

$$|a_2| \leq \frac{2\beta}{\sqrt{(1+\lambda)^2 + |1+2\lambda - \lambda^2|}\beta}$$

and

$$|a_3| \leq \begin{cases} \frac{2\beta^2[2(1+2\lambda)+|1+2\lambda-\lambda^2|]}{(1+2\lambda)(|1+2\lambda-\lambda^2|\beta+(1+\lambda)^2)} & , \frac{(1+\lambda)^2}{2(1+2\lambda)} < \beta \leq 1 \\ \frac{2\beta}{1+2\lambda} & , 0 < \beta \leq \frac{(1+\lambda)^2}{2(1+2\lambda)} \end{cases} .$$

Remark 16 The estimates for $|a_2|$ and $|a_3|$ given by Corollary 15 are more accurate than those given by Theorem 2.2 of Frasin and Aouf [9].

ii) For the class of starlike functions of order α , the function φ is given by $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$. Theorem 4 reduces the corresponding corollary of Tang et al. [19] (see Corollary 28, p. 1087 in [19], which we recall here as Corollary 17:

Corollary 17 [19] Let f given by (5) be in the class $\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda, 1)$. Then

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{(1+\lambda)^2 + |2(1+2\lambda)(1-\alpha) - (1+\lambda)^2|}}$$

and

$$|a_3| \leq \begin{cases} 2(1-\alpha) \frac{|2(1+2\lambda)(1-\alpha) - (1+\lambda)^2| + 2(1-\alpha)(1+2\lambda)}{(1+2\lambda)[|2(1+2\lambda)(1-\alpha) - (1+\lambda)^2| + (1+\lambda)^2]} & , 0 \leq \alpha < \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \frac{2(1-\alpha)}{1+2\lambda} & , \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leq \alpha < 1 \end{cases} .$$

Remark 18 The estimates for $|a_2|$ and $|a_3|$ given by Corollary 17 are more accurate than those given by Theorem 2.2 of Frasin and Aouf [9].

Remark 19 For $\mu = 1 = \lambda$ Theorem 4 reduces the corresponding corollary of Peng and Han [13], which we recall here as Corollary 20:

Corollary 20 [13] Let f given by (5) be in the class $\mathcal{S}_{\Sigma_1}^{\varphi}(1, 1)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 - 4B_2| + 4B_1}}$$

and

$$|a_3| \leq \begin{cases} \left(1 - \frac{4}{3B_1}\right) \frac{B_1^3}{|3B_1^2 - 4B_2| + 4B_1} & , B_1 \geq \frac{4}{3} \\ \frac{B_1}{3} & , B_1 < \frac{4}{3} \end{cases} .$$

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