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The spectral expansion for the Hahn–Dirac system on the whole line

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Abstract: We consider the singular Hahn–Dirac system defined by

$$-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}y_2+p\left(x\right)y_1=\lambda y_1,$$

$$D_{\omega,q}y_1+r\left(x\right)y_2=\lambda y_2,$$

where λ is a complex spectral parameter and p and r are real-valued functions defined on $(-\infty,\infty)$ and continuous at ω_0. We prove the existence of a spectral function for such a system. We also prove the Parseval equality and the spectral expansion formula in terms of the spectral function for this system on the whole line.

Key words: Hahn–Dirac system, singular point, Parseval equality, spectral function, spectral expansion

1. Introduction

The theory of Hahn difference operators $D_{\omega,q}$ (see [13,14]), defined by

$$D_{\omega,q}f\left(x\right)=\begin{cases}\frac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x\neq\omega_0,\\ f'\left(\omega_0\right), & x=\omega_0\end{cases}$$

(where $q\in(0,1)$ and $\omega>0$), is undergoing rapid development since it provides a unifying structure for the study of the forward difference operator defined by

$$\Delta_\omega f\left(x\right):=\frac{f\left(\omega+x\right)-f\left(x\right)}{\left(\omega+x\right)-x},\ x\in\mathbb{R}$$

and the quantum q-difference operator [19] defined by

$$D_qf\left(x\right):=\frac{f\left(qx\right)-f\left(x\right)}{qx-x},\ x\neq 0.$$

Hahn difference operators are also receiving increased interest due to their applications in the construction of families of orthogonal polynomials and approximation problems (see, e.g., [7,10,21–22,25] and the references therein).

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In the literature there exist some papers studying Hahn difference equations. In [16], the authors studied the theory of linear Hahn difference equations. They also studied the existence and uniqueness of the solutions of initial value problems defined via Hahn difference equations. In 2016, Hamza and Makharesh [17] investigated Leibniz's rule and Fubini's theorem in association with the Hahn difference operator. Sitthiwirattham [26] investigated the nonlocal boundary value problem for nonlinear Hahn difference equations. Recently, in [9], the regular Hahn–Sturm–Liouville problem was studied. Annaby et al. [9] defined a Hilbert space of ω, q-square summable functions. They also discussed the formulation of the self-adjoint operator and investigated the properties of the eigenvalues and the eigenfunctions. Furthermore, they constructed Green's function and gave an eigenfunction expansion theorem. In [18], the author introduced the ω, q-analogy of the Dirac system. Hira also investigated the existence and uniqueness of the solutions of this problem and gave its spectral properties. In general, when we solve a partial differential equation by the method of separation of variables, we match the problem with expanding an arbitrary function as a series of eigenfunctions. Thus, spectral expansion theorems are essential for solving various problems in mathematics. The eigenfunction expansion is obtained by several methods, such as the methods of integral equations, contour integration, and finite difference (see [2–6,11–12,23,27]).

In this work, we obtain the Parseval equality and a spectral expansion formula in terms of the spectral function for a singular one-dimensional Hahn–Dirac system defined by

$$L_1 y := -\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} y_2 + p(x) y_1 = \lambda y_1, \tag{1}$$

$$L_2 y := D_{\omega,q} y_1 + r(x) y_2 = \lambda y_2, \tag{2}$$

where λ is a complex spectral parameter and p and r are real-valued functions defined on $\mathbb{R} := (-\infty, \infty)$ and continuous at ω_0.

2. Preliminaries

In this section, we provide some preliminary material related to Hahn calculus. For more details, the reader may refer to [8,9,13,14]. For our purposes, we shall assume that $q \in (0,1)$ and $\omega > 0$.

Let us define $\omega_0 := \omega / (1-q)$ and let I be a real interval containing ω_0.

Definition 1 ([13,14 *)]Let $f : I \to \mathbb{R}$ be a function. The Hahn difference operator is defined by*

$$D_{\omega,q} f(x) = \begin{cases} \frac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0, \end{cases}$$

provided that f is differentiable at ω_0. In this case, we call $D_{\omega,q} f$ the ω, q-derivative of f.

Remark 2 *The Hahn difference operator unifies two well-known operators. When $q \to 1$, we get the forward difference operator, which is defined by*

$$\Delta_\omega f(x) := \frac{f(\omega + x) - f(x)}{(\omega + x) - x}, \ x \in \mathbb{R}.$$

When $\omega \to 0$, we get the Jackson q-difference operator, which is defined by

$$D_q f(x) := \frac{f(qx) - f(x)}{(qx) - x}, \ x \neq 0.$$

Furthermore, under appropriate conditions, we have

$$\lim_{\substack{q\to 1\\ \omega\to 0}} D_{\omega,q} f(x) = f'(x).$$

Now we will present some properties of the ω, q-derivative.

Theorem 3 ([8]) *Let $f, g : I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$ and $h(x) := \omega + qx$. Then we have:*

i) $D_{\omega,q}(af + bg)(x) = aD_{\omega,q} f(x) + bD_{\omega,q} g(x), a, b \in I,$

ii) $D_{\omega,q}(fg)(x) = D_{\omega,q}(f(x)) g(x) + f(\omega + xq) D_{\omega,q} g(x),$

iii) $D_{\omega,q}\left(\frac{f}{g}\right)(x) = \frac{D_{\omega,q}(f(x))g(x) - f(x)D_{\omega,q}g(x)}{g(x)g(\omega+xq)},$

iv) $D_{\omega,q} f\left(h^{-1}(x)\right) = D_{-\omega q^{-1}, q^{-1}} f(x),\ h^{-1}(x) = q^{-1}(x - \omega)$

for all $x \in I$.

The ω, q-integral of function f can be defined as follows.

Definition 4 (Jackson-Nörlund integral [8]) *Let $f : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define the ω, q-integral of the function f from a to b by*

$$\int_a^b f(x)\, d_{\omega,q}(x) := \int_{\omega_0}^b f(x)\, d_{\omega,q}(x) - \int_{\omega_0}^a f(x)\, d_{\omega,q}(x),$$

where

$$\int_{\omega_0}^x f(t)\, d_{\omega,q}(t) := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + xq^n\right), x \in I,$$

provided that the series converges at $x = a$ and $x = b$. In this case, f is called ω, q-integrable on $[a, b]$.

Similarly, one can define the ω, q-integral of the function f over $\mathbb{R}$ by

$$\int_{-\infty}^{\infty} f(x)\, d_{\omega,q}(x) := \lim_{a\to -\infty} \int_a^{\omega_0} f(x)\, d_{\omega,q}(x) + \lim_{b\to \infty} \int_{\omega_0}^b f(x)\, d_{\omega,q}(x).$$

The following properties of ω, q-integration can be found in [8].

Theorem 5 ([8]) *Let $f, g : I \to \mathbb{R}$ be ω, q-integrable on I. Moreover, let $a, b, c \in I$, $a < c < b$, and $\delta, \theta \in \mathbb{R}$. Then the following formulae hold:*

i) $\int_a^b \{\delta f(x) + \theta g(x)\}\, d_{\omega,q}(x) = \delta \int_a^b f(x)\, d_{\omega,q}(x) + \theta \int_a^b g(x)\, d_{\omega,q}(x),$

ii) $\int_a^a f(x)\, d_{\omega,q}(x) = 0,$

iii) $\int_a^b f(x)\, d_{\omega,q}(x) = \int_a^c f(x)\, d_{\omega,q}(x) + \int_c^b f(x)\, d_{\omega,q}(x),$

iv) $\int_a^b f(x)\, d_{\omega,q}(x) = -\int_b^a f(x)\, d_{\omega,q}(x).$

Now we present the ω, q-integration by parts.

Lemma 6 ([8]) *Let $f, g : I \to \mathbb{R}$ be ω, q-integrable on I and let $a, b \in I$ with $a < b$. Then the following formula holds:*

$$\int_a^b f(x) D_{\omega,q} g(x) d_{\omega,q}(x) + \int_a^b g(\omega + qx) D_{\omega,q} f(x) d_{\omega,q}(x)$$

$$= f(b) g(b) - f(a) g(a).$$

The next result is the fundamental theorem of Hahn calculus.

Theorem 7 ([8]) *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t) d_{\omega,q}(t), \ x \in I.$$

Then F is continuous at ω_0. Moreover, $D_{\omega,q} F(x)$ exists for every $x \in I$ and $D_{\omega,q} F(x) = f(x)$. Conversely,

$$\int_a^b D_{\omega,q} F(x) d_{\omega,q}(x) = f(b) - f(a).$$

Let $L^2_{\omega,q}(\mathbb{R})$ be the space of all complex-valued functions defined on $\mathbb{R}$ such that

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)|^2 d_{\omega,q} x \right)^{1/2} < \infty.$$

The space $L^2_{\omega,q}(\mathbb{R})$ is a separable Hilbert space with the inner product

$$(f, g) := \int_{-\infty}^{\infty} f(x) \overline{g(x)} d_{\omega,q} x, \ f, g \in L^2_{\omega,q}(\mathbb{R})$$

(see [8]).

We introduce a convenient Hilbert space $\mathcal{H} = L^2_{\omega,q}(\mathbb{R}; E)$ $(E := \mathbb{C}^2)$ of vector-valued functions, by using the inner product

$$(f, g) := \int_{-\infty}^{\infty} (f(x), g(x))_E d_{\omega,q} x,$$

where $(.,.)_E$ denotes the standard inner product in $\mathbb{C}^2$:

$$(\xi, \gamma)_E = \sum_{j=1}^{2} \xi_j \overline{\gamma}_j.$$

Let

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \ z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}.$$

Then we define the Wronskian of $y(x)$ and $z(x)$ by the formula

$$W(y, z)(x) = y_1(x) z_2\left(h^{-1}(x)\right) - z_1(x) y_2\left(h^{-1}(x)\right) \ (x \in \mathbb{R}).$$

3. Main results

First we will prove that the regular Hahn–Dirac problem defined by (1)–(2) has a compact resolvent operator; thus, it has a purely discrete spectrum.

Let $[\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}]$ be an arbitrary finite interval, where $k \in \mathbb{N} := \{1, 2, 3, ...\}$. Consider the boundary value problem defined by (1)–(2), with the boundary conditions

$$\begin{array}{l} y_2 \left(\omega_0 - q^{-\kappa}\right) \cos \alpha + y_1 \left(\omega_0 - q^{-\kappa}\right) \sin \alpha = 0, \\ y_2 \left(\omega_0 + q^{-\kappa}\right) \cos \beta + y_1 \left(\omega_0 + q^{-\kappa}\right) \sin \beta = 0, \ \ \alpha, \beta \in \mathbb{R}, \ \ \kappa \in \mathbb{N}. \end{array} \tag{3}$$

We will denote by

$$\varphi_1 (x, \lambda) = \begin{pmatrix} \varphi_{11} (x, \lambda) \\ \varphi_{12} (x, \lambda) \end{pmatrix} \text{ and } \varphi_2 (x, \lambda) = \begin{pmatrix} \varphi_{21} (x, \lambda) \\ \varphi_{22} (x, \lambda) \end{pmatrix}$$

the solution of the system (1)–(2), which satisfies the initial conditions

$$\varphi_{11} (\omega_0, \lambda) = 1, \ \ \varphi_{12} (\omega_0, \lambda) = 0, \ \ \varphi_{21} (\omega_0, \lambda) = 0, \ \ \varphi_{22} (\omega_0, \lambda) = 1. \tag{4}$$

Let us define Green's matrix by the formula

$$G (x, t, \lambda) = \frac{1}{W (\varphi_1, \varphi_2)} \begin{cases} \varphi_2 (x, \lambda) \varphi_1^T (t, \lambda), & t \leq x, \\ \varphi_1 (x, \lambda) \varphi_2^T (t, \lambda), & x < t. \end{cases} \tag{5}$$

We will show that the function

$$y (x, \lambda) = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} G (x, t, \lambda) f (t) d_{\omega,q} t \tag{6}$$

is the solution of the nonhomogeneous system

$$-q^{-1} D_{-\omega q^{-1}, q^{-1}} y_2 + \{p (x) - \lambda\} y_1 = f_1 (x), \tag{7}$$

$$D_{\omega,q} y_1 + \{r (x) - \lambda\} y_2 = f_2 (x), \tag{8}$$

where

$$f (.) = \begin{pmatrix} f_1 (.) \\ f_2 (.) \end{pmatrix} \in L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E),$$

which satisfies the boundary conditions (3).

It follows from (6) that

$$\begin{aligned} & y_1 (x, \lambda) \\ = & \frac{1}{W (\varphi_1, \varphi_2)} \varphi_{21} (x, \lambda) \int_{\omega_0 - q^{-\kappa}}^{x} \begin{pmatrix} \varphi_{11} (t, \lambda) f_1 (t) \\ + \varphi_{12} (t, \lambda) f_2 (t) \end{pmatrix} d_{\omega,q} t \\ & + \frac{1}{W (\varphi_1, \varphi_2)} \varphi_{11} (x, \lambda) \int_{x}^{\omega_0 + q^{-\kappa}} \begin{pmatrix} \varphi_{21} (t, \lambda) f_1 (t) \\ + \varphi_{22} (t, \lambda) f_2 (t) \end{pmatrix} d_{\omega,q} t, \end{aligned} \tag{9}$$

$$\begin{aligned} & y_2(x,\lambda) \\ = & \frac{1}{W(\varphi_1,\varphi_2)}\varphi_{22}(x,\lambda)\int_{\omega_0-q^{-\kappa}}^{x}\begin{pmatrix}\varphi_{11}(t,\lambda)f_1(t)\\+\varphi_{12}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t \\ & +\frac{1}{W(\varphi_1,\varphi_2)}\varphi_{12}(x,\lambda)\int_{x}^{\omega_0+q^{-\kappa}}\begin{pmatrix}\varphi_{21}(t,\lambda)f_1(t)\\+\varphi_{22}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t. \end{aligned} \tag{10}$$

From (9), we have

$$\begin{aligned} & D_{\omega,q}y_1(x,\lambda) \\ = & \frac{1}{W(\varphi_1,\varphi_2)}D_{\omega,q}\varphi_{21}(x,\lambda)\int_{\omega_0-q^{-\kappa}}^{x}\begin{pmatrix}\varphi_{11}(t,\lambda)f_1(t)\\+\varphi_{12}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t \\ & +\frac{1}{W(\varphi_1,\varphi_2)}D_{\omega,q}\varphi_{11}(x,\lambda)\int_{x}^{\omega_0+q^{-\kappa}}\begin{pmatrix}\varphi_{21}(t,\lambda)f_1(t)\\+\varphi_{22}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t \\ & +\frac{1}{W(\varphi_1,\varphi_2)}W(\varphi_1,\varphi_2)f_2(x) \\ = & -\frac{1}{W(\varphi_1,\varphi_2)}\{r(x)-\lambda\}\varphi_{22}(x,\lambda)\int_{\omega_0-q^{-\kappa}}^{x}\begin{pmatrix}\varphi_{11}(t,\lambda)f_1(t)\\+\varphi_{12}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t \\ & -\frac{1}{W(\varphi_1,\varphi_2)}\{r(x)-\lambda\}\varphi_{12}(x,\lambda)\int_{x}^{\omega_0+q^{-\kappa}}\begin{pmatrix}\varphi_{21}(t,\lambda)f_1(t)\\+\varphi_{22}(t,\lambda)f_2(t)\end{pmatrix}d_{\omega,q}t \\ & +f_2(x)=-\{r(x)-\lambda\}y_2(x)+f_2(x). \end{aligned}$$

The validity of (7) is proved similarly. Hence, the function $y(x,\lambda)$ in (6) is the solution of the system (7)–(8). We check at once that (6) satisfies the boundary conditions (3).

Now we need the following.

Definition 8 *A matrix-valued function $M(x,t)$ of two variables with $\omega_0-q^{-\kappa}\le x,t\le\omega_0+q^{-\kappa}$ is called the ω,q-Hilbert–Schmidt kernel if*

$$\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\|M(x,t)\|_E^2\,d_{\omega,q}x\,d_{\omega,q}t<+\infty,$$

where the norm $\|.\|_E$ denotes the standard norm in E.

Theorem 9 ([24]) *If*

$$\sum_{i,j=1}^{\infty}|a_{ij}|^2<+\infty, \tag{11}$$

then the operator A, *defined by the formula*

$$A\{x_i\} = \{y_i\} \ \ (i \in \mathbb{N}),$$

where $\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}} \in l^2$ *and*

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j \ \ (i \in \mathbb{N}), \tag{12}$$

is compact in the sequence space l^2.

Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Then we have

$$G(x,t) = G(x,t,0) = \frac{1}{W(\varphi_1, \varphi_2)} \begin{cases} \varphi_2(x)\,\varphi_1^T(t), & t \le x, \\ \varphi_1(x)\,\varphi_2^T(t), & x < t. \end{cases} \tag{13}$$

Theorem 10 *The function* $G(x,t)$ *defined by (13) is a* ω, q*-Hilbert–Schmidt kernel.*

Proof By the upper half of formula (13), we have

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} d_{\omega,q}x \int_{\omega_0 - q^{-\kappa}}^{x} \|G(x,t)\|_E^2 \, d_{\omega,q}t < +\infty,$$

and by the lower half of (13), we have

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} d_{\omega,q}x \int_{x}^{\omega_0 + q^{-\kappa}} \|G(x,t)\|_E^2 \, d_{\omega,q}t < +\infty$$

since the inner integral exists and is a linear combination of the products $\varphi_{ij}(x)\,\varphi_{sl}(t)$ $(i,j,s,l = 1,2)$, and these products belong to $L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E)$ because each of the factors belongs to $L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E)$. Then we obtain

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \|G(x,t)\|_E^2 \, d_{\omega,q}x d_{\omega,q}t < +\infty. \tag{14}$$

□

Theorem 11 *The operator* K *defined by the formula*

$$(Kf)(x) = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} G(x,t)\, f(t)\, d_{\omega,q}t$$

is compact in $L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E)$.

Proof Let $\psi_i = \psi_i(t)$ $(i \in \mathbb{N})$ be a complete, orthonormal basis of $L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E)$. Since $G(x,t)$ is a ω, q-Hilbert–Schmidt kernel, we can define

$$x_i = (f, \psi_i) = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (f(t), \psi_i(t))_E d_{\omega,q}t,$$

$$y_i = (g, \psi_i) = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (g(t), \psi_i(t))_E d_{\omega,q}t,$$

$$a_{ij} = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (G(x,t)\psi_i(t), \psi_j(t))_E d_{\omega,q}x d_{\omega,q}t \ (i,j \in \mathbb{N}).$$

Then $L^2_{\omega,q}((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); E)$ is mapped isometrically into l^2. Consequently, our integral operator transforms to the operator defined by formula (12) in the space l^2 by this mapping, and condition (14) is translated into condition (11). It follows from Theorem 9 that this operator is compact. Therefore, the original operator is compact and it has a purely discrete spectrum. $\square$

Let $\mu_0, \mu_{\pm1}, \mu_{\pm2}, ...$ be the eigenvalues and $\Phi_0, \Phi_{\pm1}, \Phi_{\pm2}, ...$ be the corresponding eigenfunctions of the problem defined by (1)–(3), where

$$\Phi_n(x) = \begin{pmatrix} \Phi_{n1}(x) \\ \Phi_{n2}(x) \end{pmatrix} (n \in \mathbb{Z} := (0, \pm1, \pm2, ...).$$

Since the solutions of this problem are linearly independent, we get

$$\Phi_n(x) = \rho_n \varphi_1(x, \mu_n) + \sigma_n \varphi_2(x, \mu_n) \ \ (n \in \mathbb{Z}).$$

There is no loss of generality in assuming that $|\rho_n| \le 1$ and $|\sigma_n| \le 1$. Now let us set

$$\gamma_n^2 = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \|\Phi_n(x)\|_E^2 d_{\omega,q}x.$$

Let

$$f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in L^2_{\omega,q}\left((\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}); \mathbb{R}^2\right).$$

If we apply the Parseval equality to the vector-valued function $f(.)$, then we obtain

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \|f(x)\|_E^2 d_{\omega,q}x$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\gamma_n^2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \Phi_n\left(x\right)\right)_E d_{\omega,q}x \right\}^2$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\gamma_n^2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \rho_n \varphi_1\left(x, \mu_n\right) + \sigma_n \varphi_2\left(x, \mu_n\right)\right)_E d_{\omega,q}x \right\}^2$$

$$= \sum_{n=-\infty}^{\infty} \frac{\rho_n^2}{\gamma_n^2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \varphi_1\left(x, \mu_n\right)\right)_E d_{\omega,q}x \right\}^2 \tag{15}$$

$$+ 2 \sum_{n=-\infty}^{\infty} \frac{\rho_n \sigma_n}{\gamma_n^2} \prod_{j=1}^{2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \varphi_j\left(x, \mu_n\right)\right)_E d_{\omega,q}x \right\}$$

$$+ \sum_{n=-\infty}^{\infty} \frac{\sigma_n^2}{\gamma_n^2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \varphi_2\left(x, \mu_n\right)\right)_E d_{\omega,q}x \right\}^2 .$$

Now we will define the nondecreasing step function $\Omega_{ij,q^{-\kappa}}$ $(i, j = 1, 2)$ on $(\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa})$ by

$$\Omega_{11,q^{-\kappa}}\left(\mu\right) = \begin{cases} -\sum_{\mu < \mu_n < 0} \frac{\rho_n^2}{\gamma_n^2}, & \text{for } \mu \leq 0 \\ \sum_{0 \leq \mu_n < \mu} \frac{\rho_n^2}{\gamma_n^2} & \text{for } \mu > 0, \end{cases}$$

$$\Omega_{12,q^{-\kappa}}\left(\mu\right) = \begin{cases} -\sum_{\mu < \mu_n < 0} \frac{\rho_n \sigma_n}{\gamma_n^2}, & \text{for } \mu \leq 0 \\ \sum_{0 \leq \mu_n < \mu} \frac{\rho_n \sigma_n}{\gamma_n^2} & \text{for } \mu > 0, \end{cases}$$

$$\Omega_{12,q^{-\kappa}}\left(\mu\right) = \Omega_{21,q^{-\kappa}}\left(\mu\right),$$

$$\Omega_{22,q^{-\kappa}}\left(\mu\right) = \begin{cases} -\sum_{\mu < \mu_n < 0} \frac{\sigma_n^2}{\gamma_n^2}, & \text{for } \mu \leq 0 \\ \sum_{0 \leq \mu_n < \mu} \frac{\sigma_n^2}{\gamma_n^2} & \text{for } \mu > 0. \end{cases}$$

It follows from (15) that

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left\| f\left(x\right)\right\|_E^2 d_{\omega,q}x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i\left(\mu\right) F_j\left(\mu\right) d\Omega_{ij,q^{-\kappa}}\left(\mu\right), \tag{16}$$

where

$$F_i\left(\mu\right) = \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} \left(f\left(x\right), \varphi_i(x, \mu\right)_E d_{\omega,q}x \;\; (i = 1, 2).$$

Now we recall some definitions.

Definition 12 *A function f defined on an interval $[a,b]$ is said to be of bounded variation if there is a constant $C>0$ such that*

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\leq C$$

for every partition

$$a=x_0<x_1<...<x_n=b \tag{17}$$

of $[a,b]$ by points of subdivision $x_0,x_1,...,x_n$. see ([20]).

Definition 13 *Let f be a function of bounded variation. Then, by the total variation of f on $[a,b]$, denoted by $\overset{b}{\underset{a}{V}}(f)$, we mean the quantity*

$$\overset{b}{\underset{a}{V}}(f):=\sup\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (17) of the interval $[a,b]$ (see [20]).

Lemma 14 *There exists a positive constant $\Lambda=\Lambda(\xi)$, $\xi>0$, such that*

$$\overset{\xi}{\underset{-\xi}{V}}\left\{\Omega_{ij,q^{-\kappa}}(\mu)\right\}<\Lambda\ \ (i,j=1,2), \tag{18}$$

where Λ does not depend on $q^{-\kappa}$.

Proof By equality (4), we deduce that

$$\varphi_{ij}(\omega_0,\mu)=\delta_{ij}\ \ (i,j=1,2),$$

where δ_{ij} is the Kronecker delta. Thus, there exists a $k>0$ such that

$$|\varphi_{ij}(\omega_0,\mu)-\delta_{ij}|<\varepsilon,\ \varepsilon>0,\ |\mu|<\xi,\ x\in\left[\omega_0-q^{-\kappa},\omega_0+q^{-\kappa}\right]. \tag{19}$$

Let $f_k(x)=\begin{pmatrix} f_{k1}(x)\\ f_{k2}(x)\end{pmatrix}$ be a nonnegative vector-valued function such that $f_{k1}(x)$ vanishes outside the interval $[\omega_0-q^{-\kappa},\omega_0+q^{-\kappa}]$ with

$$\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}} f_{k1}(x)\,d_{\omega,q}x=1, \tag{20}$$

and $f_{k2}(x)=0$. Set

$$F_{ik}(\mu)=\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}(f_k(x),\varphi_i)_E\,d_{\omega,q}x$$

$$=\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}} f_{k1}(x)\,\varphi_{i1}(x,\mu)\,d_{\omega,q}x\ \ (i=1,2).$$

By using (19) and (20), we obtain

$$|F_{1k}(\mu) - 1| < \varepsilon,\ |F_{2k}(\mu)| < \varepsilon,\ |\mu| < \xi. \tag{21}$$

Applying the Parseval equality to $f_k(x)$, we obtain

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} f_{k1}^2(x)\, d_{\omega,q}x \geq \int_{-\xi}^{\xi} F_{1k}^2(\mu)\, d\Omega_{11,q^{-\kappa}}(\mu)$$

$$+ 2\int_{-\xi}^{\xi} F_{1k}(\mu) F_{2k}(\mu)\, d\Omega_{12,q^{-\kappa}}(\mu) + \int_{-\xi}^{\xi} F_{2k}^2(\mu)\, d\Omega_{22,q^{-\kappa}}(\mu)$$

$$\geq \int_{-\xi}^{\xi} F_{1k}^2(\mu)\, d\Omega_{11,q^{-\kappa}}(\mu) - 2\int_{-\xi}^{\xi} |F_{1k}(\mu)| |F_{2k}(\mu)| \left| d\Omega_{12,q^{-\kappa}}(\mu) \right|.$$

By virtue of (21), we get

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} f_{k1}^2(x)\, d_{\omega,q}x$$

$$> \int_{-\xi}^{\xi} (1-\varepsilon)^2\, d\Omega_{11,q^{-\kappa}}(\mu) - 2\int_{-\xi}^{\xi} \varepsilon(1+\varepsilon) \left| d\Omega_{12,q^{-\kappa}}(\mu) \right|$$

$$= (1-\varepsilon)^2 \left(\Omega_{11,q^{-\kappa}}(\xi) - \Omega_{11,q^{-\kappa}}(-\xi) \right)$$

$$- 2\varepsilon(1+\varepsilon) \overset{\xi}{\underset{-\xi}{V}} \left\{ \Omega_{12,q^{-\kappa}}(\mu) \right\}.$$

Since

$$\overset{\xi}{\underset{-\xi}{V}} \left\{ \Omega_{12,q^{-\kappa}}(\mu) \right\} \leq \frac{1}{2} \left[\Omega_{11,q^{-\kappa}}(\xi) - \Omega_{11,q^{-\kappa}}(-\xi) + \Omega_{22,q^{-\kappa}}(\xi) - \Omega_{22,q^{-\kappa}}(-\xi) \right], \tag{22}$$

we have

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} f_{k1}^2(x)\, d_{\omega,q}x > (1-3\varepsilon) \left\{ \Omega_{11,q^{-\kappa}}(\xi) - \Omega_{11,q^{-\kappa}}(-\xi) \right\}$$
$$- \varepsilon(1+\varepsilon) \left\{ \Omega_{22,q^{-\kappa}}(\xi) - \Omega_{22,q^{-\kappa}}(-\xi) \right\}. \tag{23}$$

Let

$$g_k(x) = \begin{pmatrix} g_{k1}(x) \\ g_{k2}(x) \end{pmatrix}$$

be a nonnegative vector-valued function such that $g_{k2}(x)$ vanishes outside the interval $[\omega_0 - q^{-\kappa}, \omega_0 + q^{-\kappa}]$ with

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} g_{k2}(x)\, d_{\omega,q}x = 1,$$

and $g_{k1}(x)=0$. Similar arguments apply to the function $g_k(x)$, and we obtain

$$\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}} g_{k2}^2(x)\,d_{\omega,q}x > (1-3\varepsilon)\left\{\Omega_{22,q^{-\kappa}}(\xi)-\Omega_{22,q^{-\kappa}}(-\xi)\right\} \\ -\varepsilon(1+\varepsilon)\left\{\Omega_{11,q^{-\kappa}}(\xi)-\Omega_{11,q^{-\kappa}}(-\xi)\right\}. \tag{24}$$

Adding inequalities (23) and (24), we conclude that

$$\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}} \left\{f_{k1}^2(x)+g_{k2}^2(x)\right\}d_{\omega,q}x \\ > \left(1-4\varepsilon-\varepsilon^2\right)\left\{\begin{array}{c}\Omega_{11,q^{-\kappa}}(\xi)-\Omega_{11,q^{-\kappa}}(-\xi)\\ +\Omega_{22,q^{-\kappa}}(\xi)-\Omega_{22,q^{-\kappa}}(-\xi)\end{array}\right\}.$$

If we choose $\varepsilon>0$ such that $1-4\varepsilon-\varepsilon^2>0$, then we obtain the assertion of the lemma for the functions $\Omega_{11,q^{-\kappa}}(-\xi)$ and $\Omega_{22,q^{-\kappa}}(-\xi)$, relying on their monotonicity. From (22), we obtain the assertion of the lemma for the function $\Omega_{12,q^{-\kappa}}(-\xi)$. □

Now, for the convenience of the reader, we recall the following.

Theorem 15 ([20]) *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real, nondecreasing functions on a finite interval $a\le\mu\le b$. Then there exists a subsequence $(w_{n_s})_{s\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{s\to\infty} w_{n_s}(\mu)=w(\mu),\ \ a\le\mu\le b.$$

Theorem 16 ([20]) *Assume that $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded sequence of nondecreasing functions on a finite interval $a\le\mu\le b$, and suppose that*

$$\lim_{n\to\infty} w_n(\mu)=w(\mu),\ \ a\le\mu\le b.$$

If f is any continuous function on $a\le\mu\le b$, then

$$\lim_{n\to\infty}\int_a^b f(\mu)\,dw_n(\mu)=\int_a^b f(\mu)\,dw(\mu).$$

Now let ϱ be any nondecreasing function on $-\infty<\mu<\infty$. Denote by $L_\varrho^2(\mathbb{R})$ the Hilbert space of all functions $f:\mathbb{R}\to\mathbb{R}$ measurable with respect to the Lebesgue–Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\mu)\,d\varrho(\mu)<\infty,$$

with the inner product

$$(f,g)_\varrho := \int_{-\infty}^{\infty} f(\mu)\,g(\mu)\,d\varrho(\mu).$$

The main results of this paper are the following three theorems.

Theorem 17 *Let*

$$f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2).$$

Then there exist monotonic functions $\Omega_{11}(\mu)$ and $\Omega_{22}(\mu)$, which are bounded over every finite interval, and a function $\Omega_{12}(\mu)$, which is of bounded variation over every finite interval with the property

$$\int_{-\infty}^{\infty} \|f(x)\|_E^2 \, d_{\omega,q}x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\mu) F_j(\mu) \, d\Omega_{ij}(\mu), \tag{25}$$

where

$$F_i(\mu) = \lim_{k\to\infty} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (f(x), \varphi_i(x,\mu))_E \, d_{\omega,q}x \ (i = 1, 2).$$

The matrix-valued function $\Omega = (\Omega_{ij})_{i,j=1}^2$ $(\Omega_{12} = \Omega_{21})$ is called *a spectral function* for the system (1)–(2).

Proof Assume that the real-valued function $f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}$ satisfies the following conditions:

1) $f_n(x)$ vanishes outside the interval $[\omega_0 - q^{-n}, \omega_0 + q^{-n}]$, where $q^{-n} < q^{-\kappa}$.

2) The functions $f_n(x)$ and $D_{\omega,q} f_n(x)$ are continuous at ω_0.

If we apply the Parseval equality to $f_n(x)$, then we get

$$\int_{\omega_0 - q^{-n}}^{\omega_0 + q^{-n}} \|f_n(x)\|_E^2 \, d_{\omega,q}x = \sum_{s=-\infty}^{\infty} \frac{1}{\gamma_k^2} \left\{ \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (f_n(x), \Phi_s(x))_E \, d_{\omega,q}x \right\}^2. \tag{26}$$

Via the ω, q-integration by parts, we see that

$$\int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (f_n(x), \Phi_s(x))_E \, d_{\omega,q}x$$

$$= \frac{1}{\mu_s} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} f_{1n}(x) (L_1\Phi_s)(x) \, d_{\omega,q}x$$

$$+ \frac{1}{\mu_s} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} f_{2n}(x) (L_2\Phi_s)(x) d_{\omega,q}x$$

$$= \frac{1}{\mu_s} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (L_1 f_n)(x) \, \Phi_{s1}(x) \, d_{\omega,q}x$$

$$+ \frac{1}{\mu_s} \int_{\omega_0 - q^{-\kappa}}^{\omega_0 + q^{-\kappa}} (L_2 f_n)(x) \, \Phi_{s2}(x) \, d_{\omega,q}x.$$

Then we have

$$\sum_{|\mu_s|\geq\xi}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left(f_n\left(x\right),\Phi_s\left(x\right)\right)_E d_{\omega,q}x\right\}^2$$

$$\leq\frac{1}{\xi^2}\sum_{|\mu_s|\geq\xi}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left\{\begin{array}{c}\left(L_1f_n\right)\left(x\right)\Phi_{s1}\left(x\right)\\+(L_2f_n)\Phi_{s2}\left(x\right)\end{array}\right\}d_{\omega,q}x\right\}^2$$

$$\leq\frac{1}{\xi^2}\sum_{s=-\infty}^{\infty}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left\{\begin{array}{c}\left(L_1f_n\right)\left(x\right)\Phi_{s1}\left(x\right)\\+\left(L_2f_n\right)\left(x\right)\Phi_{s2}\left(x\right)\end{array}\right\}d_{\omega,q}x\right\}^2$$

$$=\frac{1}{\xi^2}\int_{-q^{-n}}^{q^{-n}}\left\{\left(L_1f_n\right)^2\left(x\right)+\left(L_2f_n\right)^2\left(x\right)\right\}d_{\omega,q}x.$$

By virtue of (26), we obtain

$$\left|\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}}\left\|f_n\left(x\right)\right\|_E^2 d_{\omega,q}x-\sum_{-\xi\leq\mu_s\leq\xi}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left(f_n\left(x\right),\Phi_s\left(x\right)\right)_E d_{\omega,q}x\right\}^2\right|$$

$$\leq\frac{1}{\xi^2}\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}}\left[-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}f_{2n}\left(x\right)+p\left(x\right)f_{1n}\left(x\right)\right]^2 d_{\omega,q}x$$

$$+\frac{1}{\xi^2}\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}}\left[D_{\omega,q}f_{1n}\left(x\right)+r\left(x\right)f_{2n}\left(x\right)\right]^2 d_{\omega,q}x.$$

Furthermore, we have

$$\sum_{-\xi\leq\mu_s\leq\xi}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left(f_n\left(x\right),\Phi_s\left(x\right)\right)_E d_{\omega,q}x\right\}^2$$

$$=\sum_{-\xi\leq\mu_s\leq\xi}\frac{1}{\gamma_s^2}\left\{\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left(f_n\left(x\right),\rho_s\varphi_1\left(x,\mu_s\right)+\sigma_s\varphi_2\left(x,\mu_s\right)\right)_E d_{\omega,q}x\right\}^2$$

$$=\int_{-\xi}^{\xi}\sum_{i,j=1}^{2}F_{in}\left(\mu\right)F_{jn}\left(\mu\right)d\Omega_{ij,q^{-\kappa}}\left(\mu\right),$$

where

$$F_{in}\left(\mu\right)=\int_{\omega_0-q^{-\kappa}}^{\omega_0+q^{-\kappa}}\left(f_n\left(x\right),\varphi_i\left(x,\mu\right)\right)_E d_{\omega,q}x\ \ \left(i=1,2\right).$$

Therefore, we have

$$\left|\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \|f_n(x)\|_E^2 \, d_{\omega,q}x - \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_{in}(\mu) F_{jn}(\mu) \, d\Omega_{ij,q^{-\kappa}}(\mu)\right|$$

$$\leq \frac{1}{\xi^2} \int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \left[-\frac{1}{q} D_{-\omega q^{-1},q^{-1}} f_{2n}(x) + p(x) f_{1n}(x)\right]^2 d_{\omega,q}x \tag{27}$$

$$+\frac{1}{\xi^2} \int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \left[D_{\omega,q} f_{1n}(x) + r(x) f_{2n}(x)\right]^2 d_{\omega,q}x.$$

By Lemma 14 and Theorems 15 and 16, we can find a sequence $\{q^{-\kappa_i}\}$ such that the functions $\Omega_{ij,q^{-\kappa_i}}(\mu)$ converge to a monotone function $\Omega_{ij}(\mu)$ as $\kappa_i \to \infty$. Passing to the limit with respect to $\{q^{-\kappa_i}\}$ (as $\kappa_i \to \infty$) in (27), we get

$$\left|\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \|f_n(x)\|_E^2 \, d_{\omega,q}x - \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_{in}(\mu) F_{jn}(\mu) \, d\Omega_{ij}(\mu)\right|$$

$$\leq \frac{1}{\xi^2} \int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \left[-\frac{1}{q} D_{-\omega q^{-1},q^{-1}} f_{2n}(x) + p(x) f_{1n}(x)\right]^2 d_{\omega,q}x$$

$$+\frac{1}{\xi^2} \int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \left[D_{\omega,q} f_{1n}(x) + r(x) f_{2n}(x)\right]^2 d_{\omega,q}x.$$

As $\xi \to \infty,$ we get

$$\int_{\omega_0-q^{-n}}^{\omega_0+q^{-n}} \|f_n(x)\|_E^2 \, d_{\omega,q}x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{in}(\mu) F_{jn}(\mu) \, d\Omega_{ij}(\mu).$$

Now let $f(.) \in L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2)$. Let us choose functions $\{f_\eta(x)\}$ satisfying conditions 1 and 2 and such that

$$\lim_{\eta\to\infty} \int_{-\infty}^{\infty} \|f(x) - f_\eta(x)\|_E^2 \, d_{\omega,q}x = 0.$$

Let

$$F_{i\eta}(\mu) = \int_{-\infty}^{\infty} (f_\eta(x), \varphi_i(x,\mu))_E \, d_{\omega,q}x \ \ (i=1,2).$$

Then we have

$$\int_{-\infty}^{\infty} \|f_\eta(x)\|_E^2 \, d_{\omega,q}x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{i\eta}(\mu) F_{j\eta}(\mu) \, d\Omega_{ij}(\mu).$$

Since

$$\int_{-\infty}^{\infty} \|f_{\eta_1}(x) - f_{\eta_2}(x)\|_E^2 \, d_{\omega,q}x \to 0 \text{ as } \eta_1, \eta_2 \to \infty,$$

we get

$$\int_{-\infty}^{\infty}\sum_{i=1}^{2}\left(F_{i\eta_1}(\mu)F_{j\eta_1}(\mu)-F_{i\eta_2}(\mu)F_{j\eta_2}(\mu)\right)d\Omega_{ij}(\mu)$$

$$=\int_{-\infty}^{\infty}\left\|f_{\eta_1}(x)-f_{\eta_2}(x)\right\|_E^2 d_{\omega,q}x\to 0$$

as $\eta_1,\eta_2\to\infty$. Therefore, there exists a limit function $F_i\ \ (i=1,2)$, which satisfies

$$\int_{-\infty}^{\infty}\|f(x)\|_E^2 d_{\omega,q}x=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\mu)F_j(\mu)d\Omega_{ij}(\mu),$$

by the completeness of the space $L_{\Omega}^2(\mathbb{R})$.

Now we will show that the sequence $(K_{\eta i})\ \ (i=1,2)$ given by

$$K_{\eta i}(\mu)=\int_{\omega_0-q^{-\eta}}^{\omega_0+q^{-\eta}}(f(x),\varphi_i(x,\mu))_E d_{\omega,q}x$$

converges to $F_i\ \ (i=1,2)$ as $\eta\to\infty$, in the metric of the space $L_{\Omega}^2(\mathbb{R})$. Let g be another function in $L_{\omega,q}^2(\mathbb{R};\mathbb{R}^2)$. By similar arguments, $G_i\ \ (i=1,2)$ can be defined by g.

It is obvious that

$$\int_{-\infty}^{\infty}\|f(x)-g(x)\|_E^2 d_{\omega,q}x$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{(F_i(\mu)-G_i(\mu))(F_j(\mu)-G_j(\mu))\right\}d\Omega_{ij}(\mu).$$

Let

$$g(x)=\begin{cases} f(x), & x\in[\omega_0-q^{-\eta},\omega_0+q^{-\eta}] \\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{(F_i(\mu)-K_{\eta i}(\mu))(F_j(\mu)-K_{\eta j}(\mu))\right\}d\Omega_{ij}(\mu)$$

$$=\int_{-\infty}^{\omega_0-q^{-\eta}}\|f(x)\|_E^2 d_{\omega,q}x+\int_{\omega_0+q^{-\eta}}^{\infty}\|f(x)\|_E^2 d_{\omega,q}x\to 0\ \ (\eta\to\infty),$$

which proves that $(K_{\eta i})$ converges to $F_i\ \ (i=1,2)$ in $L_{\Omega}^2(\mathbb{R})$ as $\eta\to\infty$. $\square$

Theorem 18 *Suppose that*

$$f(.)=\begin{pmatrix} f_1(.) \\ f_2(.)\end{pmatrix},\ g(.)=\begin{pmatrix} g_1(.) \\ g_2(.)\end{pmatrix}\in L_{\omega,q}^2(\mathbb{R};\mathbb{R}^2),$$

and $F_i(\mu)$, $G_i(\mu)$ $(i=1,2)$ *are the Fourier transforms of* f *and* g, *respectively. Then we have*

$$\int_{-\infty}^{\infty}(f(x),g(x))_E\,d_{\omega,q}x=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\mu)\,G_j(\mu)\,d\Omega_{ij}(\mu),$$

which is called the generalized Parseval equality.

Proof It is clear that $F\mp G$ are the transforms of $f\mp g$. Therefore, we have

$$\int_{-\infty}^{\infty}\|f(x)+g(x)\|_E^2\,d_{\omega,q}x$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}(F_i(\mu)+G_i(\mu))(F_j(\mu)+G_j(\mu))\,d\Omega_{ij}(\mu)$$

and

$$\int_{-\infty}^{\infty}\|f(x)-g(x)\|_E^2\,d_{\omega,q}x$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}(F_i(\mu)-G_i(\mu))(F_j(\mu)-G_j(\mu))\,d\Omega_{ij}(\mu).$$

By these equalities, we get the desired result. □

Theorem 19 *Let*

$$f(.)=\begin{pmatrix} f_1(.)\\ f_2(.)\end{pmatrix}\in L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2).$$

Then the integrals

$$\int_{-\infty}^{\infty}F_i(\mu)\,\varphi_j(x,\mu)\,d\Omega_{ij}(\mu)\ \ (i,j=1,2)$$

converge in $L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2)$. *Consequently, we have*

$$f(x)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\mu)\,\varphi_j(x,\mu)\,d\Omega_{ij}(\mu),$$

which is called the spectral expansion formula.

Proof Take any function $f_m\in L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2)$ and any positive number m, and set

$$f_m(x)=\int_{-m}^{m}\sum_{i,j=1}^{2}F_i(\mu)\,\varphi_j(x,\mu)\,d\Omega_{ij}(\mu).$$

Let

$$g(.)=\begin{pmatrix} g_1(.)\\ g_2(.)\end{pmatrix}\in L^2_{\omega,q}(\mathbb{R};\mathbb{R}^2)$$

be a vector-valued function that is equal to zero outside the finite interval $[\omega_0 - q^{-\tau}, \omega_0 + q^{-\tau}]$, where $q^{-\tau} < q^{-\kappa}$. Thus, we obtain

$$\int_{\omega_0 - q^{-\tau}}^{\omega_0 + q^{-\tau}} (f_m(x), g(x))_E \, d_{\omega,q} x$$

$$= \int_{\omega_0 - q^{-\tau}}^{\omega_0 + q^{-\tau}} \left(\int_{-m}^{m} \sum_{i,j=1}^{2} F_i(\mu) \varphi_j(x,\mu) \, d\Omega_{ij}(\mu), g_1(x) \right)_E d_{\omega,q} x$$

$$= \int_{-m}^{m} \sum_{i,j=1}^{2} F_i(\mu) \left\{ \int_{\omega_0 - q^{-\tau}}^{\omega_0 + q^{-\tau}} (g(x), \varphi_j(x,\mu))_E \, d_{\omega,q} x \right\} d\Omega_{ij}(\mu)$$

$$= \int_{-m}^{m} \sum_{i,j=1}^{2} F_i(\mu) G_j(\mu) \, d\Omega_{ij}(\mu). \tag{28}$$

From Theorem 5, we get

$$\int_{-\infty}^{\infty} (f(x), g(x))_E \, d_{\omega,q} x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\mu) G_j(\mu) \, d\Omega_{ij}(\mu). \tag{29}$$

By (28) and (29), we get

$$(f - f_m, g)_{\mathcal{H}} = \int_{|\mu| > m} \sum_{i,j=1}^{2} F_i(\mu) G_j(\mu) \, d\Omega_{ij}(\mu).$$

If we apply this equality to the function

$$g(x) = \begin{cases} f(x) - f_m(x), & x \in [\omega_0 - q^{-m}, \omega_0 + q^{-m}] \\ 0, & \text{otherwise,} \end{cases}$$

then we get

$$\|f - f_m\|_{\mathcal{H}}^2 \leq \sum_{i,j=1}^{2} \int_{|\mu| > m} F_i(\mu) F_j(\mu) \, d\Omega_{ij}(\mu).$$

Letting $m \to \infty$ yields the expansion result. □

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