

1-1-2019

On null submanifolds of generalized Robertson-Walker space forms

FORTUNE MASSAMBA

Mohamed H. A. Hamed

SAMUEL SSEKAJJA

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)



Recommended Citation

MASSAMBA, FORTUNE; Hamed, Mohamed H. A.; and SSEKAJJA, SAMUEL (2019) "On null submanifolds of generalized Robertson-Walker space forms," *Turkish Journal of Mathematics*: Vol. 43: No. 3, Article 40. <https://doi.org/10.3906/mat-1812-20>

Available at: <https://journals.tubitak.gov.tr/math/vol43/iss3/40>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

On null submanifolds of generalized Robertson–Walker space forms

Mohamed H. A. HAMED , Fortuné MASSAMBA* , Samuel SSEKAJJA School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Scottsville,
South Africa

Received: 06.12.2018

Accepted/Published Online: 15.04.2019

Final Version: 29.05.2019

Abstract: We investigate totally umbilical r -null submanifolds of generalized Robertson–Walker space forms. Using generalized Newton transformations, we obtain new geometric configurations for the mean curvature functions which generalize many well-known results on null geometry.

Key words: Generalized Robertson–Walker space form, totally umbilical null submanifold

1. Introduction

In [5, 7] and [21], the authors initiated the study of null geometry of submanifolds in semi-Riemannian manifolds. The null submanifolds are interesting objects with some applications to mathematical physics and general relativity. In particular, they are known to represent various types of black hole horizons (see [5, 7] and other references cited therein for more details) in general relativity. Motivated by the above, many other researchers are actively exploring these submanifolds, for instance see [6, 8, 9, 11, 12, 16–20]. In all the studies cited above, the corresponding authors adopt an extrinsic approach to the study of null geometry introduced in the books of Duggal, Bejancu, and Sahin [5, 7]. On the other hand, Kupeli uses an intrinsic approach via a vector bundle (see [15] for more details). In [13] and [14], using the idea of Duggal–Bejancu [5, p. 107] and Duggal–Jin [10], the author studied null submanifolds of generalized Robertson–Walker (GRW) space time manifolds, and presented interesting partial differential equations concerning such submanifolds which are totally umbilical.

Applying the new objects given by Massamba and Ssekajja in [19], namely generalized Newton transformations [2, 3], we get new differential equations that generalize those obtained by Kanga in [13] and [14]. The rest of the paper is organized as follows. In Section 2, we introduce basic notions on null submanifolds used in this paper. In Section 3, we present the general concept of generalized Newton transformations for null submanifolds. In Section 4, we recall some notions on GRW space-time manifolds, and obtain new geometric configurations on totally umbilical null submanifolds of GRW space forms which generalize many well-known results for null hypersurfaces and r -null submanifolds of GRW space forms.

2. Null submanifolds

Let $(\overline{M}, \overline{g})$ be an a real $(m + n)$ -dimensional semi-Riemannian manifold of the constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$ and (M, g) an m -dimensional submanifold of \overline{M} . In case g is degenerate on

*Correspondence: massfort@yahoo.fr

2010 *AMS Mathematics Subject Classification:* Primary 53C25; Secondary 53C40, 53C50

the tangent bundle TM of M we say that M is null submanifold of \overline{M} [8]. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(\Xi)$ the $\mathcal{F}(M)$ module of smooth sections of a vector bundle Ξ (same notation for any other vector bundle) over M . For a degenerate tensor field g on M , there exists locally a vector field $E \in \Gamma(TM)$, $E \neq 0$, such that $g(E, X) = 0$, for any $X \in \Gamma(TM)$. Then, for each tangent space T_xM we have $T_xM^\perp = \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_xM\}$, which is a degenerate n -dimensional subspace of $T_x\overline{M}$. The radical (null) subspace of T_xM , denoted by $\text{Rad } T_xM$, is defined by $\text{Rad } T_xM = \{E_x \in T_xM : g(E_x, X) = 0, \forall X \in T_xM\}$. The dimension of $\text{Rad } T_xM = T_xM \cap T_xM^\perp$ depends on $x \in M$. The submanifold M of \overline{M} is said to be r -null submanifold if the mapping $\text{Rad } TM : x \in M \rightarrow \text{Rad } T_xM$ defines a smooth distribution on M of rank $r > 0$, where $\text{Rad } TM$ is called the *radical* (or *null*) distribution on M . Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $\text{Rad } TM$ in TM , and is given by

$$TM = \text{Rad } TM \perp S(TM). \tag{2.1}$$

Note that $S(TM)$ is not unique and canonically isomorphic to the factor vector bundle $TM/\text{Rad } TM$ [5]. Choose a screen transversal bundle $S(TM^\perp)$, which is semi-Riemannian complementary to $\text{Rad } TM$ in TM^\perp . Since, for any local basis $\{E_i\}$ of $\text{Rad } TM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $S(TM)^\perp$ such that $\overline{g}(E_i, N_j) = \delta_{ij}$, it follows that there exists a null transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [5]. Let $\text{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$. Then,

$$\text{tr}(TM) = ltr(TM) \perp S(TM^\perp), \tag{2.2}$$

$$\begin{aligned} T\overline{M} &= TM \oplus \text{tr}(TM) \\ &= S(TM) \perp S(TM^\perp) \perp \{\text{Rad } TM \oplus ltr(TM)\}. \end{aligned} \tag{2.3}$$

We say that a null submanifold M of \overline{M} is

- (i) r -null if $1 \leq r < \min\{m, n\}$,
- (ii) coisotropic if $1 \leq r = n < m$, $S(TM^\perp) = \{0\}$,
- (iii) isotropic if $1 \leq r = m < n$, $S(TM) = \{0\}$,
- (iv) totally null if $r = n = m$, $S(TM) = S(TM^\perp) = \{0\}$.

The details on the above classes of submanifolds with examples are found in [5]. Consider a local quasiorthonormal fields of frames of \overline{M} along M , on \mathcal{U} as

$$\{E_1, \dots, E_r, N_1, \dots, N_r, Z_{r+1}, \dots, Z_m, W_{1+r}, \dots, W_n\},$$

where $\{Z_{r+1}, \dots, Z_m\}$ and $\{W_{1+r}, \dots, W_n\}$ are respectively orthogonal bases of $\Gamma(S(TM)|_{\mathcal{U}})$ and $\Gamma(S(TM^\perp)|_{\mathcal{U}})$ and that $\epsilon_a = g(Z_a, Z_a)$ and $\epsilon_\alpha = \overline{g}(W_\alpha, W_\alpha)$ are the signatures of $\{Z_a\}$ and $\{W_\alpha\}$ respectively. The following range of indices will be used. $i, j, k \in \{1, \dots, r\}$; $\alpha, \beta, \mu \in \{r + 1, \dots, n\}$; $a, b, c \in \{r + 1, \dots, m\}$.

Let P be the projection morphism of TM onto $S(TM)$. Then,

$$X = PX + \sum_{i=1}^r \eta_i(X)E_i, \tag{2.4}$$

for any $X \in \Gamma(TM)$, where the 1-forms η_i are given by

$$\eta_i(X) = \bar{g}(X, N_i), \quad \forall X \in \Gamma(TM). \tag{2.5}$$

the Gauss-Weingarten equations [7] of an r -null submanifold M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^r h_i^l(X, Y)N_i + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)W_\alpha, \tag{2.6}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^r \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^n \rho_{i\alpha}(X)W_\alpha, \tag{2.7}$$

$$\bar{\nabla}_X W_\alpha = -A_{W_\alpha}X + \sum_{i=1}^r \varphi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^n \theta_{\alpha\beta}(X)W_\beta, \tag{2.8}$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^r h_i^*(X, PY)E_i, \tag{2.9}$$

$$\nabla_X E_i = -A_{E_i}^*X - \sum_{j=1}^r \tau_{ji}(X)E_j, \quad \forall X, Y \in \Gamma(TM), \tag{2.10}$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$ respectively, h_i^l and h_α^s are symmetric bilinear forms known as *local null* and *screen fundamental* forms of TM respectively. Also h_i^* are the *second fundamental forms* of $S(TM)$. A_{N_i} , $A_{E_i}^*$, and A_{W_α} are linear operators on TM while τ_{ij} , $\rho_{i\alpha}$, $\varphi_{\alpha i}$, and $\theta_{\alpha\beta}$ are 1-forms on TM . It is easy to see from (2.6) that

$$h_i^l(X, Y) = \bar{g}(\bar{\nabla}_X Y, E_i), \quad \forall X, Y \in \Gamma(TM), \tag{2.11}$$

from which we deduce the independence of h_i^l on the choice of $S(TM)$, and that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection and satisfies the relation

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^r \{h_i^l(X, Y)\eta_i(Z) + h_i^l(X, Z)\eta_i(Y)\}, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$.

The above three local second fundamental forms are related to their shape operators by the following set of equations

$$g(A_{E_i}^*X, Y) = h_i^l(X, Y) + \sum_{j=1}^r h_j^l(X, E_i)\lambda_j(Y), \quad \bar{g}(A_{E_i}^*X, N_j) = 0, \tag{2.13}$$

$$g(A_{W_\alpha}X, Y) = \epsilon_\alpha h_\alpha^s(X, Y) + \sum_{i=1}^r \nu_{\alpha i}(X)\lambda_i(Y), \tag{2.14}$$

$$\bar{g}(A_{W_\alpha}X, N_i) = \epsilon_\alpha \tau_{i\alpha}(X), \quad g(A_{N_i}X, Y) = h_i^*(X, PY), \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$.

Definition 2.1 A null submanifold (M, g) of a semi-Riemannian manifold (\bar{M}, \bar{g}) is said to be totally umbilical in \bar{M} if there is a smooth transversal vector field $H \in \Gamma(\text{tr}(TM))$ on M , called the transversal curvature vector field of M , such that, for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = H\bar{g}(X, Y). \tag{2.16}$$

Moreover, it is easy to see that M is totally umbilical, if and only if on each coordinate neighborhood \mathcal{U} there exist smooth vector fields $H^l \in \Gamma(\text{ltr}(TM))$ and $H^s \in \Gamma(S(TM^\perp))$, and smooth functions $H_i^l \in \mathcal{F}(\text{ltr}(TM))$ and $H_i^s \in \mathcal{F}(S(TM^\perp))$ such that, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} h^l(X, Y) &= H^l\bar{g}(X, Y), & h^s(X, Y) &= H^s\bar{g}(X, Y), \\ h_i^l(X, Y) &= H_i^l\bar{g}(X, Y), & h_\alpha^s(X, Y) &= H_\alpha^s\bar{g}(X, Y). \end{aligned} \tag{2.17}$$

It is well known in [5] that the above definition does not depend on the screen distribution and the transversal vector bundle of M .

Let $(M, g, S(TM), S(TM^\perp))$ be an m -dimensional r -null submanifold of a $(m + n)$ -dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) . Let \bar{R} and R denote the curvature tensors of $\bar{\nabla}$ and ∇ , respectively. The following identities are needed in this paper (see [5] for details on a complete set of equations):

$$\begin{aligned} \bar{R}(X, Y, PZ, PU) &= g(R(X, Y)PZ, PU) + \bar{g}(h^*(Y, PU), h^l(X, PZ)) \\ &\quad - \bar{g}(h^*(X, PU), h^l(Y, PZ)) + \bar{g}(h^s(Y, PU), h^s(X, PZ)) \\ &\quad - \bar{g}(h^s(X, PU), h^s(Y, PZ)), \end{aligned} \tag{2.18}$$

$$\begin{aligned} \bar{R}(X, Y, E, PU) &= g(R(X, Y)E, PU) + \bar{g}(h^*(Y, PU), h^l(X, E)) \\ &\quad - \bar{g}(h^*(X, PU), h^l(Y, E)) + \bar{g}(h^s(Y, PU), h^s(X, E)) \\ &\quad - \bar{g}(h^s(X, PU), h^s(Y, E)), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \bar{R}(X, Y, N, PU) &= -g(R(X, Y)PU, N) + \bar{g}(A_N Y, h^l(X, PU)) \\ &\quad - \bar{g}(A_N X, h^l(Y, PU)) + \bar{g}(h^s(Y, PU), D^s(X, N)) \\ &\quad - \bar{g}(h^s(X, PU), D^s(Y, N)), \end{aligned} \tag{2.20}$$

$$g(R(X, Y)E, PU) = g((\nabla_Y A^*)(E, X) - (\nabla_X A^*)(E, Y), PU). \tag{2.21}$$

For structures of case (ii), one needs to delete all the components involving $S(TM^\perp)$. Similarly, one can find the structure equations of the other two cases.

3. Generalized Newton transformations

Motivated by the fact that r -null submanifolds are endowed with a variety of shape operators, we apply the notion of generalized Newton transformations to a system of the above operators in this section. For extended details on generalized Newton transformations, see [2] and [3].

Let $(M, g, S(TM), S(TM^\perp))$ be an r -null submanifold of (\bar{M}, \bar{g}) . Notice that the operators $A_{E_1}^*, \dots, A_{E_r}^*$ are self-adjoint on $S(TM)$, and hence diagonalizable on $S(TM)$. Let $\mathbb{Z}^+(r)$ denote the set of all sequences

$u = (u_1, \dots, u_r)$, with $u_i \in \mathbb{Z}^+$, where \mathbb{Z}^+ is the set of positive integers. Then the length of u is denoted by $|u|$ and given by $|u| = \sum_{i=1}^r u_i$. Let us define an operator $\mathcal{A}^* \in \text{End}^r(M)$ by $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$, where $\text{End}^r(M)$ is the vector space $\text{End}(M) \times \dots \times \text{End}(M)$ (r -times). Furthermore, let $t = (t_1, \dots, t_r) \in \mathbb{R}^r$ and set $t^u = t_1^{u_1} \dots t_r^{u_r}$ and $t\mathcal{A}^* = \sum_{i=1}^r t_i A_{E_i}^*$. Then, the Newton polynomial of \mathcal{A}^* is denoted by $P_{\mathcal{A}^*}$ and defined by $P_{\mathcal{A}^*} : \mathbb{R}^r \rightarrow \mathbb{R}$, $P_{\mathcal{A}^*}(t) = \det(\mathbb{I} + t\mathcal{A}^*) = \sum_{|u| \leq p} \sigma_u^* t^u$, where the coefficients $\sigma_u^* = \sigma_u^*(\mathcal{A}^*)$ (the symmetric functions or mean curvatures) depend only on \mathcal{A}^* . We note that $\sigma_{(0, \dots, 0)}^* = 1$. We suppose further that $\sigma_u^* = 0$ for all $|u| > p$. Consider the functions $\varrho^\sharp : \mathbb{Z}^+(r) \rightarrow \mathbb{Z}^+(r)$ and $\varrho_b : \mathbb{Z}^+(r) \rightarrow \mathbb{Z}^+(r)$, given by $\varrho^\sharp(s_1, \dots, s_r) = (s_1, \dots, s_{i-1}, s_i + 1, s_{i+1}, \dots, s_r)$ and $\varrho_b(s_1, \dots, s_r) = (s_1, \dots, s_{i-1}, s_i - 1, s_{i+1}, \dots, s_r)$. We can see that ϱ^\sharp increases the value of the i -th element by 1 and ϱ_b decreases the value of i -th element by 1. It is also clear that ϱ^\sharp is the inverse map to ϱ_b .

The *generalized Newton transformation* [3] of $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$ is a system of endomorphisms $T_u^* = T_u^*(\mathcal{A}^*)$, $u \in \mathbb{Z}^+(r)$, satisfying the following condition. For every smooth curve $\gamma \mapsto \mathcal{A}^*(\gamma)$ in $\text{End}^r(M)$ such that $\mathcal{A}^*(0) = \mathcal{A}^*$, we have $\frac{d}{d\gamma} \sigma_u^*(\gamma)_{\gamma=0} = \sum_i \text{tr}(\frac{d}{d\gamma} A_{E_i}^*(\gamma)_{\gamma=0} \circ T_{i_b(u)}^*)$. For a fixed system \mathcal{A}^* of endomorphisms $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$, the object T_u^* is unique (see [2] and [3]). However, it is important to note that T_u^* depend on the choice of chosen screen distribution $S(TM)$. This is due to the fact that the object $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$ is dependent on $S(TM)$. In fact, let us consider two quasiorthonormal frames $\{E_i, N_i, Z_\alpha, W_\alpha\}$ and $\{E_i, N'_i, Z'_\alpha, W'_\alpha\}$ induced on \mathcal{U} by $\{S(TM), S(TM^\perp), F\}$ and $\{S'(TM), S'(TM^\perp), F'\}$, respectively. In this case, F and F' are the complementary vector bundles of $\text{Rad}TM$ in $S(TM^\perp)^\perp$ and $S'(TM^\perp)^\perp$, respectively. Setting $Y = E_i$ in (5.2.20) of [7, p. 208]) and using $h_i^l(E_i, X) = 0$, we have

$$\begin{aligned}
 A_{E_i}^* X &= A_{E_i}^{\prime*} X + \sum_{j=1}^r \left\{ \sum_{\alpha, \beta=r+1}^n \varepsilon_\beta h_\alpha^{\prime s}(X, E_i) W_\alpha^\beta Q_{j\beta} \right\} E_j \\
 &\quad - \sum_{j=1}^r \tau_{ji}(X) E_j + \sum_{j=1}^r \tau_{ji}'(X) E_j, \quad \forall X \in \Gamma(TM),
 \end{aligned}
 \tag{3.1}$$

where W_α^β and $Q_{j\beta}$ are smooth functions on \mathcal{U} . Notice from (3.1) that the operators $A_{E_i}^*$ depend on the chosen screen distribution, $S(TM)$, and so \mathcal{A}^* and T_u^* .

Let $T^* = (T_u^* : u \in \mathbb{Z}^+(r))$ be the generalized Newton transformation of \mathcal{A}^* . Then for every $u \in \mathbb{Z}^+(r)$ of length greater or equal to p we have $T_u^* = 0$ (Cayley-Hamilton Theorem). Moreover, T_u^* satisfy the following recurrence relation

$$\begin{aligned}
 T_0^* &= \mathbb{I}, & \text{where } 0 &= (0, \dots, 0), \\
 T_u^* &= \sigma_u^* \mathbb{I} - \sum_{i=1}^r A_{E_i}^* \circ T_{i_b(u)}^*, & \text{where } |u| &\geq 1,
 \end{aligned}
 \tag{3.2}$$

where \mathbb{I} denotes the identity on M . We also have [2]:

$$\text{tr}(T_u^*) = (m - r - |u|)\sigma_u^*, \quad \sum_{i=1}^r \text{tr}(A_{E_i}^* \circ T_{i_b(u)}^*) = |u|\sigma_u^*, \tag{3.3}$$

$$\text{and } \sum_{i,j=1}^r \text{tr}(A_{E_i}^* \circ A_{E_j}^* \circ T_{j_b i_b(u)}^*) = -|u|\sigma_u^* + \sum_{i=1}^r \text{tr}(A_{E_i}^*)\sigma_{i_b(u)}^*, \tag{3.4}$$

where trace is taken with respect to $S(TM)$. If $r = 1$, i.e. M is a null hypersurface or a half-null submanifold, then $u = (u_1, 0 \dots, 0)$ and thus $|u| = u_1$, from which $\sigma_u = S_{u_1}$ (the well-known symmetric polynomials of one shape operator). Furthermore, $\mathcal{A}^* = (A_E^*)$, where E is the only section spanning $\text{Rad}TM$. Thus, T_u^* in this case coincides with that already known for one shape operator (see [2]). The operator $\mathcal{A} = (A_{N_1}, \dots, A_{N_r})$. It is important to note that the screen local second fundamental forms h_i^* are not generally symmetric. This makes the operators A_{N_i} , for $i \in \{1, \dots, r\}$ nonsymmetric (not self-adjoint on $S(TM)$) with respect to g . However, when $S(TM)$ is integrable then it is well known, see Theorem 2.5 of [5, p. 161], that h_i^* are symmetric and all the operators A_{N_i} become symmetric (or self-adjoint) on $S(TM)$. Moreover, each 1-form $\text{tr}(\tau_{ij})$ induced by $S(TM)$ is closed, i.e. $d\text{tr}(\tau_{ij}) = 0$. Thus, each operator A_{N_i} is diagonalizable on $S(TM)$. For this case, let us consider $\mathcal{A} = (A_{N_1}, \dots, A_{N_r}) \in \text{End}^r(M)$. Then its corresponding symmetric function $\sigma_u = \sigma_u(\mathcal{A})$ and generalized Newton transformation T_u satisfy the following recurrence relation

$$T_0 = \mathbb{I}, \quad \text{where } 0 = (0, \dots, 0),$$

$$T_u = \sigma_u \mathbb{I} - \sum_{i=1}^r A_{N_i} \circ T_{i_b(u)}, \quad \text{where } |u| \geq 1. \tag{3.5}$$

The above objects also satisfy relations (3.3) and (3.4) in which $\{\sigma_u^*, T_u^*\}$ is replaced with $\{\sigma_u, T_u\}$, where \mathbb{I} denotes the identity on M .

Example 3.1 (Null cone of \mathbb{R}_1^{n+2}) Let \mathbb{R}_1^{n+2} be the space \mathbb{R}^{n+2} endowed with a semi-Euclidean metric

$$\bar{g}(x, y) = -x_0 y_0 + \sum_{a=0}^{n+1} x_a y_a, \quad (x = \sum_{A=0}^{n+1} x^A \partial x_A),$$

where $\partial x_A := \frac{\partial}{\partial x_A}$. Then, the null cone Λ_0^{n+1} is given by the equation $x_0^2 = \sum_{a=1}^{n+1} x_a^2$, $x_0 \neq 0$. It is well known (for example see the books [5, 7]) that Λ_0^{n+1} is a null hypersurface of \mathbb{R}_1^{n+2} , in which the radical distribution is spanned by a global vector field $E = \sum_{A=0}^{n+1} x_A \partial x_A$ on Λ_0^{n+1} . The transversal bundle is spanned by a global section N given by $N = \frac{1}{2x_0^2} \{-x_0 \partial x_0 + \sum_{a=1}^{n+1} x_a \partial x_a\}$. Moreover, E being the position vector field, one gets $\bar{\nabla}_X E = \nabla_X E = X$, for any $X \in \Gamma(TM)$. Consequently, $A_E^* X + \tau(X)E + X = 0$. Noticing that the operator A_E^* is screen-valved, we infer from the last relation that

$$A_E^* X = -PX, \quad \tau(X) = -\bar{g}(X, N) = -\lambda(X), \tag{3.6}$$

for any $X \in \Gamma(TM)$. Next, any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ is expressed as $X = \sum_{a=1}^{n+1} \tilde{X}_a \partial x_a$, where $\{\tilde{X}_1, \dots, \tilde{X}_{n+1}\}$ satisfy $\sum_{a=1}^{n+1} x_a \tilde{X}_a = 0$. From the the second relation (3.6) we can clearly see that $\tau(X) = 0$ for any

$X \in \Gamma(S(TM))$. Consider the null cone in this example. By straightforward calculation, one gets $\bar{g}(\nabla_E X, E) = -\sum_{a=1}^{n+1} x_a X_a = 0$, which implies that $\nabla_E X \in \Gamma(S(T\Lambda_0^{n+1}))$. Hence, $A_N E = 0$. Using Gauss–Codazzi equations, we calculate $C(X, Y) = \bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\frac{1}{2x_0^2}g(X, Y)$, for any $X, Y \in \Gamma(S(T\Lambda_0^{n+1}))$. Consequently,

$$A_N X = -\frac{1}{2x_0^2}PX, \quad \forall X, Y \in \Gamma(S(T\Lambda_0^{n+1})). \tag{3.7}$$

Considering (3.6) and (3.7), we deduce that $A_N X = \frac{1}{2x_0^2}A_E^* X$, for any $\forall X \in \Gamma(S(T\Lambda_0^{n+1}))$. Hence, Λ_0^{n+1} is screen global conformal null hypersurface \mathbb{R}_1^{n+2} , with a positive conformal factor $\psi = \frac{1}{2x_0^2}$ globally define on Λ_0^{n+1} . Also, by simple calculations the eigenvalues of A_N with respect to eigenvectors E, X_1, \dots, X_n , respectively, are $k_0 = 0, k_1 = k_2 = \dots = k_n = -\psi = -\frac{1}{2x_0^2}$. Hence, $\sigma_0 = 1$ and

$$\sigma_q = \sigma_q(A_N) = \sigma_q(k_0, k_1, k_2, \dots, k_n),$$

for $q = 1, 2, \dots, n$, which is the usual symmetric function of the single operator A_N . Then, it follows that $T_0 = \mathbb{I}$ and

$$T_q = \sigma_q(k_0, k_1, k_2, \dots, k_n)\mathbb{I} - T_{q-1} \circ A_N,$$

for $q = 1, 2, \dots, n$.

The operator $\hat{\mathcal{A}} = (A_{W_{r+1}}, \dots, A_{W_n})$. Observe from (2.8) that the operators A_{W_α} , for $\alpha \in \{r + 1, \dots, n\}$ are each self-adjoint on $S(TM)$ since h_α^s are symmetric, and thus diagonalizable on $S(TM)$. Let us consider an operator $\hat{\mathcal{A}} = (A_{W_{r+1}}, \dots, A_{W_n}) \in \text{End}^{n-r}(M)$ and let $\hat{\sigma}_u$, for $u \geq 1$, be its corresponding symmetric function. Furthermore, let \hat{T}_u denote its generalized Newton transformation. Then, $\hat{\sigma}$ and \hat{T}_u satisfy the following recurrence relation

$$\begin{aligned} \hat{T}_0 &= \mathbb{I}, \quad \text{where } 0 = (0, \dots, 0), \\ \hat{T}_u &= \hat{\sigma}_u \mathbb{I} - \sum_{\alpha=r+1}^n A_{W_\alpha} \circ \hat{T}_{\alpha, (u)}, \quad \text{where } |u| \geq 1, \end{aligned} \tag{3.8}$$

where \mathbb{I} denotes the identity on M . It is easy to show that the above objects also satisfy relations (3.3) and (3.4) in which $\{\sigma_u^*, T_u^*\}$ is replaced with $\{\hat{\sigma}_u, \hat{T}_u\}$ and all the sums taken within $(r + 1)$ to n . Let us consider a quasiorthonormal basis $\{X_1, \dots, X_m\}$ adapted to TM . Then, the divergence [5] of a $(1, p)$ -tensor \mathbf{T} is a $(1, p - 1)$ -tensor $(\text{div}^\nabla \mathbf{T})$ given by

$$(\text{div}^\nabla \mathbf{T})(\omega_1, \dots, \omega_{p-1}) = \sum_{e=1}^m (\nabla_{X_e} \mathbf{T})(X_e, \omega_1, \dots, \omega_{p-1}). \tag{3.9}$$

We will need the following results.

Proposition 3.2 ([19]) *Let $(M, g, S(TM), S(TM^\perp))$ be an m -dimensional r -null submanifold of a $(m+n)$ -dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) . Then,*

$$\begin{aligned} \bar{g}(\operatorname{div}^\nabla(T_u^*), X) &= -\sum_{i=1}^r E_i(\sigma_u^*)\eta_i(X) - \sum_{i=1}^r g((\nabla_{E_i} \sum_{j=1}^r A_{E_j}^* \circ T_{j_b(u)}^*)E_i, X) \\ &\quad - \sum_{j=1}^r g(\operatorname{div}^{\nabla^*}(T_{j_b(u)}^*), A_{E_j}^* X) + \sum_{i,j=1}^r \operatorname{tr}(A_{E_i}^* \circ A_{E_j}^* \circ T_{j_b(u)}^*)\eta_i(X) \\ &\quad + \sum_{j,a} \bar{R}(Z_a, X, E_j, T_{j_b(u)}^* Z_a) - \sum_{i,j=1}^r \operatorname{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\rho_{ij}(X) \\ &\quad + \sum_{j,\alpha} \varepsilon_\alpha \operatorname{tr}(A_{W_\alpha} \circ T_{j_b(u)}^*)\nu_{\alpha j}(X) + \sum_{j,a} \sum_{i=1}^r h_i^l(X, T_{j_b(u)}^* Z_a)\rho_{ij}(Z_a) \\ &\quad - \sum_{j,a} \sum_{\alpha=r+1}^n h_\alpha^s(X, T_{j_b(u)}^* Z_a)\nu_{\alpha j}(Z_a), \quad \forall X \in \Gamma(TM), \end{aligned}$$

where $\operatorname{div}^{\nabla^*}(\cdot)$ denotes the divergence operator on $S(TM)$.

4. Generalized Robertson–Walker space-times

In general relativity, a space time is a four-dimensional differentiable manifold equipped with a Lorentzian metric. One of the important cosmological models in general relativity is the family of Robertson–Walker space-time:

$$\mathbb{L}_1^4(c, f) := (I \times_f F, \bar{g}), \quad \bar{g} = -dt^2 + f^2(t)g_c.$$

Explicitly, $\mathbb{L}_1^4(c, f)$ is warped product with Lorentzian metric \bar{g} of an open interval I and three-dimensional Riemannian manifold (F, g_c) of constant curvature c with warping function $f > 0$, which defined on an open interval I in \mathbb{R}_1^1 . Recently, Chen and van der Veken [4] studied nondegenerate surfaces (i.e. spatial or Lorentzian) of a Robertson–Walker space-time from differential geometry view point. In [13], the author studied null (degenerate, lightlike) hypersurfaces of generalized Robertson–Walker space time (GRW), which also defined as a warped product $\mathbb{L}_1^{n+1}(c, f) = I \times_f F$, where F is an n -dimensional Riemannian manifold of constant curvature c . In [13], Chen and Wei provided a general study of submanifolds in the Riemannian warped product $I \times_f F$, $\bar{g} = dt^2 + f^2(t)g_c$. In the present paper, we study null submanifolds of a GRW space-time $\mathbb{L}_1^{n+1}(c, f)$. In particular, we investigate null submanifolds with curvature invariance and parallel second fundamental forms, totally umbilical null submanifolds, null sectional and Ricci curvatures, respectively.

In this section, we review some results of the connection and curvature of GRW space-time, which follow from general results on warped product [21]. Consider a GRW space time

$$\mathbb{L}_1^{n+1}(c, f) = (I \times_f F, \bar{g}), \quad \bar{g} = -dt^2 + f^2(t)g_c,$$

where f is a smooth positive function on I , and (F, g_c) is an n -dimensional Riemannian manifold of constant sectional curvature c . The standard choices for F are S^n , \mathbb{E}^n and \mathbb{H}^n , with curvature $1, 0, -1$, respectively. Let π and σ be the natural projections of $I \times F$ onto I and F , respectively. Let $\mathcal{L}(I)$ and $\mathcal{L}(F)$ be

the set of horizontal and vertical lifts of vector field on I and F to $I \times_f F$, respectively. Let $\partial_t \in \mathcal{L}(I)$ denote the horizontal lift vector field to $I \times_f F$ of the standard vector field $\frac{d}{dt}$ on I . By a spacelike slice of $\mathbb{L}_1^{n+1}(c, f) = (I \times_f F, \bar{g})$ we mean a hypersurface of $\mathbb{L}_1^{n+1}(c, f)$ given by a fibre $S(t_0) := \pi^{-1}(t_0)$ with metric $f^2(t_0)g_c$. For each vector X tangent to $\mathbb{L}_1^{n+1}(c, f)$, we put

$$X = \phi(X)\partial_t + \widehat{X}, \tag{4.1}$$

where $\phi(X) = -\bar{g}(X, \partial_t)$ and \widehat{X} is the vertical component of X .

The following Proposition was proved in [21].

Proposition 4.1 *For any vector fields X, Y, Z on $\mathbb{L}_1^{n+1}(c, f)$,*

$$\begin{aligned} \bar{R}(X, Z)Y &= \lambda\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + \mu\{\phi(X)\phi(Z)Y \\ &\quad - \phi(Y)\phi(Z)X + (\phi(X)\bar{g}(Y, Z) - \phi(Y)\bar{g}(X, Z))\partial_t\}, \end{aligned}$$

where $\lambda = \frac{f'^2+c}{f^2}$, $\mu = \frac{ff''-(f'^2+c)}{f^2}$.

Next, we generalize the results of [13] and [14] regarding totally umbilical null hypersurfaces and submanifolds of Generalized Robertson Walker space forms.

Theorem 4.2 *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical r -null submanifold of a GRW space $\mathbb{L}_1^{n+1}(c, f)$. Then, the generalized mean curvature functions $\sigma_u^* : u \geq 1$ of $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$ satisfy the following partial differential equations*

$$\begin{aligned} E_k(\sigma_u^*) - \sum_{j=1}^r \text{tr}(A_{E_k}^* \circ A_{E_j}^* \circ T_{j_b(u)}^*) + \sum_{i,j=1}^r \text{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\rho_{ij}(E_k) \\ + \sum_{j=1}^r \mu\phi(E_j)\phi(E_k)\text{tr}(T_{j_b(u)}^*) = 0, \\ PX(\sigma_u^*) + \sum_{i,j=1}^r \text{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\rho_{ij}(PX) + \sum_{j=1}^r \mu\phi(PX)\phi(E_i)\text{tr}(T_{j_b(u)}^*) = 0. \end{aligned}$$

Moreover, the curvature tensor of M satisfies the following relation

$$\begin{aligned} R(X, Y)Z &= \{\lambda X + \mathcal{H}_i^l A_{N_i} X + \sum_{\alpha=r+1}^n \mathcal{H}_\alpha^s A_{W_\alpha} X\}g(Y, Z) \\ &\quad - \{\lambda Y + \mathcal{H}_i^l A_{N_i} Y + \sum_{\alpha=r+1}^n \mathcal{H}_\alpha^s A_{W_\alpha} Y\}g(X, Z) \\ &\quad + \mu\{\phi(X)\phi(Z)Y - \phi(Y)\phi(Z)X - \{\phi(X)\bar{g}(Y, Z) \\ &\quad - \phi(Y)\bar{g}(X, Z)\}\partial_t^T\}, \end{aligned} \tag{4.2}$$

where ∂_t^T denotes the tangential projection of ∂_t with respect to the decomposition (2.3).

Proof Considering Propositions 3.2 and 4.1, we have

$$\begin{aligned}
 \bar{g}(\operatorname{div}^\nabla(T_u^*), X) &= -\sum_{i=1}^r E_i(\sigma_u^*)\eta_i(X) - \sum_{i=1}^r g((\nabla_{E_i} \sum_{j=1}^r A_{E_j}^* \circ T_{j_b(u)}^*)E_i, X) \\
 &- \sum_{j=1}^r g(\operatorname{div}^{\nabla^*}(T_{j_b(u)}^*), A_{E_j}^* X) + \sum_{i,j=1}^r \operatorname{tr}(A_{E_i}^* \circ A_{E_j}^* \circ T_{j_b(u)}^*)\eta_i(X) \\
 &- \mu \sum_{j=1}^r \phi(E_j)\phi(X)\operatorname{tr}(T_{j_b(u)}^*) - \mu \sum_{j,a} \phi(E_j)g(Z_a, X)\phi(T_{j_b(u)}^* Z_a) \\
 &- \sum_{i,j=1}^r \operatorname{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\rho_{ij}(X) + \sum_{j,\alpha} \varepsilon_\alpha \operatorname{tr}(A_{W_\alpha} \circ T_{j_b(u)}^*)\nu_{\alpha j}(X) \\
 &+ \sum_{i,j,a} h_i^l(X, T_{j_b(u)}^* Z_a)\rho_{ij}(Z_a) - \sum_{j,a,\alpha} h_\alpha^s(X, T_{j_b(u)}^* Z_a)\nu_{\alpha j}(Z_a), \tag{4.3}
 \end{aligned}$$

for any $X \in \Gamma(TM)$. As M is totally umbilical, we have $\nu_{\alpha j}(X) = \nu_{\alpha j}(Z_a) = 0$, $h_i^l(X, T_{j_b(u)}^* Z_a) = 0$ and $h_\alpha^s(X, T_{j_b(u)}^* Z_a) = 0$, for any $X \in \Gamma(\operatorname{Rad}TM)$. Setting $X = E_k$ in (4.3), then $\operatorname{div}^\nabla(T_u^*)$ belongs to TM^\perp , we obtain the first relation of the theorem. For second relation of the theorem, we put $X = PX$ in the last equation in the proof of proposition 4.2 of [19, p. 71], we have

$$PX(\sigma_u^*) = \sum_{j=1}^r \operatorname{tr}(T_{j_b(u)}^*(\nabla_{PX} A_{E_j}^*)) = \sum_{j,a} g((\nabla_{PX} A_{E_j}^*)Z_a, T_{j_b(u)}^* Z_a). \tag{4.4}$$

Using the definition of covariant derivative together with (2.4) and the fact that M is totally umbilical in $\mathbb{L}_1^{n+1}(c, f)$, we derive

$$(\nabla_{PX} A_{E_j}^*)Z_a = PX(\mathcal{H}_j^l)Z_a + \mathcal{H}_j^l \sum_{i=1}^r \eta_i(\nabla_{PX} Z_a)E_i. \tag{4.5}$$

Next, considering (4.4) and (4.5) we have

$$\begin{aligned}
 PX(\sigma_u^*) &= \sum_{j,a} g\left(PX(\mathcal{H}_j^l)Z_a + \mathcal{H}_j^l \sum_{i=1}^r \eta_i(\nabla_{PX} Z_a)E_i, T_{j_b(u)}^* Z_a\right) \\
 &= \sum_{j=1}^r \operatorname{tr}(PX(\mathcal{H}_j^l) \cdot T_{j_b(u)}^*). \tag{4.6}
 \end{aligned}$$

Replacing $PX(\mathcal{H}_j^l)$ in (4.6) using Theorem 5.3 of [14, p. 303], we derive

$$PX(\sigma_u^*) = -\sum_{i,j=1}^r \operatorname{tr}(A_{E_i} \circ T_{j_b(u)}^*)\rho_{ij}(PX) - \mu \sum_{j=1}^r \operatorname{tr}(\phi(PX)\phi(E_i)T_{j_b(u)}^*),$$

from which we get

$$PX(\sigma_u^*) + \sum_{i,j=1}^r \operatorname{tr}(A_{E_i} \circ T_{j_b(u)}^*)\rho_{ij}(PX) + \mu \sum_{j=1}^r \phi(PX)\phi(E_i)\operatorname{tr}(T_{j_b(u)}^*) = 0,$$

proving the second relation of the theorem. Finally, the last relation follows from simple calculations as in [14], which ends the proof. \square

Corollary 4.3 *From proposition (5.4) in [14, p. 304], if $\partial_t \in \text{Rad } TM \oplus \text{ltr}(TM)$, then the partial differential equations become*

$$E_k(\sigma_u^*) - \sum_{j=1}^r \text{tr}(A_{E_k}^* \circ A_{E_j}^* \circ T_{j_b}^*(u)) + \sum_{j=1}^r \mu \phi(E_j) \phi(E_k) \text{tr}(T_{j_b}^*(u)) = 0,$$

$$PX(\sigma_u^*) + \sum_{j=1}^r \mu \phi(PX) \phi(E_j) \text{tr}(T_{j_b}^*(u)) = 0.$$

Theorem 4.4 *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical r -null submanifold of a GRW space $\mathbb{L}_1^{n+1}(c, f)$. Then, the generalized mean curvature functions $\hat{\sigma}_u : u \geq 1$ of $\hat{\mathcal{A}} = (A_{W_{r+1}}, \dots, A_{W_n})$ satisfy the following partial differential equations*

$$E_j(\hat{\sigma}_u) - \sum_{\alpha=r+1}^n \text{tr}(A_{E_j}^* \circ A_{W_\alpha} \circ \hat{T}_{\alpha_b}(u)) + \sum_{\alpha,\beta=r+1}^n \text{tr}(A_{W_\alpha} \circ \hat{T}_{\alpha_b}(u)) \theta_{\alpha\beta}(E_j)$$

$$+ \sum_{\alpha=r+1}^n \mu \phi(W_\alpha) \phi(E_j) \text{tr}(\hat{T}_{\alpha_b}(u)) = 0, \tag{4.7}$$

$$PX(\hat{\sigma}_u) + \sum_{\alpha,\beta=r+1}^n \epsilon_\alpha \text{tr}(A_{W_\alpha} \circ \hat{T}_{\alpha_b}(u)) \theta_{\alpha\beta}(PX) + \sum_{i,\alpha}^n \epsilon_\alpha \text{tr}(A_{E_i}^* \circ \hat{T}_{\alpha_b}(u)) \tau_{i,\alpha}(PX)$$

$$+ \mu \sum_{\alpha=r+1}^n \epsilon_\alpha \phi(PX) \phi(W_\alpha) \text{tr}(\hat{T}_{\alpha_b}(u)) = 0. \tag{4.8}$$

Proof By the method of Theorem 4.2 while considering Propostion 4.1, we have

$$\bar{g}(\text{div}^\nabla(\hat{T}_u), X)$$

$$= - \sum_{i=1}^r E_i(\hat{\sigma}_u) \eta_i(X) - \sum_{i=1}^r g((\nabla_{E_i} \sum_{\alpha=r+1}^n A_{W_\alpha} \circ \hat{T}_{\alpha_b}(u)) E_i, X)$$

$$- \sum_{\alpha=r+1}^n g(\text{div}^{\nabla^*}(\hat{T}_{\alpha_b}(u)), A_{W_\alpha} X) + \sum_{i,\alpha} \text{tr}(A_{E_i}^* \circ A_{W_\alpha} \circ \hat{T}_{\alpha_b}(u)) \eta_i(X)$$

$$+ \sum_{\alpha,a} \bar{R}(Z_a, X, W_\alpha, \hat{T}_{\alpha_b}(u) Z_a) - \sum_{\alpha,\beta} \text{tr}(A_{W_\beta} \circ \hat{T}_{\alpha_b}(u)) \theta_{\alpha\beta}(X)$$

$$+ \sum_{\alpha,\alpha,\beta} g(A_{W_\beta} X, \hat{T}_{\alpha_b}(u) Z_a) \theta_{\alpha\beta}(Z_a) - \sum_{i,\alpha,a} h_i^*(X, \hat{T}_{\alpha_b}(u) Z_a) \nu_{\alpha i}(Z_a)$$

$$+ \sum_{i,\alpha} \text{tr}(A_{N_i} \circ \hat{T}_{\alpha_b}(u)) \nu_{\alpha i}(X), \tag{4.9}$$

$$\text{and } X(\hat{\sigma}_u) = \sum_{\alpha=r+1}^n \text{tr}(\hat{T}_{\alpha_b}(u) (\nabla_X P A_{W_\alpha})), \quad \forall X \in \Gamma(TM), \tag{4.10}$$

which completes the proof. □

From (4.9) and Proposition 4.1, we obtain

$$\begin{aligned}
 & \bar{g}(\operatorname{div}^\nabla(\widehat{T}_u), X) \\
 &= - \sum_{i=1}^r E_i(\widehat{\sigma}_u)\eta_i(X) - \sum_{i=1}^r g((\nabla_{E_i} \sum_{\alpha=r+1}^n A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})E_i, X) \\
 & - \sum_{\alpha=r+1}^n g(\operatorname{div}^{\nabla^*}(\widehat{T}_{\alpha_b(u)}), A_{W_\alpha}X) + \sum_{i,\alpha} \operatorname{tr}(A_{E_i}^* \circ A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})\eta_i(X) \\
 & - \sum_{\alpha=r+1}^n \mu\phi(W_\alpha)\phi(E_j)\operatorname{tr}(\widehat{T}_{\alpha_b(u)}) - \sum_{\alpha,\beta=r+1}^n \operatorname{tr}(A_{W_\beta} \circ \widehat{T}_{\alpha_b(u)})\theta_{\alpha\beta}(X) \\
 & + \sum_{a,\alpha,\beta} g(A_{W_\beta}X, \widehat{T}_{\alpha_b(u)}Z_a)\theta_{\alpha\beta}(Z_a) - \sum_{i,\alpha,a} h_i^*(X, \widehat{T}_{\alpha_b(u)}Z_a)\nu_{\alpha i}(Z_a) \\
 & + \sum_{i,\alpha} \operatorname{tr}(A_{N_i} \circ \widehat{T}_{\alpha_b(u)})\nu_{\alpha i}(X), \quad \forall X \in \Gamma(TM). \tag{4.11}
 \end{aligned}$$

Setting $X = E_j$ in (4.11) and using the Definition 2.1 of totally umbilicity of M and Theorem 4.4, we obtain the first relation (4.7). Using (4.10), we derive

$$PX(\widehat{\sigma}_u) = \sum_{\alpha=r+1}^n g\left((\nabla_{PX}PA_{W_\alpha})Z_a, \widehat{T}_{\alpha_b(u)}Z_a\right).$$

Using the definition of covariant derivative and the fact that M is totally umbilical in $\mathbb{L}_1^{n+1}(c, f)$, we derive

$$(\nabla_{PX}PA_{W_\alpha})Z_a = \nabla_{PX}PA_{W_\alpha}Z_a - PA_{W_\alpha}\nabla_{PX}Z_a = \epsilon_\alpha PX(\mathcal{H}_\alpha^s)Z_a. \tag{4.12}$$

Replacing $PX(\mathcal{H}_\alpha^s)$ in (4.12) using Theorem 5.3 of [14, p. 303], we derive

$$\begin{aligned}
 PX(\widehat{\sigma}_u) &= \sum_{\alpha,a} \epsilon_\alpha g(PX(\mathcal{H}_\alpha^s)Z_a, \widehat{T}_{\alpha_b(u)}Z_a) \\
 &= - \sum_{\alpha,\beta=r+1}^n \epsilon_\alpha \operatorname{tr}(A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})\theta_{\alpha\beta}(PX) - \sum_{i,\alpha} \epsilon_\alpha \operatorname{tr}(A_{E_i}^* \circ \widehat{T}_{\alpha_b(u)})\tau_{i,\alpha}(PX) \\
 & - \sum_{\alpha=r+1}^n \epsilon_\alpha \mu\phi(PX)\phi(W_\alpha)\operatorname{tr}(\widehat{T}_{\alpha_b(u)}),
 \end{aligned}$$

from which we get

$$\begin{aligned}
 & PX(\widehat{\sigma}_u) + \sum_{\alpha,\beta=r+1}^n \epsilon_\alpha \operatorname{tr}(A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})\theta_{\alpha\beta}(PX) + \sum_{i,\alpha} \epsilon_\alpha \operatorname{tr}(A_{E_i}^* \circ \widehat{T}_{\alpha_b(u)})\tau_{i,\alpha}(PX) \\
 & + \mu \sum_{\alpha=r+1}^n \epsilon_\alpha \phi(PX)\phi(W_\alpha)\operatorname{tr}(\widehat{T}_{\alpha_b(u)}) = 0.
 \end{aligned}$$

proving the second relation (4.8). Finally, the last relation follows from simple calculations as in [14], which ends the proof.

Definition 4.5 Let (M, g) be an r -null submanifold of $(\overline{M}, \overline{g})$. We say that M is a u -constant mean curvature submanifold if σ_u^* and $\widehat{\sigma}_u$ are constant functions on M . In particular, M will be called u -minimal if σ_u^* and $\widehat{\sigma}_u$ vanishes on M .

As an example, we have the following.

Lemma 4.6 Any totally geodesic null submanifold is u -minimal.

The proof of Lemma 4.6 follows from a straightforward calculation. For a nontrivial example, we have the following.

Example 4.7 Consider a submanifold M in \mathbb{R}_2^5 given by the equations (see [7])

$$x_4 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad x_3 = (1 - x_5^2)^{\frac{1}{2}}, \quad x_5, x_1, x_2 > 0.$$

Then we have

$$\begin{aligned} TM &= \text{Span} \left\{ E = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_4}, \quad U = x_4 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_4}, \right. \\ &\quad \left. V = -x_5 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_5} \right\}, \\ TM^\perp &= \text{Span} \left\{ E, W = x_3 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_5} \right\}. \end{aligned}$$

Thus, $\text{Rad } TM = \text{Span}\{E\}$ is a distribution on M and $S(TM^\perp) = \text{Span}\{W\}$. Hence, M is a half-null submanifold of \mathbb{R}_2^5 , with $S(TM) = \text{Span}\{U, V\}$. The null transversal bundle $\text{tr}(TM)$ is spanned by

$$N = \frac{1}{2x_2^2} \left\{ x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_4} \right\}.$$

By direct calculations, we get $\overline{\nabla}_U E = U$, $\overline{\nabla}_V E = 0$, $\overline{\nabla}_E E = E$, $\overline{\nabla}_U N = \frac{1}{2x_2^2} U$, $\overline{\nabla}_V N = 0$, $\overline{\nabla}_E N = -N$. Then, applying the Gauss–Weingarten formulae, we get $A_E^* U = -U$, $A_E^* V = 0$, $A_N U = -\frac{1}{2x_2^2} U$, $\rho(U) = 0$, $\tau(U) = 0$, $A_N V = 0$, $\rho(V) = 0$, $\tau(V) = 0$, $A_N E = 0$, $\rho(E) = -1$, $\tau(E) = 0$. Hence, from the above calculations, we infer that $A_N X = \frac{1}{2x_2^2} A_E^* X$, for all $X \in \Gamma(TM)$, which shows that M is a screen conformal half null submanifold. Now, by direct calculations, we obtain $\overline{\nabla}_U V = \overline{\nabla}_V U = 0$, $\overline{\nabla}_U U = \frac{1}{2} E + x_2^2 N$, $\overline{\nabla}_V V = -W$, $\overline{\nabla}_E E = E$, $\overline{\nabla}_U W = 0$, $\overline{\nabla}_V W = V$, $\overline{\nabla}_E W = 0$. Then, by the Gauss–Weingarten formulae, we get

$$\begin{aligned} \nabla_U U &= \frac{1}{2} E, \quad E(U, U) = \frac{1}{2}, \quad A_W U = 0, \quad A_W V = -V, \quad A_W E = 0, \\ h_1^l(U, U) &= x_2^2, \quad h_1^l(V, V) = 0, \quad h_2^s(U, U) = 0, \quad h_2^s(V, V) = -1, \\ h_2^s(X, E) &= 0, \quad \nu(X) = 0, \quad \forall X \in \Gamma(TM). \end{aligned}$$

Let us denote by k_0^*, k_1^*, k_2^* and $\widehat{k}_0, \widehat{k}_1, \widehat{k}_2$ the eigenvalues of A_E^* and A_W with respect to $\{E, U, V\}$, respectively. In view of the previous calculations, $k_0^* = 0, k_1^* = -1, k_2^* = 0$ and $\widehat{k}_0 = 0, \widehat{k}_1 = 0, \widehat{k}_2 = -1$. The corresponding

symmetric functions $\sigma_u^*(A_E^*)$ and $\widehat{\sigma}_u(A_W)$ are actually the symmetric functions for single shape operators, given by $\sigma_u^*(A_E^*) = \sigma_l^*(0, -1, 0)$ and $\widehat{\sigma}_u(A_W) = \widehat{\sigma}_l(0, 0, -1)$, for all $l = 0, 1, 2$. Clearly, the two functions above are constant along M . Thus, M is u -constant mean curvature half null submanifold of \mathbb{R}_2^5 .

Remark 4.8 ([14]) Let M be a null submanifold of $\mathbb{L}_1^{n+1}(c, f)$. Then

1. ∂_t cannot be tangent to M ;
2. ∂_t cannot be orthogonal to M .

Next, we apply Theorems 4.2 and 4.4 to deduce the following result.

Corollary 4.9 Under the assumptions of Theorems 4.2 and 4.4, If h^l is parallel and ∂_t belongs to $tr(TM) \oplus \text{Rad}(TM)$, then M is a u -constant mean curvature null submanifold if $A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)}$ is trace-free on $S(TM)$.

Proof From Theorem (5.7) in [14, p. 305], if h^l is parallel then, we get $H_i^l = 0$. This implies that $A_{E_i}^* = 0$ on $S(TM)$. Thus, $\sigma^* = 0$. On the other hand, the partial differential equations of Theorem 4.4 reduces to

$$E_j(\widehat{\sigma}_u) + \sum_{\alpha, \beta=r+1}^n \text{tr}(A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})\theta_{\alpha\beta}(E_j) = 0,$$

and

$$PX(\widehat{\sigma}_u) + \sum_{\alpha, \beta=r+1}^n \epsilon_\alpha \text{tr}(A_{W_\alpha} \circ \widehat{T}_{\alpha_b(u)})\theta_{\alpha\beta}(PX) = 0.$$

which completes the proof. □

In Theorem 5.1 of [13, p. 870] the author showed that for a totally umbilical null hypersurface of $\mathbb{L}_1^{n+1}(c, f)$, the function ρ such that $B(X, Y) = \rho g(X, Y)$, where $X, Y \in \Gamma(TM)$, satisfies the following differential equations

$$E(\rho) - \rho^2 + \rho\tau(E) + \mu\phi(E)^2 = 0, \tag{4.13}$$

$$PX(\rho) + \rho\tau(PX) + \mu\phi(E)\phi(PX) = 0. \tag{4.14}$$

Moreover, it can easily be shown that equations (4.13) and (4.14) hold for a half null submanifold of $\mathbb{L}_1^{n+1}(c, f)$. In the case of a totally umbilical r -null submanifold with $r > 1$, Kang [14, p. 303] showed, in Theorem 5.3 therein, that the functions \mathcal{H}_i^l and \mathcal{H}_α^s in (2.17) satisfy the differential equations

$$E_j(\mathcal{H}_i^l) - \mathcal{H}_i^l \mathcal{H}_j^l + \sum_{k=1}^r \mathcal{H}_k^l \tau_{ki}(E_j) + \mu\phi(E_i)\phi(E_j) = 0, \tag{4.15}$$

$$E_j(\mathcal{H}_\alpha^s) - \mathcal{H}_\alpha^s \mathcal{H}_j^l + \sum_{\beta=r+1}^n \mathcal{H}_\beta^s \theta_{\beta\alpha}(E_j) + \mu\phi(W_\alpha)\phi(E_j) = 0, \tag{4.16}$$

and

$$PX(\mathcal{H}_i^l) + \sum_{k=1}^r \mathcal{H}_k^l \rho_{ki}(PX) + \mu\phi(E_i)\phi(PX) = 0, \tag{4.17}$$

$$PX(\mathcal{H}_\alpha^s) + \sum_{\alpha=r+1}^n \mathcal{H}_\beta^s \theta_{\beta\alpha}(PX) + \sum_{i=1}^r \mathcal{H}_i^l \tau_{i\alpha}(PX) + \mu\phi(W_\alpha)\phi(PX) = 0. \tag{4.18}$$

Hence, we can say that Theorems 4.2 and 4.4 are generalizations of differential equations (4.13)–(4.18) above.

Let $(M, g, S(TM), S(TM^\perp))$ be an r -null submanifold of a semi-Riemannian manifold (\bar{M}, \bar{g}) , the screen distribution $S(TM)$ is said to be totally umbilical in M [5] if there is a smooth vector field \mathbf{K} of $\text{Rad}TM$ on M , such that $h^*(X, PY) = g(X, PY)\mathbf{K}$, for any $X, Y \in \Gamma(TM)$. Moreover, $S(TM)$ is totally umbilical, if and only if, on any coordinate neighborhood $\mathcal{U} \subset M$, there exist smooth functions \mathcal{K}_i such that $h_i^*(X, PY) = \mathcal{K}_i g(X, PY)$, for any $X, Y \in \Gamma(TM)$. It is also easy to see that for an umbilical $S(TM)$ one gets $P(A_{N_i}X) = \mathcal{K}_i PX$, $h^*(E, PX) = 0$, $\forall X \in \Gamma(TM)$, where $E \in \Gamma(\text{Rad}TM)$.

Theorem 4.10 *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical r -null submanifold of a GRW space $\mathbb{L}_1^{n+1}(c, f)$. Then, the generalized mean curvature functions $\sigma_u : u \geq 1$ of $\mathcal{A} = (A_{N_1}, \dots, A_{N_r})$ satisfy the following partial differential equations*

$$E_k(\sigma_u) - \sum_{j=1}^r \text{tr}(A_{E_k}^* \circ A_{N_j} \circ T_{j_b(u)}) - \sum_{i,j=1}^r \text{tr}(A_{N_i} \circ T_{j_b(u)}) \tau_{ij}(E_k) - \lambda \sum_{j=1}^r \text{tr}(T_{j_b(u)}) + \mu \sum_{j=1}^r \phi(N_j)\phi(E_k)\text{tr}(T_{j_b(u)}) = 0,$$

Proof By the method of theorem 4.2 and theorem 4.4 with recurrence (3.5) and (2.20) we obtain

$$\begin{aligned} & \bar{g}(\text{div}^\nabla(T_u), X) \\ &= - \sum_{i=1}^r E_i(\sigma_u)\eta_i(X) - \sum_{i=1}^r g((\nabla_{E_i} \sum_{j=1}^r A_{N_j} \circ T_{\alpha_b(u)})E_i, X) \\ & - \sum_{j=1}^r g(\text{div}^{\nabla^*}(T_{\alpha_b(u)}, A_{N_j}X) + \sum_{i,j} \text{tr}(A_{E_i}^* \circ A_{N_j} \circ T_{\alpha_b(u)})\tau_{ij}(X) \\ & - \sum_{j,\alpha} \mu\phi(N_j)\phi(E_i)\text{tr}(T_{j_b(u)}) - \sum_{i,j} \text{tr}(A_{N_i} \circ T_{j_b(u)})\rho_{ji}(X) \\ & + \sum_{a,j,i} g(A_{N_i}X, T_{j_b(u)}Z_a)\theta_{ji}(Z_a) - \sum_{i,j,a} h_i^*(X, T_{j_b(u)}Z_a)\nu_{ji}(Z_a) \\ & - \sum_{j,a} \bar{g}(h^s(X, T_{j_b(u)}Z_a, D^s(Z_a, N_j)), \quad \forall X \in \Gamma(TM). \end{aligned} \tag{4.19}$$

Now, replacing X by E_k in above equation and using the fact of M and $S(TM)$ are totally umbilical we get the result, which completes the proof. □

By virtue of Theorem 4.10, we deduce the following result.

Corollary 4.11 *Under the hypotheses of Theorem 4.10, ∇ on M is metric connection if and only if the mean curvature functions $\sigma_u; u \geq 1$ are a solution of the following partial differential equations*

$$E_k(\sigma_u) - \sum_{i,j=1}^r \text{tr}(A_{N_i} \circ T_{j_b(u)})\tau_{ij}(E_k) - \lambda \sum_{j=1}^r \text{tr}(T_{j_b(u)}) + \mu \sum_{j=1}^r \phi(N_j)\phi(E_k)\text{tr}(T_{j_b(u)}) = 0.$$

Let $x \in M$ and E be a null vector of $T_x\bar{M}$. A plane Π of $T_x\bar{M}$ is called a null plane directed by E if it contains E , $\bar{g}(E, W) = 0$ for any $W \in \Pi$ and there exists $W_0 \in \Pi$ such that $\bar{g}(W_0, W_0) \neq 0$. Then, from [5, p. 95], we define the null sectional curvature of Π with respect to E and $\bar{\nabla}$ as the real number

$$\bar{K}_E(\Pi) = \frac{\bar{R}(W, E, E, W)}{g(W, W)}, \tag{4.20}$$

where W is an arbitrary nonnull vector in Π . Similarly, we define the null sectional curvature $K_E(\Pi)$ of the null plane Π of the tangent space T_xM with respect to E and ∇ as a real number

$$K_E(\Pi) = \frac{R(W, E, E, W)}{g(W, W)}. \tag{4.21}$$

Using the fact that both the null sectional curvatures in (4.20) and (4.21) are independent of $W \in \Pi$, we derive from (2.19) and (2.21) that

$$\begin{aligned} \bar{K}_E(\Pi_j) &= E_j(\sigma_u^*) - \sum_{j=1}^r \text{tr}(A_{E_j}^{*2} \circ T_{j_b(u)}^*) + \sum_{i,j=1}^r \text{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\tau_{ij}(E_j) \\ &+ \sum_{j=1}^r \mu\phi(E_j)^2\text{tr}(T_{j_b(u)}^*) = K_E(\Pi_j). \end{aligned} \tag{4.22}$$

Therefore, we have:

Theorem 4.12 *Let $(M, g, S(TM), S(TM^\perp))$ be either an r -null or a co-isotropic submanifold of GRW space $\mathbb{L}_1^{n+1}(c, f)$. Then, both the null sectional curvature $K_E(\Pi_j)$ and $\bar{K}_E(\Pi_j)$ vanish, if and only if, $\sigma_u^* : u \geq 1$ of $\mathcal{A}^* = (A_{E_1}^*, \dots, A_{E_r}^*)$ is a solution of the partial differential equations*

$$E_j(\sigma_u^*) - \sum_{j=1}^r \text{tr}(A_{E_j}^{*2} \circ T_{j_b(u)}^*) + \sum_{i,j=1}^r \text{tr}(A_{E_i}^* \circ T_{j_b(u)}^*)\tau_{ij}(E_j) + \sum_{j=1}^r \mu\phi(E_j)^2\text{tr}(T_{j_b(u)}^*) = 0.$$

Proof The proof follows immediately from (4.22). □

From Theorem 4.2 and equation (4.22), we deduce the following:

Theorem 4.13 *Let (M, g) be a totally umbilical null submanifold of GRW space $\mathbb{L}_1^{n+1}(c, f)$. Then, both the null sectional curvature functions $\overline{K}_E(\Pi_i)$ and $K_E(\Pi_i)$ vanish.*

Note that the null sectional curvature of a null plane Π_i is independent of the choice of nonnull vectors in Π_i , but depends quadratically on the null on the null vectors. A geometric interpretation of the null sectional curvature can be found in [1], where the authors showed that a three-dimensional, conformally flat Lorentzian manifold has isotropic and spatially constant null sectional curvature if and only if it is locally a Robertson–Walker manifold.

Acknowledgments

Mohamed H. A. Hamed and Samuel Ssekajja would like to thank the Simons Foundation through the RGSM-Project for financial support. The authors thank the referee for his/her valuable comments and suggestions. This work is based on the research supported wholly/in part by the National Research Foundation of South Africa (Grant Numbers: 95931 and 106072).

References

- [1] Albuje AL, Haesen S. A geometrical interpretation of the null sectional curvature. *Journal of Geometry and Physics* 2010; 60: 471-476.
- [2] Andrzejewski K, Kozłowski W, Niedziałomski K. Generalized Newton transformation and its applications to extrinsic geometry. *Asian Journal of Mathematics* 2016; 20: 293-322.
- [3] Andrzejewski K, Walczak PG. The Newton transformation and new integral formulae for foliated manifolds. *Annals of Global Analysis and Geometry* 2010; 3: 103-111.
- [4] Chen BY, Van der Veken J. Spatial and Lorentzian Surfaces in Robertson-Walker space-times. *Journal of Mathematical Physics* 2007; 48: 1-12.
- [5] Duggal KL, Bejancu A. *Lightlike Submanifold of Semi-Riemannian Manifolds and Applications*. Dordrecht, the Netherlands: Kluwer Academic Publishers, 1996.
- [6] Duggal KL, Sahin B. Lightlike submanifolds of indefinite Sasakian manifolds. *International Journal of Mathematics and Mathematical Sciences* 2007: Art. ID 57585: 21 pp.
- [7] Duggal KL, Sahin B. *Differential Geometry of Lightlike Submanifolds*, *Frontiers in Mathematics*. Basel, Switzerland: Birkhäuser Verlag, 2010.
- [8] Duggal KL, Jin DH. Geometry of null curves. *Mathematics Journal of Toyama University* 1999; 22: 95-120.
- [9] Duggal KL, Jin DH. Half lightlike submanifolds of codimension 2. *Mathematics Journal of Toyama University* 1999; 22: 121-161.
- [10] Duggal KL, Jin DH. Totally umbilical lightlike submanifolds. *Kodai Mathematical Journal* 2003; 26: 49-68.
- [11] Erdogan FE, Güneş R, Şahin B. Half-lightlike submanifolds with planar normal sections in R_2^4 . *Turkish Journal of Mathematics* 2014; 38: 764-777.
- [12] Gupta RS, Upadhyay A, Shaffuddin A. Lightlike submanifolds of indefinite cosymplectic manifolds. *Analele științifice ale Universității "Al. I. Cuza" din Iași Matematică (NS)* 2012; 58: 157-180.
- [13] Kang TH. On Lightlike hypersurfaces of a GRW space-time. *Bulletin of the Korean Mathematical Society* 2012; 49: 863-874.
- [14] Kang TH. On Lightlike submanifolds of a GRW space-time. *Communications of the Korean Mathematical Society* 2014; 29: 295-310.

- [15] Kupeli DN. Singular Semi-Riemannian Geometry, Mathematics and Its Applications. Kluwer Academic Publishers, 1996.
- [16] Massamba F. Totally contact umbilical lightlike Hypersurfaces of indefinite Sasakian manifolds. Kodai Mathematical Journal 2008; 31: 338-358.
- [17] Massamba F. Almost Weyl structures on null geometry in indefinite Kenmotsu manifolds. Mathematica Slovaca 2016; 66: 1443-1458.
- [18] Massamba F. A conformal connection on null hypersurfaces of indefinite Kenmotsu manifolds. Bulletin of the Malaysian Mathematical Sciences Society 2017; 40: 1559-1575
- [19] Massamba F, Ssekajja S. A New approach to totally umbilical null submanifolds. Novi Sad Journal of Mathematics 2017; 47: 63-76.
- [20] Perktas SY, Erdogan FE. On generalized CR-lightlike submanifolds. Palestine Journal of Mathematics 2019; 8: 200-208.
- [21] O'Neill B. Semi-Riemannian Geometry, with Applications to Relativity. Pure and Applied Mathematics. Academic Press, 1983.