[Turkish Journal of Mathematics](https://dctubitak.researchcommons.org/math)

Volume 43 | Number 3 Article 38

1-1-2019

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1626 – 1639
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doi:10.3906/mat-1808-75

# Converse theorems in Lyapunov's second method and applications for fractional order systems

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**Received:** 14.08.2018 • **Accepted/Published Online:** 11.04.2019 • **Final Version:** 29.05.2019

**Abstract:** We establish a characterization of the Lyapunov and Mittag-Leffler stability through (fractional) Lyapunov functions, by proving converse theorems for Caputo fractional order systems. A hierarchy for the Mittag-Leffler order convergence is also proved which shows, in particular, that fractional differential equation with derivation order lesser than one cannot be exponentially stable. The converse results are then applied to show that if an integer order system is (exponentially) stable, then its corresponding fractional system, obtained from changing its differentiation order, is (Mittag-Leffler) stable. Hence, available integer order control techniques can be disposed to control nonlinear fractional systems. Finally, we provide examples showing how our results improve recent advances published in the specialized literature.

**Key words:** Fractional differential equations, Lyapunov stability, nonlinear systems

## 1. Introduction

The Lyapunov stability concept is central in the qualitative study of dynamical systems [14, 20, 23]. There are two main reasons explaining its importance: first, this concept allows to formalize practical goals (e.g., to obtain a desired response of a system reached with a quantifiable degree of accuracy); and second, the powerful method of Lyapunov functions, which allows asserting the stability property of a wide class of systems without using their analytic solutions, which are not easy to obtain due to the complexity of many differential equations, including for instance many fractional differential equations.

The second reason is strengthened in converse theorems that prove the existence of Lyapunov functions for stable systems (see [12] for a survey of results in integer order systems). This class of theorems is analogous to existence results for differential equations; before trying any function as a solution, to know if there exists such a function is a commonsense requirement. As a consequence, a characterization of stability through Lyapunov functions is obtained and the term stability usually means the Lyapunov one.

Although converse results may seem purely mathematical ones, some practical applications of them can be found. With their application, a linearization equivalence to study the asymptotic stability of nonlinear systems was shown [14, Theorem 3.13]. Also, they provide tools to analyze perturbed systems [23, Theorem 23.1].

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2010 *AMS Mathematics Subject Classification:* 34K37, 34D05, 34D45

Our contributions are summarized as follows. In Section 2, we recall some stability notions and show that the suited one for fractional systems is weaker than the one for integer systems. Indeed, if the latter stability is assumed, main open problems [6, 22] can be easily solved. Then, we provide a hierarchy for the order of the Mittag-Leffler convergence, which precludes that systems of a given differentiation order to have arbitrarily fast convergence. In particular, we show that fractional systems of order lesser than one cannot be exponentially stable. It follows that some stability proofs for fractional systems—e.g., those based on fractional generalizations of Gronwall Lemma—cannot be right since they yield exponential convergence (see e.g., [4]).

In Section 3, we establish converse Lyapunov theorems for (Mittag-Leffler) stable local and nonlocal systems. We use the 2-flow characterization of fractional systems provided in [1]. The converse is stated for a weaker stability notion which is suited for fractional systems as well as time-varying integer order systems.

In Section 4, we use the converse results to show that if an integer order system is (exponential) stable, then its corresponding fractional system, obtained by changing the derivation order, is (Mittag-Leffler) stable. The importance of this result is that to prove the stability of fractional systems we can use known integer order methods [14]. We also use converse results to show a perturbed-type result involving the differentiation order. Finally, in Section 5, we provide several examples showing how our results can, in addition, improve recent progresses in the specialized literature.

**Notation.** Let $X \subseteq \mathbb{R}^n$. $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. $\|\cdot\|$ also denotes a matrix norm on $\mathbb{R}^{n\times n}$ compatible with the Euclidean norm. For $x, y \in X$, $x^T y$ denotes the inner product. $\mathbb{R}_{\geq}0$ is the set of nonnegative real numbers.

A $\mathcal{C}^k$ function is a $n-$times continuously differentiable function. A class $\mathcal{K}$ function is a strictly increasing continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(0) = 0$. $V : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is radially unbounded if $\|x\| \to \infty$ implies $V(x,t) \to \infty$ for all $t > 0$.

The Mittag-Leffler function is defined as $E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, where $z \in \mathbb{C}$, $\alpha > 0$ and $\Gamma$ is the gamma function [5]. Note that $E_{\alpha=1}(z) = \exp(z)$.

$D^\alpha$ stands for the Caputo derivative defined by $D^\alpha_{0+} f(t) := \big(I^{m-\alpha}_{0+} D^m f\big)(t)$ [5], where $\alpha > 0$, $m = \lceil \alpha \rceil$ and

$$I^\alpha_{0+} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

We omit the subindex 0 in $D^\alpha$ since wlog all the results are referred to this initial time.

## 2. Stability preliminaries

In this section, we recall stability notions to show a subtlety in their applications to fractional order systems and to establish a convergence's order hierarchy.

The stability concepts are referred to the following system

$$D^\alpha x(t) = f(x,t), \tag{2.1}$$

where $x(t), f(x(t),t) \in \mathbb{R}^n$ for all $t \geq 0$, $f(0,\cdot) \equiv 0$ and $\alpha \in (0,1]$. We consider $X \subseteq \mathbb{R}^n$ a neighborhood of $x = 0$ such that $\forall x_0 \in X$, the initial value problem associated to (2.1) has a unique continuous solution $x : [0,\infty) \to X$ satisfying $x(0) = x_0$. Such $X$ is called a positively invariant set of (2.1). A sufficient condition to have $X = \mathbb{R}^n$ is that $f$ be continuous and Lipschitz continuous in its first argument [3, 5]. Note that for the uniqueness, $x_0$ must be specified at the initial time of the fractional derivative ($t = 0$ in our case).

Let $\varphi(\tau; t, x)$ be the solution of (2.1) at time $\tau \geq 0$, given that it takes the value $x$ at time $t \geq 0$. This function is well defined on every $x \in X$ for local and nonlocal dynamical systems [1, Theorem 20, 22]. Main examples of nonlocal dynamics are linear and nonlinear one-dimensional and triangular $n-$dimensional Caputo fractional systems.

**Definition 2.1** *Consider a point $x_e \in X$ and any solution $x(\cdot)$ of (2.1) starting at $X$*

*(i)* $x_e$ *is Lyapunov stable at* $t_0 \geq 0$ *if* $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon, t_0) > 0$ *such that if* $||x(t_0) - x_e|| < \delta$, *then* $||x(t) - x_e|| < \epsilon$ *for any* $t \geq t_0$. $x_e$ *is Lyapunov stable if it is Lyapunov stable for any* $t_0$. *If* $\delta = \delta(\epsilon)$, *then* $x_e$ *is uniformly Lyapunov stable.*

*(ii)* $x_e$ *is asymptotically Lyapunov stable at* $t_0 \geq 0$ *if it is stable at* $t_0$ *and* $\exists r = r(t_0)$ *such that if* $||x(t_0) - x_e|| < r$ *then* $\lim_{t\to\infty} ||x(t) - x_e|| = 0$. *If* $r$ *does not depend on* $t_0$, $x_e \in X$ *is uniformly asymptotically stable.*

*(iii)* $x = 0$ *is Mittag-Leffler stable of order* $\alpha$ *if there exist numbers* $a, \lambda, m, \alpha > 0$ *and a neighborhood of the origin* $B \subset X$ *such that*

$$\|x(t)\|^a \leq mE_\alpha(-\lambda t^\alpha)\|x_0\| \quad \forall t > 0, \forall x(0) = x_0 \in B. \tag{2.2}$$

*In particular,* $x = 0$ *is exponentially stable if it is Mittag-Leffler stable with* $\alpha = 1$.

The (asymptotic) stability concept is standard in the literature and can be found, e.g., in [14, Definition 4.1, p.112]. The Mittag-Leffler stability as stated here was already proposed in [15]. In general, the above definitions concern an equilibrium point, i.e. a point such that if $x(0) = x_e$ then $x(t) = x_e$, $\forall t > t_0$.

When $\alpha = 1$ and $f = f(x)$, stability at $t_0$ is equivalent to stability. Since fractional systems have a fixed initial time and have a nonlocal behavior [5], the stability at $t = 0$ appears to be suited for them. In fact, we will see that (fractional) Lyapunov functions yield this weak type of stability. First, to stress the weakness of the stability at $t_0$, the following result presents a fractional integral version of the Lyapunov direct method, based critically on stability. It can be seen also as a sufficient condition for a fractional like Barbalat Lemma. Both problems have remained open (see [6, 22]).

**Theorem 2.2** *Consider that for system (2.1) there exist* $\beta > 0$ *and* $\gamma \in \mathcal{K}$ *satisfying*

*(I.1)* $\left(I^\beta \gamma(\|x\|)\right)(t) < C < \infty \quad \forall t \geq 0$ *and for any* $x(\cdot)$ *solution of (2.1) with* $x(0) \in X$, *where* $C \in \mathbb{R}$ *is independent of time, although it can depend on* $x_0$.

*(I.2)* $x = 0$ *is uniformly Lyapunov stable.*

*Then* $x = 0$ *is uniformly asymptotically stable.*

**Proof** From hypothesis (I.1), we have $\liminf_{t\to\infty} \gamma(||x(t)||) = 0$ according to [6, Proposition 15]. Since $\gamma \in \mathcal{K}$, $\liminf_{t\to\infty} ||x(t)|| = 0$.

If $x$ does not converge to zero, there exist $\epsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$, with $t_n \to \infty$ as $n \to \infty$, such that $||x(t_n)|| > \epsilon$ for each $n \in \mathbb{N}$. Since $x = 0$ is uniformly Lyapunov stable at every $t_0$, there exists $\delta_0 = \delta(\epsilon)$ such that $||x(t)|| > \delta_0, \forall t \leq t_n$ and $\forall n \in \mathbb{N}$, where $\delta$ is the function in Definition (2.1)(i). Since $t_n \to \infty$ as $n \to \infty$, it follows that $||x(t)|| > \delta_0, \forall t \geq 0$. However, this contradicts that $\liminf_{t\to\infty} ||x(t)|| = 0$.

Therefore, $x$ converges to zero and $x = 0$ is uniformly asymptotically stable. □

**Corollary 2.3** *Consider that for system* (2.1) $x = 0$ *is uniformly stable and there exist a continuous function* $V : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, *a number* $\beta \in (0, 1)$ *and* $\gamma_1, \gamma_2, \gamma_3$ *class-*$\mathcal{K}$ *functions satisfying*

*(V.1)* $\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|) \quad \forall t \geq 0$ *and* $\forall x \in X$,

*(V.2)* $V(t) = V(x(t), t)$ *satisfies* $D^\beta V(t) \leq -\gamma_3(\|x(t)\|)$ $\forall t \geq 0$ *and for all solution* $x(\cdot)$ *of* (2.1) *with* $x(0) \in X$.

*Then* $x = 0$ *is uniformly asymptotically stable.*

**Proof** Using hypothesis $(V.2)$, we obtain

$$V(t) - V(0) \leq -\big(I^\beta \gamma(V)\big)(t),$$

where we have used the comparison Lemma [3, Proposition 23] for the inequality $D^\beta V(t) \leq -\gamma_3(\|x(t)\|) \leq -\gamma_3(\gamma_2^{-1}(V(t))) =: -\gamma(V(t))$ and the fact that the continuous solution of the equation $D^\beta V(t) = -\gamma(V(t))$ with initial condition $V(0)$ is given by $V(t) = V(0) + \big(I^\beta \gamma(V)\big)(t)$. For the latter, note that $\gamma$ is also a class-$\mathcal{K}$ function and, by taking $\beta-$derivative of the continuous function $V(t) = V(0) + \big(I^\beta \gamma(V)\big)(t)$, we obtain $D^\beta V = \gamma(V)$ using [5, Theorem 3.7]. For the uniqueness of the continuous solution, we can take $\gamma$ local Lipschitz without loss of generality since if it is not the case, one can take enough smooth class-$\mathcal{K}$ function $\gamma'$ such that $\gamma' \leq \gamma$ since $\gamma$ is strictly increasing and continuous or to note that the equation $D^\beta V_1 - D^\beta V_2 = \gamma(V_1) - \gamma(V_2)$ with $V_1(0) = V_2(0)$ implies $D^\beta V_1(0) = D^\beta V_2(0)$.

Using this together with hypothesis $(V.1)$, we get $I^\beta \gamma(\gamma_1(\|x\|)) \leq V(0) < \infty$. The claim $x = 0$ uniformly asymptotically stable follows from applying Theorem 2.2 (note that $\gamma(\gamma_1(\cdot))$ is also a class-$\mathcal{K}$ function). □

$V$ is called fractional Lyapunov function if it satisfies conditions $(V.1)$ and $D^\beta V \leq 0$ for some $\beta > 0$ and for any solution of (2.1) that starts in $X$, in the same sense (i.e. along the solutions of (2.1)) that for $(V.1)$ and $\dot{V} \leq 0$ it is called a Lyapunov function. Note that the existence of a fractional Lyapunov function (and therefore, the function $V$ in Corollary 2.3) assures uniform stability only at $t = 0$ [7]. Therefore, the uniform stability of the origin must be asserted separately when $\beta < 1$, whereas for $\beta = 1$ it is implied by condition $(V.2)$ [14].

The invariant condition on $X$, which obviously includes the global case $X = \mathbb{R}^n$, is reasonable as a local statement, for the existence of such an invariant neighborhood of $x = 0$ can be deduced from a local condition along the following lines. Suppose that $(D^\alpha V)(t) \leq g(x(t))$, where $g(x) \leq 0$ for any $x \in B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. Let $c < \gamma_1(r)$. Then $\{x \in B_r : \gamma_1(x) < c\}$ is contained in the interior of $B_r$. Let $X_c = \{x \in B_r : V(x) \leq c\}$. Then $\{x \in B_r : \gamma_2(x) < c\} \subseteq X_c \subseteq \{x \in B_r : \gamma_1(x) < c\}$ (see p.151 in [14]). Since any solution of (2.1) starting at $X_c$ must move along $X_c$ when it is in $B_r$, because $(D^\alpha V)(t) \leq 0$, whenever $x(t) \in B_r$, implies $V(x(t)) \leq V(x(0)) \leq c$, and since $X_c$ is contained in the interior of $B_r$, any continuous solution never can leave $B_r$. Therefore, $X_c$ is invariant.

The following result shows a hierarchy for the Mittag-Leffler stability,

**Theorem 2.4** *Consider system (2.1) with* $f : \Omega_r \times [0,\infty) \to \mathbb{R}^n$ *a continuously differentiable function,* $\Omega_r = \{x \in \mathbb{R}^n | \|x\| < r\}$ *for some* $r > 0$*,* $f(0,t) = 0$ *for all* $t \geq 0$ *and the Jacobian matrix* $\big(\partial f/\partial x\big)(x,t)$ *is bounded and Lipschitz continuous on* $\Omega_r$*, uniformly in* $t$*. Then* $x = 0$ *cannot be Mittag-Leffler stable of order* $\beta \in (\alpha, 1]$*.*

**Proof** By contradiction, suppose that $x = \mathcal{O}(E_\beta(-\lambda t^\beta))$ for some $\beta > \alpha$ and $\lambda > 0$, where $\mathcal{O}$ stands for the big O notation. According to Definition (2.1) and the fact that $E_\beta(-\lambda t^\beta)$ is monotonically decreasing as a time function, we can choose $\delta$ such that for any $\|x(0)\| < \delta$, $x(t) \in \Omega_r$ for any $t > 0$. Let $A(t) := \big(\partial f/\partial x\big)(0,t)$. From Taylor's Theorem and the fact that $f(0,t) = 0$ for all $t \geq 0$, we can rewrite (2.1) as

$$D^\alpha x = A(t)x + q(x;t)$$

where $q(x;t) = [\big(\partial f/\partial x\big)(\xi,t) - \big(\partial f/\partial x\big)(0,t)]x$ and $\xi$ is a point in the ball of radius $\|x(0)\|$ (see [14, p. 166]). By $\alpha-$integration or using the integral equivalence in [5], the solution of equation (2.1) can be written as

$$x(t) = x(0) + \big(I^\alpha Ax + q\big)(t).$$

By the Lipschitz assumption on the Jacobian, we have $\|q(x,t)\| \leq L\|x\|^2$ for some constant $L > 0$ (see [14, p. 161]). Since $x = \mathcal{O}(E_\beta(-\lambda t^\beta)$ and $E_\beta(-\lambda t^\beta) < 1$ is decreasing, $||x||^2 = \mathcal{O}(E_\beta(-\lambda t^\beta))$. By the bounded hypothesis on the Jacobian matrix, $\|Ax\| \leq \|A\|\|x\|$. Hence, there exists a constant number $C$ such that

$$\|\big(I^\alpha Ax + q\big)(t)\| \leq C\big(I^\alpha E_\beta(-\lambda\tau^\beta)\big)(t) \quad \forall t > 0.$$

On the other hand, $\lim_{t\to\infty}\big(I^\alpha E_\beta(-\lambda\tau^\beta)\big)(t) = 0$, since $E_\beta(-\lambda t^\beta) \leq C_0 t^{-\beta}$ [19, Theorem 1.6] and $\big(I^\alpha \tau^{-\beta}\big)(t) = C_1 t^{\alpha-\beta}$ [19, §2.2.4]. It follows that $\lim_{t\to\infty}\|\big(I^\alpha Ax+q\big)(t)\| = 0$ and, in particular, $\lim_{t\to\infty}\big(I^\alpha Ax+q\big)(t) = 0$, when $\beta > \alpha$. Thus, $\lim_{t\to\infty} x(t) = x(0)$, which is a contradiction whenever $x(0) \neq 0$, as required to determine the stability. Therefore, $x = 0$ cannot be a Mittag-Leffler stable point of (2.1) of order $\beta > \alpha$. □

**Remark 2.5** *(i) As a consequence, fractional systems of derivation order lesser than one cannot be exponentially stable. This invalidates some proofs of asymptotic stability, which conclude an exponential order of convergence for fractional systems. The mistake could be related to the use of an improper generalization of the Gronwall's inequality (see e.g., [4]).*

*(ii) After sending this article, we have taken note of an alternative version of Theorem 2.4 segregated in the results of [3, Theorem 6, Lemma 5, Remark 10, Remark 7], where no differentiability but just a local Lipschitz condition is required on* $f$*.*

### 3. Existence of Lyapunov functions

In this section, converse results are presented as a characterization of stability concepts through (fractional) Lyapunov functions.

### 3.1. Stability

We will start proving the existence of (fractional) Lyapunov functions for stable systems. We included also a result for globally bounded systems, i.e. systems of type (2.1) where for any $a > 0$ and $\|x(0)\| \leq a$ there exists $b \in \mathbb{R}$ s.t. $\|x(t)\| \leq b$ for all $t \geq 0$ (see [14, Definition 4.6]).

**Theorem 3.1** *Suppose that $x = 0$ is an equilibrium point of the (nonlocal) dynamical system (2.1).*

*(i) $x = 0$ is Lyapunov stable at $t = 0$ if and only if there exist a function $V : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a number $\beta \in (0, 1]$ and a class $\mathcal{K}$ function $\gamma$ such that $V(0, \cdot)$ is continuous, $V(0, 0) = 0$, $V(x, t) \geq \gamma(\|x\|)$ for any $x \in X$, and $V(t) = V(x(t), t)$ satisfies $D^\beta V \leq 0$ for all solution $x(\cdot)$ of (2.1) with $x(0) \in X$. When $f = f(x)$ and $\alpha = 1$, $0$ is Lyapunov stable if and only if there exists a positive definite continuous function $V : X \to \mathbb{R}_{\geq 0}$ such that $\dot{V} \leq 0$.*

*(ii) The solutions are globally bounded if and only if there exist a number $\beta \in (0, 1]$ and a nonnegative, radially unbounded function $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $V(t) = V(x(t), t)$ satisfies $D^\beta V \leq 0$ for all solution $x(\cdot)$ of (2.1).*

**Proof** (i) *Necessity.* For any smooth nondecreasing function $W : \mathbb{R} \to \mathbb{R}$ such that $W(r) = 0$ for all $r \leq 0$, define $V(x, t) := W(||\varphi(0; t, x)||)$, where $\varphi(0; t, x)$ is the solution of (2.1) at time 0 given that it takes the value $x$ at time $t > 0$. This function is well defined on $X \times \mathbb{R}_{\geq 0}$ since $\varphi$ is defined by a (nonlocal) dynamical system. In addition, $V(0, \cdot)$ is continuous from the continuous property of $\varphi$ and the choice of $W$.

Then $V(x, t) \geq 0$, $V(0, 0) = 0$ and along the trajectories of (2.1) that start at $X$, $V(x, t)$ is constant. Hence, $\dot{V} = 0$ and $D^\beta V = \left(I^{1-\beta}\dot{V}\right)(t) \leq 0$ for any $0 < \beta \leq 1$.

From Definition (2.1), for any $\epsilon > 0$, there exists $\delta(\epsilon)$ such that if $\epsilon = ||x||$, then $||\varphi(0; t, x)|| \geq \delta$. Hence, we have that $V(x, t) \geq W(\delta(\epsilon))$ for any $x \in X$ and the existence of the class $\mathcal{K}$ function $\gamma(\cdot)$ follows.

If $f = f(x)$ and $\alpha = 1$, we have $V(x, t) = V(x)$ is smooth if $f$ does it (see also [23, Theorem 18.6, p. 91]).

*Sufficiency.* By assumption, there exists $\beta \in (0, 1]$ such that $D^\beta V \leq 0$. A direct application of the comparison Lemma [3, Proposition 23] yields $0 \leq V(x(t), t) \leq V(x(0), 0)$, $\forall t > 0$.

Using that $\gamma$ is a class $\mathcal{K}$ function and the hypothesis $V(x, t) \geq \gamma(\|x\|)$ for any $x \in X$ and $t > 0$, we have $0 < \gamma(\epsilon) \leq V(x, t)$ for any $\epsilon > 0$ such that $x \in X$ and $\|x\| = \epsilon$. Since $V(\cdot, 0)$ is continuous and $V(0, 0) = 0$, there exists $\delta(\epsilon)$ such that $||x_0|| < \delta$ implies $V(x_0, 0) < \gamma(\epsilon)$. Suppose that $x = 0$ is not stable. Hence, there exists $t_1$, such that $||x_0|| < \delta$ implies $||x(t_1; 0, x_0)|| = \epsilon$. Since $V(x(t_1; 0, x_0), t_1) \leq V(x_0, 0)$, we have $\gamma(\epsilon) \leq V(x(t_1; 0, x_0), t_1) \leq V(x_0, 0) < \gamma(\epsilon)$. This is a contradiction and hence, $x = 0$ is stable.

(ii) *Sufficiency.* By assumption, there exists $\beta \in (0, 1]$ such that $D^\beta V \leq 0$. Let $V(t) = V(x(t), t)$. Hence, $0 \leq V(t) \leq V(0)$, $\forall t > 0$. By the radially unbounded hypothesis, $V(0) < \infty$ whenever $x(0) \in \mathbb{R}^n$. Therefore, $V(t) < V(0) < \infty$, $\forall t > 0$. Applying again the radially unbounded hypothesis, it follows that $x$ must be globally bounded.

*Necessity.* Consider function $V$ of the part (i). Since the system is globally bounded, this function is well defined on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Moreover, $V(x, t) \to \infty$ as $\|x\| \to \infty$, for all $t \geq 0$. Hence, $V$ is radially unbounded and a nonnegative function. The rest of the proof is similar to the one of the part (i). □

**Remark 3.2** *(i) The weakness of the notion stability at $t = 0$ is reflected in the fact that a Lyapunov function is also a fractional Lyapunov function ($\dot{V} \leq 0$ implies $D^\beta V = I^{1-\beta}\dot{V} \leq 0$) but the converse is not necessarily true (see e.g. [7, Proposition 3]).*

*(ii) An alternative to the radially unbounded hypothesis is to use a finite function $V$ s.t. $V(x,t) \geq \gamma(\|x\|)$, since $V(t) < V(0)$ implies $\|x\| \leq \gamma^{-1}(V(0))$ (see [13]).*

### 3.2. Asymptotic stability

We will show a characterization of the Mittag-Leffler stability through fractional Lyapunov functions.

**Theorem 3.3** *Consider the (non-local) dynamical system (2.1). The equilibrium point $x = 0$ is Mittag-Leffler stable if and only if there exist neighborhoods of the origin $B \subseteq X$, a set $G \subseteq X \times \mathbb{R}_{\geq 0}$, a function $V : G \to \mathbb{R}$ and constant numbers $c_1, c_2, c_3, \lambda > 0$ such that for any solution $x(\cdot)$ of (2.1) with $x(0) \in B$ the following hold*

*(i)* $c_1\|x(t)\| \leq V(x(t),t)$ *and* $V(x(t),0) \leq c_2\|x(t)\|^{c_3} \quad \forall t \geq 0$,

*(ii)* $D^\alpha V(t) \leq -\lambda V(t) \quad \forall t \geq 0$, *where* $V(t) := V(x(t),t)$.

**Proof**

*Necessity.* Since $x = 0$ is Mittag-Leffler stable, there exist numbers $a, m, \lambda > 0$ and a neighborhood of the origin $B$ satisfying Definition (2.1)(iii). Consider the function

$$V(x,t) := \{mE_\alpha(-\lambda t^\alpha)\|\varphi(0;t,x)\|\}^{1/a}. \tag{3.1}$$

defined on $G := \{(x,t) \in X \times \mathbb{R}_{\geq 0} : \varphi(0;t,x) \in B\}$. $V$ is well defined by the bijection property of the flow $\varphi$ [1] and the fact that $V(x,t) \leq \{m||x(0)||\}^{1/a} < \infty$ for all $t > 0$ and $x \in B$, since $E_\alpha(0) = 1$ and $E_\alpha(-\lambda t^\alpha)$ is nonincreasing [11].

From Mittag-Leffler stability (Definition (2.1)(iii)), $\|x(t)\| \leq V(x(t),t)$ for all $t > 0$ when $x(0) \in B$. Since $x = 0$ is an equilibrium point, $V(0,t) = 0$. Since $E_\alpha(0) = 1$, $V(x,0) \leq \{m\|x\|\}^{1/a}$. Putting $c_3 = 1/a$, $c_2 = m^{1/a}$ and $c_1 = 1$, the part (i) is obtained.

Note that along trajectories of system (2.1), $\varphi(0;\cdot,x) \equiv x(0)$. Since $E_\alpha(-\lambda t^\alpha)$ is a solution of $D^\alpha y = -\lambda y$ when $y(0) = 1$, it follows that $D^\alpha V(t) = -\lambda V(t)$ and the part (ii) is proven.

*Sufficiency.* From (i), (ii) and the comparison Lemma [3, Proposition 23] or applying Gronwall inequality [5, Lemma 6.19], $c_1\|x(t)\| \leq V(x(t),t) \leq V(x(0),0)E_\alpha(-\lambda t^\alpha) \leq c_2\|x(0)\|E_\alpha(-\lambda t^\alpha)$. Therefore, $x = 0$ is Mittag-Leffler stable. □

**Remark 3.4** *(i) In the necessity part of Theorems 3.1 and 3.3, the dynamical character of the flow $\varphi$ was used only to obtain the initial value $\varphi(0;t,x)$. Putting aside transitive or even continuity properties of the flow, this function exists provided that there is no intersection of trajectories, i.e. if for every $(t_1, x_1)$ there is only one initial condition $x_0$ such that the system reaches $x_1$ at time $t_1$. Although not all fractional systems hold this (see [1, Theorem 22]), one could define $V$ excluding the set of intersection points and consider the inequality involving $D^\alpha V$ almost everywhere for $t > 0$, to get a more general result.*

*(ii) In the sufficient part of Theorems 3.1 and 3.3, the assumption on the dynamical character of the flow $\phi$, whereby they are valid for every fractional system, is not necessary. Moreover, it must be noted that*

*similar statements have already been proven in the literature [3, 7, 9, 15]. The novelty relies in the bounded case (Theorem 3.1(ii)) and the underscore that the stability is respect to the initial time of the derivative ($t = 0$).*

We recall converse results for the exponential and the uniform asymptotic stability, which we will use in the next section.

**Theorem 3.5** *Consider system (2.1) with $\alpha = 1$ and $f = f(x)$ a continuously differentiable function with Jacobian matrix bounded on $X$. If the equilibrium point $x = 0$ is exponentially stable, then there exist a neighborhood of the origin $B \subset X$, a smooth function $V : B \to \mathbb{R}$ and constant numbers $c_1, c_2, c_3, c_4 > 0$ such that for any $x \in B$*

*(i)* $c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$,

*(ii)* $\nabla V^T f(x) \leq -c_3 V(x)$,

*(iii)* $\|\nabla V(x)\| \leq c_4\|x\|$.

*Moreover, for items (i), (ii) the function $V$ can be chosen (enough) smooth, provided that $f$ is (enough) smooth.*

**Proof** Items (i), (ii), and (iii) follow from [14, Theorem 4.14, p.162], particularized to autonomous dynamics. The smooth part follows from [18, Theorem 11]. □

**Theorem 3.6** *Consider system (2.1) with $\alpha = 1$ and $f = f(x)$ a continuously differentiable function with Jacobian matrix bounded on $X$. If the equilibrium point $x = 0$ is uniformly asymptotically stable, then there exist a neighborhood of the origin $B \subset X$, a continuously differentiable function $V : B \to \mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that for any $x \in B$*

*(i)* $\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$,

*(ii)* $\nabla V^T f(x) \leq -\gamma_3(\|x\|)$,

*(iii)* $\|\nabla V(x)\| \leq \gamma_4(\|x\|)$.

*Moreover, for items (i), (ii), the function $V$ can be chosen smooth provided that $f$ is smooth.*

**Proof** Items (i), (ii), and (iii) follow from [14, Theorem 4.16, p.162] when particularized to autonomous dynamics. The smooth claim follows from [16, Proposition 4.2]. □

## 4. Applications

Converse results are mainly applied in perturbed systems (see e.g., [14, 23]): by assuming that the unperturbed system is (asymptotically) stable, a Lyapunov function exists, which is used in the perturbed case to prove (asymptotic) stability. The following results can be seen as an extension of these applications, in which the perturbation (or rather the uncertainty) occurs in the derivation order.

**Proposition 4.1** *Consider the system*

$$\dot{x} = f(x, t) \tag{4.1}$$

*where $f : X \times \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable function with $f(0,t) = 0$ for any $t \geq 0$, and the system*

$$D^\alpha x = f(x,t), \tag{4.2}$$

*where $\alpha \in (0,1]$ is a constant number. Assume that the Lyapunov functions obtained from the application of the converse results in Section 3 are differentiable, locally convex around $x = 0$ and does not depend on time.*

*(i) If $x = 0$ is a Lyapunov stable equilibrium point of (4.1), then $x = 0$ is a Lyapunov stable point at time $t = 0$ for (4.2).*

*(ii) If $x = 0$ is exponentially stable for system (4.1), then it is Mittag-Leffler stable for (4.2).*

*(iii) If $x = 0$ is uniformly asymptotically stable for system (4.1), then it is uniformly asymptotically stable for (4.2) whenever it is uniformly Lyapunov stable.*

**Proof** (i). Since $x = 0$ is a stable point for system (4.1), it has a Lyapunov function according to Theorem 3.1. This function is differentiable and locally convex $V = V(x)$ by the additional hypothesis in the statement. Then, we have in some neighborhood of $x = 0$

$$\dot{V} = \frac{\partial V}{\partial x}^T \dot{x} = \frac{\partial V}{\partial x}^T f(x,t) \leq 0.$$

By using (4.2), we get for any $t > 0$

$$\frac{\partial V}{\partial x}^T D^\alpha x(t) \leq 0,$$

where we have passed to an invariant neighborhood of $x = 0$ if necessary (see the paragraph before Theorem 2.4).

Since $f$ is locally Lipschitz continuous, the solutions of (4.1) are continuous. Using [22, Theorem 3] (or [10, Lemma 3]) which states that $D^\alpha V(t) \leq \frac{\partial V}{\partial x}^T D^\alpha x$ when $V$ is convex and differentiable, it follows that

$$D^\alpha V(t) \leq 0, \quad \forall t \geq 0.$$

By Theorem 3.1 and Remark 3.4(ii), the origin of system (4.2) is stable at time $t = 0$.

(ii). If $x = 0$ is exponentially stable for system (4.1), by Theorem 3.5, there exist a neighborhood $B$ around $x = 0$ and a positive definite smooth function $V = V(x)$ such that

$$\frac{\partial V}{\partial x}^T f(x,t) \leq -\lambda \|x\|^2,$$

for all $x \in B$.

Then, using (4.2), we have

$$\frac{\partial V}{\partial x}^T D^\alpha x \leq -\lambda \|x\|^2.$$

Since $f$ is Lipschitz continuous, the solutions of (4.2) are continuous. Using [22, Theorem 3] (or [10, Lemma 3]) which states that $D^\alpha V(t) \leq \frac{\partial V}{\partial x}^T D^\alpha x$ when $V$ is convex and differentiable, it follows that

$$\big(D^\alpha V(x)\big)(t) \leq -\lambda \|x(t)\|^2, \quad \forall t \geq 0.$$

Since $V$ is convex by assumption and where we have passed to an invariant neighborhood of $x=0$ if necessary (see the paragraph before Theorem 2.4). Therefore, for all $t \geq 0$,

$$D^{\alpha}V(t) \leq -\frac{\lambda}{c_2}V(t),$$

where $c_2 > 0$ is obtained according to Theorem 3.5. Using a similar reasoning to that in the proof of Theorem 3.3 and Remark 3.4, it follows that the origin of system (4.2) is Mittag-Leffler stable for $||x||^2$. Note that, without loss of generality, $B$ can be assumed bounded. Thus, $f$, a continuously differentiable function, has Jacobian matrix bounded on $B$.

(iii). The proof is similar to part (i) by using Theorem 3.6 instead of Theorem 3.5 and Corollary 2.3 instead of Theorem 3.3. □

The additional assumption in Proposition 4.1 of convexity, which also appears in [3, Theorem 25] and [22, Theorem 3] for the case when $f = f(x)$, can be omitted when exponential stability of $x = 0$ for (4.1) is assumed and $f = f(x)$.

**Theorem 4.2** *Consider the system*

$$\dot{x} = f(x) \tag{4.3}$$

*where* $f : X \to X$ *is a* $\mathcal{C}^2$ *function with* $f(0) = 0$*, and the system*

$$D^{\alpha}x = f(x) \tag{4.4}$$

*where* $\alpha \in (0, 1]$ *is a constant number. If* $x = 0$ *is exponentially stable for system* (4.3)*, then it is Mittag-Leffler stable for* (4.4).

**Proof** Since $x = 0$ is exponentially stable for system (4.1), by Theorem 3.5 there exists $V = V(x)$ a $\mathcal{C}^2$ positive definite function. Hence, $x = 0$ is a local minimum of $V$ and $\nabla V(0) = 0$. By a Taylor's expansion around $x = 0$, we have

$$V(x) = \frac{1}{2}x^T H_V(0)x + \mathcal{O}(\|x\|^3)$$

where $H_V$ is the Hessian of $V$. Since $V \geq 0$, it follows that $H_V(0) \geq 0$ (otherwise, one can choose a sufficiently small $\epsilon > 0$ and an eigenvector $\xi$ of $H_V(0)$ with negative eigenvalue obtaining $V(\epsilon\xi) < 0$).

Consider the function $W(x) := V(x) + \varepsilon x^T x$, where $\varepsilon > 0$ is a constant to be precised. Clearly, $W$ is $\mathcal{C}^2$ positive definite function such that $\nabla W(0) = 0$ and $H_W(0) > 0$. Therefore, $W$ is locally convex i.e. there exists a neighborhood around $x = 0$ where $W$ is convex. Moreover, for any $x$ around zero

$$\begin{aligned}\nabla W^T(x)f(x) &= \nabla V^T(x)f(x) + 2\varepsilon x^T f(x)\\ &\leq -c_3c_1\|x\|^2 + 2L\varepsilon\|x\|^2\end{aligned}$$

where we have used 3.5 to get the constants $c_3, c_1$, that $f$ is locally Lipschitz with constant $L$ (due to its smoothness), $f(0) = 0$ and the Cauchy-Schwartz inequality. Pick any $\varepsilon \in (0, \frac{c_3c_1}{2L})$. Then, $\nabla W^T(x)f(x) \leq -\lambda\|x\|^2$ for some $\lambda = \lambda(\varepsilon) > 0$. Using the same arguments of Proposition 4.1(ii) for $W$ instead of $V$, the claim follows. □

The next result deals with the stability under perturbations in the field as well as in the derivation order.

**Theorem 4.3** *Consider the system*

$$\dot{x} = f(x) \tag{4.5}$$

*where* $f : X \to X$ *is a continuously differentiable function and* $x = 0$ *is an exponentially stable point (in particular,* $f(0) = 0$*). Consider the following perturbed system*

$$D^\alpha x = f(x) + g(x,t) \tag{4.6}$$

*where* $\alpha \in (0,1]$ *is a constant number, and* $g$ *is a locally Lipszchitz function on its first argument, with local Lipschitz constant* $L(r)$ *when* $||x|| < r$ *such that* $\lim_{r\to 0} L(r) = 0$ *uniformly in* $t$ *and such that* $g(0,t) = 0$ *for all* $t \geq 0$*. Then,* $x = 0$ *is Mittag-Leffler stable for* (4.6)*.*

**Proof** Since $x = 0$ is exponentially stable for system (4.5) and using the convex construction as in the proof of Theorem 4.2 if necessary, there exist a neighborhood $B$ around $x = 0$, a positive definite convex continuously differentiable function $V$, a number $K > 0$ such that $||\frac{\partial V}{\partial x}|| \leq K||x||$, $\forall x \in B$, and a number $\lambda > 0$ such that $\nabla V^T f(x) \leq -\lambda ||x||^2$, $\forall x \in B$, according to Theorem 3.5(i) and Theorem 3.5(ii). Then, by using the same function $V$ but evaluated along the trajectories of system (4.6), we have

$$\begin{aligned} D^\alpha V(x(t)) &\leq \frac{\partial V}{\partial x}^T D^\alpha x \\ &= \frac{\partial V}{\partial x}^T [f(x) + g(x,t)] \\ &\leq -\lambda ||x||^2 + KL(r)||x||^2. \end{aligned}$$

Since $\lim_{r\to 0} L(r) = 0$, there exists $r_0$ sufficiently small so that $\lambda - KL(r_0) > 0$. Hence, there exists a neighborhood $B$ around the origin, such that $D^\alpha V(x) \leq -(\lambda - KL(r_0))V(x)$ for any $x \in B$. Using a similar reasoning as in the proof of Theorem 3.3, it follows that the origin of system (4.6) is Mittag-Leffler stable for $||x||^2$. □

We say that a system is robustly stable if its stability is not affected with field perturbation like the one of Theorem 4.3 (this concept has been used with a slightly different meaning in [21]). The next result is an important consequence for the control of fractional order systems, which allows employing the variety of control techniques developed for integer order systems [14, 20]. The control is robust if the controlled system is robustly stable. The proof is omitted since it is a straightforward consequence from Theorem 4.3.

**Corollary 4.4** *To robustly control the system*

$$D^\alpha x = f(x,u) \tag{4.7}$$

*where* $f$ *is a continuously differentiable function,* $f(0,0) = 0$ *and* $0 < \alpha < 1$*, it is enough to find a control* $u = u(x)$ *that exponentially stabilizes the system* $\dot{x} = f(x,u)$*. Similarly, to robustly control the system*

$$D^\alpha x = f(x,t,u) \tag{4.8}$$

*where* $f$ *is a continuous function and* $0 < \alpha < 1$*, it is enough to find a control* $u = u(x,t)$ *such that the system* $\dot{x} = f(x,t,u)$ *is exponentially stable with a Lyapunov function* $V = V(x)$*.*

**5. Examples**

This section presents illustrative examples showing as our results improve recent progresses in the literature. In the first one, we will see the difficulty in finding Lyapunov functions (even for linear time-invariant systems) and therefore the necessity of converse theorems to assure their existence.

**Example 5.1** *Consider the fractional system*

$$D^\alpha x = Ax, \tag{5.1}$$

*where* $0 < \alpha \leq 1$. *The eigenvalues of matrix* $A \in \mathbb{R}^{n\times n}$ *belong to* $\Lambda_\alpha := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2}\}$. *Lyapunov stability of* $x = 0$ *at* $t = 0$ *has been proved using the explicit solution to system* (5.1) *(see e.g., [2]).*

*To find a (fractional) Lyapunov function for* (5.1) *has proved to be a hard problem and currently, there is not a general solution available that take into account all cases associated to* $\Lambda_\alpha$. *In the scalar case, the function* $V = x^2$ *is enough, since (5.1) takes the form*

$$D^\alpha x = -\lambda x \tag{5.2}$$

*with* $\lambda > 0$ *and its solution* $|x(t)| = |x(0)|E_\alpha(-\lambda t^\alpha)$ *is known to be decreasing for* $t > 0$ *(see e.g. [11, equation A.30]). Then,* $\frac{d}{dt}V = \frac{d}{dt}x^2 = 2x\frac{d}{dt}x \leq 0$ *for* $t > 0$. *Hence,* $x = 0$ *is a Lyapunov stable point of system* (5.2).

*For the vector case, fractional Lyapunov functions can be obtained. Consider the function* $V(x) = x^T Px$, *where* $P > 0$ *is a symmetric constant matrix. Then,*

$$D^\alpha V \leq 2x^T PD^\alpha x = x^T PAx + x^T A^T Px$$

*where the inequality is due to [22, Theorem 3], since* $V$ *is convex and differentiable. If*

$$PA + A^T P < 0, \tag{5.3}$$

*then* $V$ *is a fractional Lyapunov function. For* $\alpha = 1$ *this condition is also necessary for asymptotic (exponential) stability [14, Theorem 4.6, p. 136]. Such a matrix* $P$ *exists if and only if the eigenvalues of* $A$ *belong to* $\Lambda_1$. *However,* (5.1) *is asymptotically stable if the eigenvalues of* $A$ *belong to* $\Lambda_\alpha$ *(see e.g. [2]). Then, condition* (5.3) *is not necessary for* $\alpha < 1$ *and it is unclear if functions of type* $V(x) = x^T Px$ *covers all the stable cases. From Theorem 4.2(ii), condition* (5.3) *is sufficient for Mittag-Leffler stability of* $x = 0$ *for system* (5.1). *Since* $\Lambda_1 \subsetneq \Lambda_\alpha$, *the set of stable matrices is enlarged as* $\alpha$ *decreases and thus, a converse of Theorem 4.2 is not true. This is intuitive since the stability used for the latter is weaker than stability for integer order systems.*

*Consider now the nonlinear triangular system*

$$\begin{cases} D^{\alpha_1} x_1 = -x_1 + x_2^2 \\ D^{\alpha_2} x_2 = -x_2, \end{cases} \tag{5.4}$$

*where* $0 < \alpha_1 < \alpha_2 \leq 1$ *and* $x_1(t), x_2(t) \in \mathbb{R}$ *for all* $t \geq 0$. *For a sufficiently close to zero initial condition,* $(x_1, x_2) = (0, 0)$ *is a stable point, since* $x_2 = 0$ *is asymptotically stable, and then* $x_1 = 0$ *is also stable (moreover it is also asymptotically stable).*

*There is no known method to get a (fractional) Lyapunov function for system* (5.4). *The difficulty is now to find a common order of derivation which must be necessarily different from either* $\alpha_1$ *or* $\alpha_2$. *However, Caputo fractional derivative has not a semi group property, that is, in general* $D^{\alpha_1}D^{\alpha_2}f \neq D^{\alpha_1+\alpha_2}f$.

*Since (5.4) has triangular form, a similar argument as in [1] shows that it is a nonlocal dynamical system. Hence,* $V(x,t) := \sup_{\tau\in[t,\infty)} ||x(\tau;t,x)||$, *where* $x = (x_1, x_2)$, *is a fractional Lyapunov function from the arguments of the proof of Theorem 3.1(ii).*

**Example 5.2** *Consider the system*

$$D^\alpha x = -\frac{x}{1+x^2}. \tag{5.5}$$

*When* $\alpha = 1$, $x = 0$ *is exponentially stable for* (5.5), *since the linearization at the origin is* $\dot{x} = -x$ *[14, p. 177]. From Theorem 4.2,* $x = 0$ *is Mittag-Leffler stable for* (5.5) *and* $\alpha \in (0,1]$. *The alternative method of seeking a Lyapunov function may be very hard when a quadratic function does not work, as in this case.*

**Example 5.3** *Consider the system*

$$D^\alpha x(t) = -f(t)f^T(t)x(t)$$

*where* $\alpha \in (0,1]$ *and* $f, x : [0,\infty) \to \mathbb{R}^n$. *It was proven in [17] that if* $\alpha = 1$ *and* $f$ *is of persistent excitation, then the Lyapunov function* $V(x) = x^T x$ *decays exponentially. From Theorem 4.2, it follows that for the same* $f$ *and* $\alpha \in (0,1)$, $x$ *converges in Mittag-Leffler order to zero. This result improves [8] where only the convergence was established.*

**Example 5.4** *In view of Theorem 4.2* and/or *Proposition 4.1, it could be postulated that whenever an integer order system is asymptotically stable, so is its corresponding fractional system. However, a counterexample can be shown for time-varying systems. Consider the system*

$$D^\alpha x(t) = -f(t)x(t), \tag{5.6}$$

*where* $\alpha \in (0,1]$ *and* $f, x : [0,\infty) \to \mathbb{R}$. *It can be easily verified that* $x = 0$ *is a stable equilibrium point for* $\alpha \in (0,1]$ *at* $t = 0$.

*It was proven in [6] that there exists a bounded continuous nonnegative function* $f$, *such that* $I^\alpha f < C < +\infty$ *and* $\lim_{t\to\infty}\int_0^t f(\tau)d\tau = +\infty$. *Hence,* $\lim_{t\to\infty} x(t) = 0$ *when* $\alpha = 1$, *but* $\lim_{t\to\infty} x(t) \neq 0$ *when* $\alpha < 1$. *In fact, the condition* $\lim_{t\to\infty}[I^\alpha f](t) = +\infty$ *is necessary for asymptotic stability of* (5.6) *(see e.g., [9]).*

**Acknowledgment**

The first author thanks 'CONICYTPCHA/National PhD scholarship program, 2018'. The second author thanks CONICYT-Chile for the support under Grants FB0809, FONDECYT 1190959 and FONDECYT 1150488.

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