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Generalized helices in three-dimensional Lie groups

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Abstract: We introduce three types of helices in three-dimensional Lie groups with left-invariant metric and give their geometrical description similar to that of Lancret. We generalize the results known for the case of three-dimensional Lie groups with bi-invariant metric.

Key words: Slant helix, Lancret's theorem, curves in Lie groups

1. Introduction

Let γ be a C^3-regular naturally parameterized curve in the Euclidean space E^3. Denote by T, N, and B the standard Frenet frame of γ. The *generalized helix* can be defined in one of the following equivalent ways:

- T makes a constant angle with a fixed constant unit vector field on E^3;
- N is orthogonal to a fixed constant unit vector field on E^3;
- B makes a constant angle with a fixed constant unit vector field on E^3;
- the ratio of torsion $\varkappa$ and curvature k is constant (the Lancret theorem), i.e.

$$\frac{\varkappa}{k} = const.$$

The Euclidean space E^3 endowed with the usual cross-product belongs to the class of three-dimensional Lie groups G with left-invariant metric. The invariant unit vector field ξ on G is a natural analog of the constant unit vector field on E^3. It is natural to define three types of generalized helices in G by one of the first three conditions and characterize them in terms similar to the fourth one. In the case of three-dimensional Lie groups with *biinvariant* metric the problem was considered in [2] and [8]. The constant angle curve was defined by the property that the tangent vector field T makes a constant angle with a fixed invariant unit vector field ξ. As a result, in [2], the following assertion was proved :

Let γ be a parameterized curve in a three-dimensional Lie group with biinvariant metric. Denote by $\langle\cdot,\cdot\rangle$ the corresponding scalar product. The necessary and sufficient condition that there is a biinvariant unit

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vector field ξ *such that* $\langle T, \xi \rangle = const$ *is*

$$\frac{\varkappa - \varkappa_G}{k} = const, \tag{1.1}$$

where $\varkappa_G = \frac{1}{2}\langle [T,N], B\rangle$ *and* $[\ ,\ ]$ *is the Lie bracket.*

A *slant helix* was defined as a curve for which the principal normal vector field makes a *constant angle* with a fixed invariant direction [8] and the following assertion was proved:

Let γ *be a parameterized curve in a three-dimensional Lie group with biinvariant metric. The necessary and sufficient condition that there is biinvariant unit vector field* ξ *such that* $\langle N, \xi \rangle = const$ is

$$\frac{\varkappa (H^2+1)^{\frac{3}{2}}}{\dot{H}} = const, \tag{1.2}$$

where $H = \dfrac{\varkappa - \varkappa_G}{k}$ and $\dot{H} = \frac{dH}{ds}$.

Observe that only two of three dimensional Lie groups can be endowed with the biinvariant metric. In this paper we define three types of helices on 3-dimensional Lie groups with *left-invariant* metric and generalize the above-mentioned assertions. The main results are Theorems 2.4, 2.6, and 2.8.

2. Generalized helices in Lie groups with left-invariant metric

Let G be a three-dimensional Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$ and let $\mathfrak{g}$ denote the Lie algebra for G which consists of the all smooth vector fields of G invariant under left translation.

Definition 2.1 *Let* G *be a three-dimensional Lie group with left-invariant metric. Denote by* $\langle \cdot, \cdot \rangle$ *the corresponding scalar product. Let* γ *be a parameterized curve with the Frenet frame* T, N*, and* B*. The curve* γ *is called the generalized helix of the first, second, or third kind with axis* ξ *if there is a left-invariant along* γ *unit vector field* ξ *such that* $\langle T, \xi \rangle = const$*,* $\langle N, \xi \rangle = const$*, or* $\langle B, \xi \rangle = const$*, respectively.*

There are two classes of three-dimensional Lie groups: unimodular and nonunimodular. In the case of the unimodular group, there is a (positively oriented) orthonormal frame of left-invariant vector fields $\{e_1, e_2, e_3\}$ such that the brackets satisfy [7]

$$[e_1, e_2] = \lambda_3\, e_3, \quad [e_1, e_3] = \lambda_2\, e_2, \quad [e_2, e_3] = \lambda_1\, e_1.$$

The constants λ_i are called *structure constants*. The constants

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$$

are called *connection coefficients*. In the case of the nonunimodular group, there is an orthonormal frame $\{e_1, e_2, e_3\}$ such that [7]

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = -\beta e_2 + \delta e_3, \quad [e_2, e_3] = 0.$$

Using the Koszul formula we can easily find the covariant derivatives $\nabla_{e_i} e_j$ that can be put in the tables

∇	e_1	e_2	e_3
e_1	0	$\mu_1 e_3$	$-\mu_1 e_2$
e_2	$-\mu_2 e_3$	0	$\mu_2 e_1$
e_3	$\mu_3 e_2$	$-\mu_3 e_1$	0

and

∇	e_1	e_2	e_3
e_1	0	βe_3	$-\beta e_2$
e_2	$-\alpha e_2$	αe_1	0
e_3	$-\delta e_3$	0	δe_1

in unimodular and nonunimodular cases, respectively.

In the three-dimensional case one can naturally define the *cross-product* by $e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2$. We introduce the following affine transformation:

$$\mu(X) = \left[\begin{array}{ll} \mu_1 X^1 e_1 + \mu_2 X^2 e_2 + \mu_3 X^3 e_3 & \text{for unimodular group,} \\ \beta X^1 e_1 + \delta X^3 e_2 - \alpha X^2 e_3 & \text{for nonunimodular group.} \end{array} \right. \tag{2.1}$$

Then for both groups we have $\nabla_{e_i} e_k = \mu(e_i) \times e_k$ and hence

$$\nabla_X e_k = \mu(X) \times e_k \tag{2.2}$$

for arbitrary vector field X.

Let $\gamma(s)$ be a naturally parameterized curve on the group and $T = \dot\gamma$ be the unit tangent vector field. Using (2.2), for arbitrary vector field $\xi \circ \gamma$ we have

$$\nabla_T \xi = T^i \nabla_{e_i}(\xi^k e_k) = T^i (e_i(\xi^k)) e_k + \xi^k \nabla_T e_k =$$
$$T(\xi^k) e_k + \xi^k \mu(T) \times e_k) = \frac{d\xi^k}{ds} e_k + \mu(T) \times \xi = \dot\xi^k e_k + \mu(T) \times \xi. \tag{2.3}$$

In what follows, we call the vector field $\dot\xi = \frac{d\xi^i}{ds} e_i$ the *dot-derivative* of the vector field ξ along the curve γ. Observe that if ξ is left-invariant along γ, then $\dot\xi = 0$ and vice versa. Since the frame (e_1, e_3, e_3) is left-invariant, *the dot-derivative is subject to the usual Leibnitz rule with respect to scalar and cross-products*, i.e. $\dot{\langle \xi, \eta \rangle} = \langle \dot\xi, \eta \rangle + \langle \xi, \dot\eta \rangle$, $(\xi \dot\times \eta) = \dot\xi \times \eta + \xi \times \dot\eta$.

Let T, N, and B be the vectors of the standard Frenet frame of γ. Using (2.3), we get

$$\nabla_T T = \dot T + \mu(T) \times T, \quad \nabla_T B = \dot B + \mu(T) \times B, \quad \nabla_T N = \dot N + \mu(T) \times N.$$

Assuming $k_0 = \mid \dot T \mid \neq 0$, we can define a new frame $\{\tau, \nu, \beta\}$ along the curve γ by

$$\tau = T, \quad \nu = \frac{1}{k_0} \dot\tau, \quad \beta = \tau \times \nu. \tag{2.4}$$

In what follows we call (2.4) the *dot-Frenet frame.* Set $\varkappa_0 = |\dot\beta|$ by definition.

Proposition 2.2 *The dot-Frenet frame $\{\tau, \nu, \beta\}$ satisfies the dot-Frenet formulas, namely*

$$\dot\tau = k_0 \nu, \quad \dot\nu = -k_0 \tau + \varkappa_0 \beta, \quad \dot\beta = -\varkappa_0 \nu. \tag{2.5}$$

The proof is straightforward and trivial. In what follows we call k_0 and $\varkappa_0$ *dot-curvature* and *dot-torsion*, respectively.

The Frenet and the dot-Frenet frames are connected by

$$\tau = T, \quad \nu = \cos\alpha\, N + \sin\alpha\, B, \quad \beta = -\sin\alpha\, N + \cos\alpha\, B, \tag{2.6}$$

where $\alpha = \alpha(s)$ is the angle function. The inverse transformation is of the form

$$T = \tau, \quad N = \cos\alpha\,\nu - \sin\alpha\,\beta, \quad B = \sin\alpha\,\nu + \cos\alpha\,\beta. \tag{2.7}$$

Proposition 2.3 *The transformation $\mu(T)$ can be given by*

$$\mu(T) = (\varkappa + \dot\alpha - \varkappa_0)T + k_0\sin\alpha\, N + (k - k_0\cos\alpha)B \tag{2.8}$$

with respect to the Frenet frame $\{T, N, B\}$.

Proof The decomposition has evident form

$$\mu(T) = \big\langle \mu(T), T\big\rangle T + \big\langle \mu(T), N\big\rangle N + \big\langle \mu(T), B\big\rangle B.$$

The first Frenet formula and (2.5) yield

$$\nabla_T T = \dot T + \mu(T)\times T = k_0\nu + \mu(T)\times T = kN.$$

Multiplying this relation by N, we get $k = k_0\cos\alpha + \big\langle \mu(T)\times T, N\big\rangle$. Observe that $\big\langle \mu(T)\times T, N\big\rangle = \big\langle \mu(T), T\times N\big\rangle = \big\langle \mu(T), B\big\rangle$. Thus, we get $k = k_0\cos\alpha + \big\langle \mu(T), B\big\rangle$ and hence $\big\langle \mu(T), B\big\rangle = k - k_0\cos\alpha$.

By the second Frenet formula,

$$\nabla_T N = \dot N + \mu(T)\times N = -kT + \varkappa B.$$

Calculating the dot-derivative of N in decomposition (2.7) and applying (2.5), we find

$$\begin{aligned}\dot N = \dot\alpha(-\sin\alpha\nu - \cos\alpha\beta) + \cos\alpha(-k_0\tau + \varkappa_0\beta) - \sin\alpha(-\varkappa_0\nu) = \\ -\dot\alpha B + \varkappa_0(\beta\cos\alpha + \nu\sin\alpha) - k_0\tau\cos\alpha = (-\dot\alpha + \varkappa_0)B - k_0\cos\alpha\, T,\end{aligned} \tag{2.9}$$

so we have $(-\dot\alpha + \varkappa_0)B - k_0\cos\alpha\, T + \mu(T)\times N = -kT + \varkappa B$ and hence $\varkappa = -\dot\alpha + \varkappa_0 + \big\langle \mu(T)\times N, B\big\rangle = -\dot\alpha + \varkappa_0 + \big\langle \mu(T), T\big\rangle$. Thus, $\big\langle \mu(T), T\big\rangle = \varkappa + \dot\alpha - \varkappa_0$.

By the third Frenet formula,

$$\nabla_T B = \dot B + \mu(T)\times B = -\varkappa N.$$

Calculating the dot-derivative of B in decomposition (2.7) and applying (2.5), we find

$$\begin{aligned}\dot B = \dot\alpha(\cos\alpha\nu - \sin\alpha\beta) + \sin\alpha(-k_0\tau + \varkappa_0\beta) - \cos\alpha(-\varkappa_0\nu) = \\ -\dot\alpha N + \varkappa_0(\beta\sin\alpha - \nu\cos\alpha) - k_0\tau\sin\alpha = (\dot\alpha - \varkappa_0)N - k_0\sin\alpha T,\end{aligned}$$

so we have $-\varkappa N = (\dot\alpha - \varkappa_0)N - k_0 \sin\alpha\, T + \mu(T) \times B$ and hence

$$0 = -k_0 \sin\alpha + \big\langle \mu(T) \times B, T \big\rangle = -k_0 \sin\alpha + \big\langle \mu(T), N \big\rangle.$$

Thus, $\big\langle \mu(T), N \big\rangle = k_0 \sin\alpha$. Collecting the results, we get

$$\mu(T) = (\varkappa + \dot\alpha - \varkappa_0)T + k_0 \sin\alpha\, N + (k - k_0 \cos\alpha)B,$$

as was claimed. □

Define *a group-curvature* k_G and *a group-torsion* $\varkappa_G$ of a curve by

$$k_G = \mid \mu(T) \times T \mid, \quad \varkappa_G = \mid \mu(T) \times B \mid,$$

respectively. As a consequence of (2.8), the dot-curvature and the dot-torsion of a curve can be expressed in terms of the group-curvature k_G, the group-torsion $\varkappa_G$ of a curve, and the angle function α by

$$k_G^2 = (k - k_0)^2 + 4kk_0 \sin^2(\alpha/2), \quad \varkappa_G^2 = k_0^2 \sin\alpha^2 + (\varkappa - \varkappa_0 + \dot\alpha)^2.$$

Theorem 2.4 *The regular curve γ in three-dimensional group Lie G with left-invariant metric is the generalized helix of the first kind if and only if*

$$\frac{\varkappa_0}{k_0} = \cot\theta \quad (k_0 \neq 0),$$

where $\theta = const$.

Proof If $\gamma : I \subset \mathbb{R} \to G$ is a parameterized helix of the first kind, then there exists a unit invariant vector field ξ such that $\big\langle T, \xi \big\rangle = \cos\theta$, where $\theta = const$. Calculating the dot-derivative we find $\big\langle \dot T, \xi \big\rangle + \big\langle T, \dot\xi \big\rangle = 0$. As $\dot\xi = 0$, we get $k_0 \big\langle \nu, \xi \big\rangle = 0$. Hence, $\big\langle \nu, \xi \big\rangle = 0$. The next dot-derivative yields $\big\langle -k_0\tau + \varkappa_0\beta, \xi \big\rangle = 0$ or $-k_0 \cos\theta + \varkappa_0 \sin\theta = 0$ and hence $\dfrac{\varkappa_0}{k_0} = \cot\theta$.

Conversely, suppose that $\dfrac{\varkappa_0}{k_0} = \cot\theta = const$. Put $\xi = \cos\theta T + \sin\theta B$. Evidently, $\big\langle T, \xi \big\rangle = \cos\theta$. Then $\dot\xi = k_0\nu\cos\theta + (-\varkappa_0\nu\sin\theta) = \nu(k_0\cos\theta - \varkappa_0\sin\theta) = 0$ and hence ξ is left-invariant. □

Remark 2.5 *If the metric is biinvariant, then $\mu_1 = \mu_2 = \mu_3 := \mu$ and hence $\mu(T) = \mu T$. As a consequence, $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$, and we get* (1.1).

Theorem 2.6 *A regular curve γ in three-dimensional group Lie G with left-invariant metric is the generalized helix of the second kind if and only if*

$$\frac{k_0 \cos\alpha (H^2+1)^{\frac{3}{2}}}{\dot H - k_0 \sin\alpha (H^2+1)} = \tan\theta,$$

where $H = \dfrac{\varkappa_0 - \dot\alpha}{k_0 \cos\alpha}$.

Proof If $\gamma : I \subset \mathbb{R} \to G$ is a parameterized helix of the second kind, then there is a unit invariant vector field ξ such that $\langle N, \xi\rangle = \cos\theta$, where $\theta = const$. As $\dot{\xi} = 0$, then using (2.7) and dot-Frenet formulas we get

$$(-\dot{\alpha} + \varkappa_0)\langle B, \xi\rangle - k_0 \cos\alpha\langle T, \xi\rangle = 0.$$

Hence,

$$\langle T, \xi\rangle = \frac{\varkappa_0 - \dot{\alpha}}{k_0 \cos\alpha}\langle B, \xi\rangle = H\langle B, \xi\rangle, \tag{2.10}$$

where $H = \dfrac{\varkappa_0 - \dot{\alpha}}{k_0 \cos\alpha}$. Therefore, $\langle \dot{T}, \xi\rangle = \dot{H}\langle B, \xi\rangle + H\langle \dot{B}, \xi\rangle$. Formulas (2.5) and (2.7) imply $k_0\langle \nu, \xi\rangle = \dot{H}\langle B, \xi\rangle + H\langle N(\dot{\alpha} - \varkappa_0) - k_0 T \sin\alpha, \xi\rangle$. Using (2.6) we get

$$k_0\langle N\cos\alpha + B\sin\alpha, \xi\rangle = \dot{H}\langle B, \xi\rangle + H(\dot{\alpha} - \varkappa_0)\langle N, \xi\rangle - k_0 H \sin\alpha\langle T, \xi\rangle.$$

By replacing $\langle T, \xi\rangle$ in accordance with (2.10) we get

$$k_0 \cos\alpha\cos\theta + k_0 \sin\alpha\langle B, \xi\rangle = \dot{H}\langle B, \xi\rangle - H^2 k_0 \cos\alpha\cos\theta - k_0 \sin\alpha H^2\langle B, \xi\rangle.$$

Thus, $k_0 \cos\alpha\cos\theta(1 + H^2) = \langle B, \xi\rangle(\dot{H} - k_0 H^2 \sin\alpha - k_0 \sin\alpha)$. It means that

$$\langle B, \xi\rangle = \frac{k_0 \cos\alpha\cos\theta(1 + H^2)}{\dot{H} - k_0 \sin\alpha(1 + H^2)}. \tag{2.11}$$

In combination with (2.10) we get

$$\xi = \left(\frac{H k_0 \cos\alpha(1 + H^2)}{\dot{H} - k_0 \sin\alpha(1 + H^2)} T + N + \frac{k_0 \cos\alpha(1 + H^2)}{\dot{H} - k_0 \sin\alpha(1 + H^2)} B\right)\cos\theta.$$

Since $|\xi| = 1$,

$$\left(H^2 \frac{k_0^2 \cos^2\alpha(1 + H^2)^2}{(\dot{H} - k_0 \sin\alpha(1 + H^2))^2} + 1 + \frac{k_0^2 \cos^2\alpha(1 + H^2)^2}{(\dot{H} - k_0 \sin\alpha(1 + H^2))^2}\right) = \frac{1}{\cos^2\theta}.$$

After simple transformation we get

$$\frac{k_0 \cos\alpha(H^2 + 1)^{\frac{3}{2}}}{\dot{H} - k_0 \sin\alpha(H^2 + 1)} = \tan\theta, \tag{2.12}$$

as was required. Moreover, (2.12) implies

$$\xi = \left(\frac{H}{\sqrt{1 + H^2}}\sin\theta T + \cos\theta N + \frac{1}{\sqrt{1 + H^2}}\sin\theta B\right). \tag{2.13}$$

Conversely, take ξ given by (2.13) with $\theta = const$ and suppose (2.12) is fulfilled. Then

$$(a)\ \ \langle \xi, N\rangle = \cos\theta; \quad (b)\ \ \langle \xi, B\rangle = \frac{1}{\sqrt{1 + H^2}}\sin\theta; \quad (c)\ \ \langle \xi, T\rangle = \frac{H}{\sqrt{1 + H^2}}\sin\theta.$$

The dot-derivative of (a) yields $\langle\dot{\xi}, N\rangle + \langle\xi, \dot{N}\rangle = 0$. Using (2.7), we obtain

$$\langle\dot{\xi}, N\rangle + (-\dot{\alpha} + \varkappa_0)\langle\xi, B\rangle - k_0\cos\alpha\langle\xi, T\rangle = 0$$

or

$$\langle\dot{\xi}, N\rangle + k_0\cos\alpha\left(H\langle\xi, B\rangle - \langle\xi, T\rangle\right) = 0.$$

From (2.10) it follows that $\langle\dot{\xi}, N\rangle = 0$.

By using (2.7) and (2.12), the dot-derivative of (b) yields

$$\langle\dot{\xi}, B\rangle = \frac{d}{ds}\langle\xi, B\rangle - \langle\xi, \dot{B}\rangle = \frac{d}{ds}\left(\frac{1}{\sqrt{1+H^2}}\sin\theta\right) + k_0\sin\alpha\langle T, \xi\rangle + Hk_0\cos\alpha\langle N, \xi\rangle =$$
$$-\frac{\dot{H}H}{(1+H^2)^{\frac{3}{2}}}\sin\theta + k_0\sin\alpha\langle T, \xi\rangle + Hk_0\cos\alpha\cos\theta.$$

We can express $\dot{H}$ from (2.11) and then, by using (c), we continue with

$$\langle\dot{\xi}, B\rangle = -\frac{H}{(1+H^2)^{\frac{3}{2}}}\sin\theta\left(\frac{k_0\cos\alpha(1+H^2)^{\frac{3}{2}}}{\tan\theta} + k_0\sin\alpha(1+H^2)\right) + k_0\sin\alpha\langle T, \xi\rangle + Hk_0\cos\alpha\cos\theta =$$
$$\left(-Hk_0\cos\alpha\cos\theta - \frac{k_0 H\sin\alpha\sin\theta}{\sqrt{1+H^2}}\right) + \frac{k_0 H\sin\alpha\sin\theta}{\sqrt{1+H^2}} + Hk_0\cos\alpha\cos\theta = 0.$$

In a similar way, by using (2.6), (2.7), and (2.12), we get

$$\langle\dot{\xi}, T\rangle = \frac{d}{ds}\langle\xi, T\rangle - \langle\xi, \dot{T}\rangle = \frac{d}{ds}\left(\frac{H}{\sqrt{1+H^2}}\sin\theta\right) + k_0\langle\cos\alpha N + \sin\alpha B, \xi\rangle =$$
$$\left(-\dot{H}\frac{H^2}{(1+H^2)^{\frac{3}{2}}} + \frac{\dot{H}}{(1+H^2)^{\frac{1}{2}}}\right)\sin\theta - k_0\cos\alpha\cos\theta - k_0\sin\alpha\sin\theta\frac{1}{\sqrt{1+H^2}}.$$

Again, we can express $\dot{H}$ from (2.12) and continue with

$$\langle\dot{\xi}, T\rangle = \sin\theta\frac{1}{(1+H^2)^{\frac{3}{2}}}\left(\frac{k_0\cos\alpha(1+H^2)^{\frac{3}{2}}}{\tan\theta} + k_0\sin\alpha(1+H^2)\right) -$$
$$k_0\cos\alpha\cos\theta - k_0\sin\alpha\sin\theta\frac{1}{\sqrt{1+H^2}} = \left(-k_0\cos\alpha\cos\theta - \frac{k_0\sin\alpha\sin\theta}{\sqrt{1+H^2}}\right)$$
$$- k_0\cos\alpha\cos\theta - k_0\sin\alpha\sin\theta\frac{1}{\sqrt{1+H^2}} = 0.$$

Since $\langle\dot{\xi}, N\rangle = 0$, $\langle\dot{\xi}, B\rangle = 0$, and $\langle\dot{\xi}, T\rangle = 0$, we have $\dot{\xi} = 0$ and hence ξ is left-invariant.

□

Remark 2.7 *If the metric is biinvariant, then* $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$, *and we get* (1.2).

Theorem 2.8 *A regular curve γ in three-dimensional group Lie G with left-invariant metric is a generalized helix of the third kind if and only if*

$$\frac{k_0 \sin\alpha (Q^2+1)^{\frac{3}{2}}}{\dot{Q} - k_0 \cos\alpha (Q^2+1)} = \tan\theta,$$

where $Q = \dfrac{\dot{\alpha} - \varkappa_0}{k_0 \sin\alpha} = -H \cot\alpha$.

Proof If $\gamma : I \subset \mathbb{R} \to G$ is a parameterized helix of the third kind, then there exists a unit invariant vector field ξ such that $\langle B, \xi \rangle = \cos\theta$, where θ is constant. Since $\dot{\xi} = 0$, we have $(\dot{\alpha} - \varkappa_0)\langle N, \xi\rangle - k_0 \sin\alpha \langle T, \xi \rangle = 0$, so

$$\langle T, \xi \rangle = \frac{\dot{\alpha} - \varkappa_0}{k_0 \sin\alpha} \langle N, \xi \rangle = Q \langle N, \xi \rangle, \tag{2.14}$$

where we put $Q = \dfrac{\dot{\alpha} - \varkappa_0}{k_0 \sin\alpha}$. Calculating the dot-derivative, we get

$$\langle \dot{T}, \xi \rangle = \dot{Q} \langle N, \xi \rangle + Q \langle \dot{N}, \xi \rangle.$$

Applying (2.5) and (2.9), we continue with

$$k_0 \langle \nu, \xi \rangle = \dot{Q} \langle N, \xi \rangle + Q(\varkappa_0 - \dot{\alpha}) \langle B, \xi \rangle - Q k_0 \cos\alpha \langle T, \xi \rangle.$$

Using (2.6), we find

$$k_0 \langle N \cos\alpha + B \sin\alpha, \xi \rangle = \dot{Q} \langle N, \xi \rangle + Q \langle N(\varkappa_0 - \dot{\alpha}) - k_0 T \cos\alpha, \xi \rangle.$$

Replacing $\langle T, \xi \rangle$ by (2.14), we continue with

$$k_0 \cos\alpha \langle N, \xi \rangle + k_0 \sin\alpha \cos\theta = \dot{Q} \langle N, \xi \rangle - Q^2 k_0 \cos\theta \sin\alpha - Q^2 k_0 \cos\alpha \langle T, \xi \rangle.$$

Hence,

$$k_0 \cos\theta \sin\alpha (1 + Q^2) = \langle N, \xi \rangle (\dot{Q} - k_0 Q^2 \cos\alpha - k_0 \cos\alpha).$$

Thus,

$$\langle N, \xi \rangle = \frac{k_0 \sin\alpha (1+Q^2)}{\dot{Q} - k_0 \cos\alpha (1+Q^2)}.$$

Since $\langle T, \xi \rangle = Q \langle N, \xi \rangle$, we get

$$\xi = \left(Q \frac{k_0 \sin\alpha (1+Q^2)}{(\dot{Q} - k_0 \cos\alpha (1+Q^2))} T + \frac{k_0 \sin\alpha (1+Q^2)}{(\dot{Q} - k_0 \cos\alpha (1+Q^2))} N + B \right) \cos\theta.$$

The condition $|\xi| = 1$ implies

$$\left(Q^2 \frac{k_0^2 \sin^2\alpha (1+Q^2)^2}{(\dot{Q} - k_0 \cos\alpha (1+Q^2))^2} + 1 + \frac{k_0^2 \sin^2\alpha (1+Q^2)^2}{(\dot{Q} - k_0 \cos\alpha (1+H^2))^2} \right) = \frac{1}{\cos^2\theta}.$$

After transformations we get

$$\frac{k_0 \sin\alpha (Q^2+1)^{\frac{3}{2}}}{\dot{Q} - k_0 \cos\alpha (Q^2+1)} = \tan\theta, \tag{2.15}$$

as was claimed.

By using (2.15) we can decompose ξ as follows:

$$\xi = \left(\frac{Q}{\sqrt{1+Q^2}} \sin\theta T + \frac{1}{\sqrt{1+Q^2}} \sin\theta N + \cos\theta B \right). \tag{2.16}$$

Conversely, take the vector field given by (2.16) with $\theta = const$ and suppose (2.15) is fulfilled. By the same procedure as in the proof of Theorem 2.6, one can check that ξ is left-invariant and $\langle B, \xi \rangle = \cos\theta$. □

3. Conclusion

We have defined three classes of slant helices in three-dimensional Lie groups with left invariant metric and obtained their description in terms of new geometric invariants of the curve. The results generalize the corresponding descriptions for helices in Euclidean 3-space and in 3-dimensional Lie groups with biinvariant metric.

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