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A fully Hadamard and Erdélyi–Kober-type integral boundary value problem of a coupled system of implicit differential equations

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Abstract: In this article, we give sufficient conditions for the existence of solutions for a new coupled system of second-order implicit differential equations with Hadamard and Erdélyi–Kober fractional integral boundary conditions and nonlocal conditions at the boundaries in Banach space. The main result is based on a Mönch fixed point theorem combined with the measure of noncompactness of Kuratowski; an example is given to illustrate our approach.

Key words: Hadamard fractional derivative, Erdélyi–Kober fractional integral, Caputo fractional derivative, measure of noncompactness of Kuratowski, Mönch fixed point theorem, coupled system of differential equations

1. Introduction

Fractional differential equations have large applications in a variety of fields such as electrical networks, signal and image processing, viscoelasticity, aerodynamics, economics, and so on, and hence has increased more attention from both theoretical and applied points of view in recent years (for further details see [12, 16]).

We note here that most of the work on the topic in the literature is based on Riemann–Liouville- and Caputo-type fractional differential equations; for this, we refer the readers to [1, 5, 6, 10, 11]. Another kind of fractional derivative that appears side by side to Riemann–Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [13], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of the Hadamard derivative) contains a logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [7–9, 15, 17, 20].

Ahmad et al. [3] considered a fully Hadamard-type integral boundary value problem of a coupled system of fractional differential equations,

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & 1 < t < e,\ 1 < \alpha \leq 2, \\ D^{\alpha}v(t) = g(t, u(t), v(t)), & 1 < t < e,\ 1 < \beta \leq 2, \\ u(1) = 0, \quad u(e) = {}_H I^{\gamma} u(\sigma_1) = \frac{1}{\Gamma(\gamma)} \int_1^{\sigma_1} \left(\log \frac{\sigma_1}{s}\right)^{\gamma-1} u(s) \frac{ds}{s}, \\ v(1) = 0, \quad v(e) = {}_H I^{\gamma} v(\sigma_2) = \frac{1}{\Gamma(\gamma)} \int_1^{\sigma_2} \left(\log \frac{\sigma_2 1}{s}\right)^{\gamma-1} v(s) \frac{ds}{s}, \end{cases}$$

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where $\gamma > 0$, $1 < \sigma_1 < e$, $1 < \sigma_2 < e$, $D^{(\cdot)}$ is the Hadamard fractional derivative of fractional order $(\cdot)$, I^γ is the Hadamard fractional integral of order γ, and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The main results are based on the classical Leray–Schauder alternative for the existence of solutions, whereas the uniqueness of a solution is established by Banach's contraction principle.

Motivated by the Ahmad paper, but reasoning quite differently, we consider a new problem which deals with a system of implicit differential equations with Hadamard and Erdélyi–Kober fractional integral boundary conditions and nonlocal conditions at the right endpoint and left endpoint of the interval, respectively,

$$\begin{cases} x''(t) = f(t, {}^c D^\alpha x(t), y(t)), & t \in J = [0, T],\ 1 < \alpha \le 2, \\ y''(t) = g(t, {}^c D^\beta y(t), x(t)), & t \in J = [0, T],\ 1 < \beta \le 2, \\ x(0) = \varphi_1(x), \qquad y(0) = \varphi_2(y), & \\ x(T) = \alpha_1\ {}_H I^\gamma y(\zeta) + \beta_1 I_\eta^{\epsilon,\lambda} x(\zeta), & \\ y(T) = \alpha_2\ {}_H I^\gamma y(\omega) + \beta_2 I_\eta^{\epsilon,\lambda} x(\omega), & \end{cases} \tag{1}$$

where ${}^cD^\alpha$ and ${}^cD^\beta$ are the Caputo fractional derivatives of orders α and β, respectively, $f, g : [0, T] \times E \times E \to E$, E is a real Banach space, $\varphi_1, \varphi_2 : C(J, E) \to E$ are given functions, $\alpha_1, \alpha_2, \beta_1$ and β_2 are real numbers, $I_\eta^{\epsilon,\lambda}$ denotes the Erdélyi–Kober fractional integral of order $\lambda > 0$, $\eta > 0$ and $\epsilon \in \mathbb{R}$. ${}_H I^\gamma$ denotes the Hadamard fractional integral of order $\gamma > 0$, and $\zeta, \omega \in (0, T)$.

The paper is organized as follows: In Section 2, we present some useful preliminaries and lemmas. Section 3 deals with the existence result for problem (1) which is obtained via the concept of Mönch's fixed theorem combined with the measure of noncompactness, and in Section 4, we give an example to illustrate our main approach.

2. Preliminaries

In this section, we recall some basic definitions.

Let E be a Banach space with the norm $|\cdot|$ and $C(J; E)$ be the Banach space of continuous functions y mapping J into E with the usual norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable.

Let $L^1(J; E)$ be the space of E-valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)| dt.$$

For $1 < \alpha, \beta \le 2$, we define spaces

$$\Delta = \{x \in C(J, E) \mid {}^cD^\alpha x \in C(J, E)\},$$

and

$$\bar{\Delta} = \left\{y \in C(J, E) \mid {}^cD^\beta y \in C(J, E)\right\},$$

equipped, respectively, with norms

$$\|x\|_\Delta = \|x\| + \|{}^cD^\alpha x\| = \sup_{t \in J} |x(t)| + \sup_{t \in J} |{}^cD^\alpha x(t)|,$$

$$\|y\|_{\bar{\Delta}} = \|y\| + \|{}^cD^\beta y\| = \sup_{t\in J}|y(t)| + \sup_{t\in J}|{}^cD^\beta y(t)|.$$

Obviously $(\Delta; \|.\|_\Delta)$ and $(\bar{\Delta}; \|.\|_{\bar{\Delta}})$ are Banach spaces. The product space $(\Delta \times \bar{\Delta}, \|(x,y)\|)$ is also a Banach space with norm

$$\|(x,y)\| = \|x\|_\Delta + \|y\|_{\bar{\Delta}} \qquad for\ all\ (x,y) \in \Delta \times \bar{\Delta}.$$

Definition 1 *Let $p, q > 0$, then the Beta function $\mathrm{B}(p,q)$ is defined as*

$$\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Remark 2 *For $p, q > 0$, the following identity holds,*

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where Γ is Gamma function defined by

$$\Gamma(q) = \int_0^\infty e^{-s}s^{q-1}ds.$$

Definition 3 *[2] Caputo's derivative of order α for a function $f : [0,\infty) \to E$ is defined as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

for $n-1 < \alpha < n$; $n \in \mathbb{N}$, and if $0 < \alpha \leq 1$, then

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f^{(1)}(s)ds.$$

Definition 4 *[3] The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$ is defined as*

$${}_HI^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s},$$

provided the integral exists.

Definition 5 *[18] The Erdélyi–Kober fractional integral of order $\lambda > 0$, with $\eta > 0$ and $\epsilon \in \mathbb{R}$, of a continuous function $f : (0,\infty) \to E$ is defined by*

$$I_\eta^{\epsilon,\lambda} f(t) = \frac{\eta t^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda)}\int_0^t \frac{s^{\eta\epsilon+\eta-1}}{(t^\eta - s^\eta)^{1-\lambda}}f(s)ds,$$

provided the right side is pointwise defined on E.

Remark 6 *For $\eta = 1$, the above operator is reduced to the Kober operator*

$$I_1^{\epsilon,\lambda} f(t) = \frac{t^{-(\lambda+\epsilon)}}{\Gamma(\lambda)} \int_0^t \frac{s^\epsilon}{(t-s)^{1-\lambda}} f(s) ds, \qquad \epsilon, \lambda > 0.$$

For $\epsilon = 0$, the Kober operator is reduced to the Riemann–Liouville fractional integral with a power weight,

$$I_1^{0,\lambda} f(t) = \frac{t^{-\lambda}}{\Gamma(\lambda)} \int_0^t \frac{f(s)}{(t-s)^{1-\lambda}} ds, \qquad \lambda > 0.$$

We now define the Kuratowski measure of noncompactness.

Definition 7 *[14]. Let M be a metric space and X be a subset of M. The Kuratowski measure of noncompactness $\nu(X)$ of the set X is defined as*

$$\nu(X) := \inf \left\{ \epsilon > 0 \ : \quad X \subseteq \bigcup_{i=1}^{n} B_i \ \text{ and } \ diam(B_i) \leq \epsilon \right\}.$$

where $diam(B_i) = \sup \{\|x-y\|; \quad x, y \in B_i\}$.

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 8 *[14] Let A and B be bounded sets of a Banach space X.*

1. *$\nu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*
2. *ν is equal to zero on every singleton set.*
3. *$\nu(B) = \nu(\overline{B}) = \nu(convB)$; where $convB$ is the convex hull of B.*
4. *$A \subset B \Longrightarrow \nu(A) \leq \nu(B)$.*
5. *$\nu(A+B) \leq \nu(A) + \nu(B)$ where $A + B = \{x + y : \quad x \in A, \quad y \in B\}$.*
6. *$\nu(\lambda B) = |\lambda|\nu(B)$, $\lambda \in \mathbb{R}$, where $\lambda B = \{\lambda x \ : \ x \in B\}$.*
7. *$\nu(A \cup B) = \max\{\nu(A), \nu(B)\}$.*
8. *$\nu(B + x_0) = \nu(B)$ for $x_0 \in X$.*

We use ν, ν_C, and $\widetilde{\nu}$ to denote the Kuratowski noncompactness measure of bounded sets in the spaces E, $C(J,E)$, and $E \times E$, respectively. The next few results involving the Kuratowski measure of noncompactness and the Mönch fixed point theorem are needed for our discussion.

Theorem 9 *(Mönch fixed point theorem) [19] Let D be a bounded, closed, and convex subset of a Banach X space such that $0 \in D$, and let T be a continuous mapping of D into itself. If the implication*

$$V = \bar{conv}T(V), \qquad or \qquad V = T(V) \cup \{0\} \Rightarrow \nu(V) = 0,$$

holds for every subset V of D, then T has a fixed point.

Lemma 10 *[14] If $V \subset C(J,E)$ is a bounded and equicontinuous set, then*

i) the function $t \to \nu(V(t))$ is continuous on J and

$$\nu_c(V) = \sup_{1 \leq t \leq T} \nu(V(t)),$$

ii)

$$\nu\left(\int_1^T x(s)ds : \quad x \in V\right) \leq \int_1^T \nu(V(s))ds,$$

where $V(s) = \{x(s) : \ x \in V\}$, $s \in J$.

Theorem 11 *[4] Suppose that $\nu_1, \nu_2, \nu_3, \ldots, \nu_n$ are the measures of noncompactness in $E_1, E_2, \ldots, E_n$, a sequence of Banach spaces, respectively. Moreover assume that the function $F : [0,+\infty)^n \to [0,+\infty)$ is convex and $F(x_1, x_2 \ldots x_n) = 0$ if and only if $x_i = 0$ for $i = 1, \ldots, n$. Then*

$$\widetilde{\nu}(V) = F(\nu(V_1), \nu(V_2), \ldots, \nu(V_n)),$$

defines a measure of noncompactness on $E_1 \times E_2 \times \cdots \times E_n$ where V_i denotes the natural projection of V onto E_i for $i = 1, \ldots, n$.

Example 12 *[4] Let ν be a measure of noncompactness. We define $F(x,y) = x + y$ for any $x, y \in [0,+\infty)$. Then F has all the properties mentioned in Theorem 11. Hence $\widetilde{\nu}(V) = \nu(V_1) + \nu(V_2)$ is a measure of noncompactness in the space $E \times E$ where V_i, $i = 1, 2$ denote the natural projections of V.*

3. Existence results

In this section, conditions are given for the existence of solutions of (1), for which Theorem 9 is applied.

Definition 13 A pair of functions $(x,y) \in \Delta \times \bar{\Delta}$, whose α-derivative exists on J, is said to be a solution of (1) if x and y satisfy the equations,

$$\begin{cases} x''(t) = f(t, {}^c D^{\alpha} x(t), y(t)) \\ y''(t) = g(t, {}^c D^{\beta} y(t), x(t)) \end{cases},$$

on J, and also satisfy the conditions,

$$\begin{cases} x(0) = \varphi_1(x), \qquad y(0) = \varphi_2(y), \\ x(T) = \alpha_1 \ {}_H I^{\gamma} y(\zeta) + \beta_1 I_{\eta}^{\epsilon,\lambda} x(\zeta), \\ y(T) = \alpha_2 \ {}_H I^{\gamma} y(\omega) + \beta_2 I_{\eta}^{\epsilon,\lambda} x(\omega). \end{cases}$$

To prove the existence of solutions to (1), we need the following auxiliary lemmas.

Lemma 14 *[2] Let $\alpha > 0$ and $n > 0$, then the following formulas hold*

$$\left({}_H I^{\alpha} s^n\right)(t) = n^{-\alpha} t^n \qquad \text{and} \qquad \left({}_H D^{\alpha} s^n\right)(t) = n^{\alpha} t^n.$$

Lemma 15 *Let $\eta, \lambda > 0$ and $\epsilon, q \in \mathbb{R}$, then we have*

$$I_\eta^{\epsilon,\lambda} t^q = \frac{t^q \Gamma\left(\epsilon + (\frac{q}{\eta}) + 1\right)}{\Gamma\left(\epsilon + (\frac{q}{\eta}) + \lambda + 1\right)}.$$

Proof By Definitions 1 and 5, we have

$$I_\eta^{\epsilon,\lambda} t^q = \frac{\eta t^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda)} \int_0^t \frac{s^{\eta\epsilon+\eta-1}}{(t^\eta - s^\eta)^{1-\lambda}} s^q ds.$$

If $u = \frac{s}{t}$, then $ds = tdu$ and $(0 < s < t \Rightarrow 0 < \frac{s}{t} < 1)$

$$\begin{aligned} \frac{\eta t^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda)} \int_0^1 \frac{t t^{\eta\epsilon+\eta+q-1} u^{\eta\epsilon+\eta+q-1}}{(t^\eta - t^\eta u^\eta)^{1-\lambda}} du &= \frac{\eta t^{-\eta(\lambda+\epsilon)} t^{\eta\epsilon+\eta+q}}{\Gamma(\lambda) t^{\eta(1-\lambda)}} \int_0^1 \frac{u^{\eta\epsilon+\eta+q-1}}{(1-u^\eta)^{1-\lambda}} du \\ &= \frac{\eta t^q}{\Gamma(\lambda)} \int_0^1 u^{\eta(\epsilon+1)+q-1} (1-u^\eta)^{\lambda-1} du. \end{aligned}$$

Let $u^\eta = w \Rightarrow \eta u^{\eta-1} du = dw$, then $du = \frac{dw}{\eta u^{\eta-1}} = \frac{dw}{\eta w^{\frac{\eta-1}{\eta}}}$, and

$$\begin{aligned} \frac{\eta t^q}{\Gamma(\lambda)} \int_0^1 \frac{1}{\eta} w^{\frac{\eta(\epsilon+1)+q-1}{\eta}} (1-w)^{\lambda-1} w^{\frac{1-\eta}{\eta}} dw &= \frac{t^q}{\Gamma(\lambda)} \int_0^1 (1-w)^{\lambda-1} w^{\frac{\eta\epsilon+\eta+q-1+1-\eta}{\eta}} dw \\ &= \frac{t^q}{\Gamma(\lambda)} \int_0^1 (1-w)^{\lambda-1} w^{\epsilon+\frac{q}{\eta}} dw \\ &= \frac{t^q}{\Gamma(\lambda)} \beta(\epsilon + \frac{q}{\eta} + 1, \lambda) \\ &= \frac{t^q}{\Gamma(\lambda)} \frac{\Gamma(\epsilon + \frac{q}{\eta} + 1)\Gamma(\lambda)}{\Gamma(\epsilon + \frac{q}{\eta} + 1 + \lambda)} \\ &= \frac{t^q \Gamma(\epsilon + \frac{q}{\eta} + 1)}{\Gamma(\epsilon + \frac{q}{\eta} + 1 + \lambda)}. \end{aligned}$$

The proof is complete. □

Lemma 16 *Let $1 < \alpha \leq 2$ and let $z : J \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$\begin{aligned} x(t) = \int_0^t (t-s) z(s) ds &+ \left(1 + t \frac{\beta\Gamma(\epsilon+1) - \Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left[T - \beta \frac{\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right]} \right) \varphi_1(x) \\ &+ \frac{\beta\eta\zeta^{-\eta(\lambda+\epsilon)} t}{\Gamma(\lambda)\left[T - \beta \frac{\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right]} \int_0^\zeta \int_0^r \frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^\eta - r^\eta)^{1-\lambda}} z(s) ds dr \qquad (2) \\ &- \frac{t}{\left[T - \beta \frac{\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right]} \int_0^T (T-s) z(s) ds, \end{aligned}$$

if and only if x is a solution of the fractional BVP

$$\begin{cases} x''(t) = z(t), & 0 < t < T, \\ x(0) = \varphi_1(x), & x(T) = \beta I_\eta^{\epsilon,\lambda} x(\zeta), \end{cases} \tag{3}$$

where

$$T - \beta \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \neq 0.$$

Proof First, if $x(t)$ is a solution of (3), then

$$x(t) = \int_0^t (t-s) z(s) ds + c_0 + c_1 t,$$

for some constants $c_0, c_1 \in \mathbb{R}$. Using the boundary conditions, we obtain

$$x(0) = c_0 \Rightarrow c_0 = \varphi_1(x)$$

$$x(T) = \int_0^T (T-s) z(s) ds + \varphi_1(x) + c_1 T,$$

and we have

$$x(T) = \beta I_\eta^{\epsilon,\lambda} x(\zeta),$$

$$x(\zeta) = \int_0^\zeta (\zeta - s) z(s) ds + \varphi_1(x) + c_1 \zeta,$$

$$\begin{aligned} \beta I_\eta^{\epsilon,\lambda} x(\zeta) &= \beta \frac{\eta \zeta^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda)} \int_0^\zeta \int_0^r \frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^\eta - r^\eta)^{1-\lambda}} z(s) ds dr \\ &\quad + \beta \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+\lambda+1)} \varphi_1(x) + \beta c_1 \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \\ &= x(T). \end{aligned}$$

Then

$$\begin{aligned} c_1 \left(T - \beta \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \right) &= \frac{\beta \eta \zeta^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda)} \int_0^\zeta \int_0^r \frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^\eta - r^\eta)^{1-\lambda}} z(s) ds dr \\ &\quad + \left(\beta \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} - 1 \right) \varphi_1(x) - \int_0^T (T-s) z(s) ds \\ &= \frac{\beta \eta \zeta^{-\eta(\lambda+\epsilon)}}{\Gamma(\lambda) \left[T - \beta \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \right]} \int_0^\zeta \int_0^r \frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^\eta - r^\eta)^{1-\lambda}} z(s) ds dr \end{aligned}$$

$$\left(\frac{\beta \Gamma(\epsilon+1) - \Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda) \left[T - \beta \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \right]} \right) \varphi_1(x) - \frac{1}{\left[T - \beta \frac{\zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)} \right]} \int_0^T (T-s) z(s) ds.$$

Hence, we get (2).

Conversely, assume that x satisfies the fractional integral equation (2). We have immediately that $x(0)=\varphi_1(x)$, and using Lemma 15 and Definition 5, we deduce easily (3). This completes the proof. □

Lemma 17 *For $\bar{z}\in C(J,E)$, y is solution of the integral equation*

$$\begin{aligned} y(t) &= \int_0^t (t-s)\bar{z}(s)ds + \left(1-\frac{t}{T-\alpha\omega}\right)\varphi_2(y) \\ &+ \frac{\alpha t}{\Gamma(\gamma)(T-\alpha\omega)}\int_0^\omega\int_0^r \left(\log\frac{\omega}{r}\right)^{\gamma-1}(r-s)\frac{\bar{z}(s)}{r}dsdr \\ &- \frac{t}{T-\alpha\omega}\int_0^T (T-s)\bar{z}(s)ds, \end{aligned} \tag{4}$$

if and only if y is a solution of the fractional BVP

$$\begin{cases} y''(t)=\bar{z}(t), & 0<t<T, \\ y(0)=\varphi_2(y), & y(T)=\alpha\ {}_H I^\gamma y(\omega), \end{cases} \tag{5}$$

where

$$T-\alpha\omega\neq 0.$$

Proof First, suppose $y(t)$ is a solution of (5). Then,

$$y(t)=\int_0^t (t-s)\bar{z}(s)ds + c_0 + c_1 t, \tag{6}$$

for some constants $c_0, c_1\in\mathbb{R}$. We obtain

$$y(0)=c_0 \Rightarrow c_0=\varphi_2(y),$$

$$y(T)=\int_0^T (T-s)\bar{z}(s)ds + \varphi_2(y) + c_1 T,$$

and from (5),

$$y(T)=\alpha\ {}_H I^\gamma y(\omega).$$

Thus, we have from (6)

$$y(\omega)=\int_0^\omega (\omega-s)\bar{z}(s)ds + c_0 + c_1\omega,$$

and so we have from Definition 4,

$$\begin{aligned} \alpha\ {}_H I^\gamma y(\omega) &= \alpha\left[\frac{1}{\Gamma(\gamma)}\int_0^\omega\int_0^r\left(\log\frac{\omega}{r}\right)^{\gamma-1}(r-s)\frac{\bar{z}(s)}{r}dsdr + c_1\omega\right] \\ &= y(T). \end{aligned}$$

Then

$$c_1(T-\alpha\omega) = \frac{\alpha}{\Gamma(\gamma)}\int_0^\omega\int_0^r \left(\log\frac{\omega}{r}\right)^{\gamma-1}(r-s)\frac{\bar{z}(s)}{r}dsdr$$
$$-\varphi_2(y) - \int_0^T (T-s)\bar{z}(s)ds,$$

or

$$c_1 = \frac{\alpha}{\Gamma(\gamma)(T-\alpha\omega)}\int_0^\omega\int_0^r \left(\log\frac{\omega}{r}\right)^{\gamma-1}(r-s)\frac{\bar{z}(s)}{r}dsdr$$
$$-\frac{1}{T-\alpha\omega}\varphi_2(y) - \frac{1}{T-\alpha\omega}\int_0^T (T-s)\bar{z}(s)ds.$$

Hence, we get (4).

Conversely, assume that y satisfies the fractional integral equation (4). We have immediately $y(0) = \varphi_2(y)$, and using Lemma 15 and Definition 5, an easy computation yields (5). This completes the proof. □

We state, without proof, a result analogous to Lemma 17 for a system.

Lemma 18 *For $f, g \in C(J,E)$, (x,y) is a solution to the system of integral equations,*

$$x(t) = \int_0^t (t-s)f(s)ds + \varphi_1(x)\left[1 + \frac{\beta_1\Gamma(\epsilon+1)-\Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}t\right]$$
$$+\frac{\alpha_1 t}{(T-\alpha_1\zeta)\Gamma(\gamma)}\int_0^\zeta\int_0^r\left(\log\frac{\zeta}{r}\right)^{\gamma-1}(r-s)\frac{g(s)}{r}dsdr$$
$$+\frac{\beta_1\eta\zeta^{-\eta(\lambda+\epsilon)}t}{\Gamma(\lambda)\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^\zeta\int_0^r\frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^\eta-r^\eta)^{1-\lambda}}f(s)dsdr$$
$$-\frac{t}{\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^T(T-s)f(s)ds - \frac{t}{T-\alpha_1\zeta}\varphi_2(y) - \frac{t}{T-\alpha_1\zeta}\int_0^T(T-s)g(s)ds,$$

$$y(t) = \int_0^t(t-s)g(s)ds + \varphi_2(y)\left[1-\frac{t}{T-\alpha_2\omega}\right] - \frac{t}{T-\alpha_2\omega}\int_0^T(T-s)g(s)ds$$
$$+\frac{\alpha_2 t}{\Gamma(\gamma)(T-\alpha_2\omega)}\int_0^\omega\int_0^r\left(\log\frac{\omega}{r}\right)^{\gamma-1}(r-s)\frac{g(s)}{r}dsdr$$
$$+\frac{\beta_2\eta\omega^{-\eta(\lambda+\epsilon)}t}{\Gamma(\lambda)\left(T-\frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^\omega\int_0^r\frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\omega^\eta-r^\eta)^{1-\lambda}}f(s)dsdr$$
$$+\frac{\beta_2\Gamma(\epsilon+1)-\Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left(T-\frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}t\varphi_1(x) - \frac{t}{T-\frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}}\int_0^T(T-s)f(s)ds,$$

if and only if (x,y) is a solution of the linear system of differential equations,

$$\begin{cases} x''(t) = f(t), & 0<t<T, \\ y''(t) = g(t), & 0<t<T, \end{cases} \tag{7}$$

supplemented with the boundary conditions,

$$\begin{cases} x(0)=\varphi_1(x), \qquad y(0)=\varphi_2(y), \\ x(T)=\alpha_1\ {}_HI^{\gamma}y(\zeta)+\beta_1 I_{\eta}^{\epsilon,\lambda}x(\zeta), \\ y(T)=\alpha_2\ {}_HI^{\gamma}y(\omega)+\beta_2 I_{\eta}^{\epsilon,\lambda}x(\omega), \end{cases} \tag{8}$$

where

$$T-\alpha_1\zeta\neq 0, \qquad T-\alpha_2\omega\neq 0,$$

$$T-\beta_1\frac{\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\neq 0, \qquad T-\beta_2\frac{\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\neq 0.$$

We make use the Mönch fixed point theorem, Theorem 9, combined with the measure of noncompactness of Kuratowski to prove our main result. We now list suitable conditions on the functions involved in this problem:

$(H1)$ $f,g:J\times E\times E\to E$ are continuous functions and there exist positive constants l_1 and l_2 such that for all $t\in J$ and $x_1,x_2,y_1,y_2\in E,$ we have

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq l_1\left(|x_1-x_2|+|y_1-y_2|\right),$$

$$|g(t,x_1,y_1)-g(t,x_2,y_2)|\leq l_2\left(|x_1-x_2|+|y_1-y_2|\right).$$

$(H2)$ $\varphi_1,\varphi_2:C(J,E)\to E$ are Lipschitz continuous with Lipschitz constants k_1 and k_2, respectively.

$(H3)$ There exist constants $C_{\varphi_1}>0$ and $C_{\varphi_2}>0,$ such that

$$\nu(\varphi_1(V_1))\leq C_{\varphi_1}\nu_c(V_1) \text{ for all subsets } V_1\subset\Delta,$$

$$\nu(\varphi_2(V_2))\leq C_{\varphi_2}\nu_c(V_2) \text{ for all subsets } V_2\subset\overline{\Delta},$$

where V_1 and V_2 are the natural projection of $V=V_1\times V_2$ over Δ and $\overline{\Delta},$ respectively.

$(H4)$ Assume that

$$\max\left\{K_{\eta}^{\epsilon,\lambda}, L_{\eta}^{\epsilon,\lambda}\right\}<1,$$

where

$$f^*=\sup_{s\in J}f(s,0,0),\ \ g^*=\sup_{s\in J}g(s,0,0),$$

$$K_\eta^{\epsilon,\lambda} := \Big| T^2 l_1 + \beta_1 M_1 T^3 l_1 \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} + T^3 M_1 l_1$$
$$+k_1 + k_1\beta_1 M_1 M_2 T + \alpha_1 T^3 M_3 l_2 \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + T^3 M_3 l_2 + T M_3 k_2 \Big|$$
$$+ \Big| \frac{T^2}{2} f^* \left[1 + \beta_1 M_1 T \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} + T M_1\right] + \varphi_1(0)\left[1 + \beta_1 M_1 M_2 T\right]$$
$$+ \frac{T^3}{2} M_3 g^* \left[\alpha_1 \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + 1\right] + T M_3 \varphi_2(0) \Big|,$$

$$L_\eta^{\epsilon,\lambda} := \Big| T^2 l_2 + \alpha_2 N_3 T^3 l_2 \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + T^3 N_3 l_2 + k_2 + k_2 T N_3$$
$$\beta_2 N_1 T^3 l_1 \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} + T^3 N_1 l_1 + \beta_2 N_2 T k_1 \Big|$$
$$+ \Big| \frac{T^2}{2} g^* \left[1 + \alpha_2 N_3 T \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + T N_3\right] + \varphi_2(0)\left[1 + T N_3\right]$$
$$+ \frac{T^3}{2} N_1 f^* \left[1 + \beta_2 \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)}\right] + \beta_2 N_2 T \varphi_1(0) \Big|,$$

$$M_1 = \frac{1}{T - \frac{\beta_1 \zeta \Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}},\ M_2 = \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} - \frac{1}{\beta_1},\ M_3 = \frac{1}{T - \alpha_1 \zeta},$$

$$N_1 = \frac{1}{T - \frac{\beta_2 \omega \Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}},\ N_2 = \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} - \frac{1}{\beta_2},\ N_3 = \frac{1}{T - \alpha_2 \omega}.$$

Remark 19 *The condition (H1) is equivalent to the inequalities,*

$$\nu\left(f(t, B_1, B_2)\right) \le l_1\left(\nu(B_1) + \nu(B_2)\right),$$
$$\nu\left(g(t, B_1, B_2)\right) \le l_2\left(\nu(B_1) + \nu(B_2)\right),$$

for any bounded sets $B_1, B_2 \subset E$ and for each $t \in J$.

Theorem 20 *Assume that assumptions (H1)-(H4) hold, where*

$$\mathcal{K}_\eta^{\epsilon,\lambda} = \Bigg(2l_1 T + C_{\varphi_1}(1 + \beta_1\ M_1\ M_2\ T) + \frac{2l_2\alpha_1 M_3 T\zeta^2}{2^\gamma} + T M_3 C_{\varphi_2}$$
$$+2l_2 M_3 T^2 + 2l_1 T^2 M_1 + \frac{2l_1 M_1 \beta_1 \eta T^3 \Gamma\left(\epsilon + \frac{2}{\eta} + 1\right)}{2\Gamma\left(\epsilon + \frac{2}{\eta} + \lambda + 1\right)} \Bigg),$$

$$\mathcal{L}_\eta^{\epsilon,\lambda} = \Bigg(\frac{2l_2\alpha_2 N_3 T\omega^2}{2^\gamma} + 2l_2 T + (1 + T\ N_3)\, C_{\varphi_2} + \beta_2 N_2 T\ C_{\varphi_1}$$
$$+2l_2 N_3 T^2 + 2l_1 T^2 N_1 + \frac{2l_1\ N_1 \beta_2 \eta\ T^3 \Gamma\left(\epsilon + \frac{2}{\omega} + 1\right)}{2\Gamma\left(\epsilon + \frac{2}{\omega} + \lambda + 1\right)} \Bigg).$$

If

$$1-\mathcal{K}_{\eta}^{\epsilon,\lambda}-\mathcal{L}_{\eta}^{\epsilon,\lambda}>0, \tag{9}$$

then the boundary value problem (1) has at least one solution.

Proof We transform the boundary value problem (1) into a fixed point problem. We consider the set

$$D_R=\left\{(x,y)\in\Delta\times\bar{\Delta}:\quad \|(x,y)\|\leq R\right\}.$$

Clearly, the subset D_R is closed, bounded and convex.

Throughout this paper, for convenience, we use

$${}_HI^{\gamma}h(s,x(s),y(s))(v)=\frac{1}{\Gamma(\gamma)}\int_0^v\left(\log\frac{v}{s}\right)^{\gamma-1}h(s,x(s),y(s))\frac{ds}{s},$$

$$I_{\eta}^{\epsilon,\lambda}h(s,x(s),y(s))(v)=\frac{\eta v^{\eta(\lambda+\epsilon)}}{\Gamma(\lambda)}\int_0^v\frac{s^{\eta\epsilon+\eta-1}}{(v^{\eta}-s^{\eta})^{1-\lambda}}h(s,x(s),y(s))ds,$$

where $h\in\{f,g\}$.

We define the operator $T:\Delta\times\bar{\Delta}\to\Delta\times\bar{\Delta}$ by

$$T(x,y)(t)=\begin{pmatrix}T_1(x,y)(t)\\ T_2(x,y)(t)\end{pmatrix},$$

where first,

$$\begin{aligned}T_1(x,y)(t):=&\int_0^t(t-s)f(s,{}^cD^{\alpha}x(s),y(s))ds+\varphi_1(x)\left[1+\frac{\beta_1\Gamma(\epsilon+1)-\Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}t\right]\\&+\frac{\alpha_1t}{(T-\alpha_1\zeta)\Gamma(\gamma)}\int_0^{\zeta}\int_0^r\left(\log\frac{\zeta}{r}\right)^{\gamma-1}(r-s)\frac{g(s,{}^cD^{\beta}y(s),x(s))}{r}dsdr\\&-\frac{t}{T-\alpha_1\zeta}\varphi_2(y)-\frac{t}{T-\alpha_1\zeta}\int_0^T(T-s)g(s,{}^cD^{\beta}y(s),x(s))ds\\&+\frac{\beta_1\eta\zeta^{-\eta(\lambda+\epsilon)}t}{\Gamma(\lambda)\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^{\zeta}\int_0^r\frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^{\eta}-r^{\eta})^{1-\lambda}}f(s,{}^cD^{\alpha}x(s),y(s))dsdr\\&-\frac{t}{\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^T(T-s)f(s,{}^cD^{\alpha}x(s),y(s))ds,\end{aligned}$$

and

$$T_2(x,y)(t) := \int_0^t (t-s) g(s,{}^c D^\beta y(s), x(s)) ds + \varphi_2(y)\left[1 - \frac{t}{T-\alpha_2\omega}\right]$$
$$+ \frac{\alpha_2 t}{\Gamma(\gamma)(T-\alpha_2\omega)} \int_0^\omega \int_0^r \left(\log\frac{\omega}{r}\right)^{\gamma-1} (r-s)\frac{g(s,{}^c D^\beta y(s), x(s))}{r} ds dr$$
$$- \frac{t}{T-\alpha_2\omega}\int_0^T (T-s) g(s,{}^c D^\beta y(s), x(s)) ds + \frac{\beta_2\Gamma(\epsilon+1) - \Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left(T - \frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)} t\varphi_1(x)$$
$$+ \frac{\beta_2\eta\omega^{-\eta(\lambda+\epsilon)} t}{\Gamma(\lambda)\left(T - \frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)} \int_0^\omega \int_0^r \frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\omega^\eta - r^\eta)^{1-\lambda}} f(s,{}^c D^\alpha x(s), y(s)) ds dr$$
$$- \frac{t}{T - \frac{\beta_2\omega\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}} \int_0^T (T-s) f(s,{}^c D^\alpha x(s), y(s)) ds.$$

Then, with M_i ad N_i, $i = 1,2,3,$ as in (H4),

$$T_1(x,y)(t) = \int_0^t (t-s) f(s,{}^c D^\alpha x(s), y(s)) ds + [1+\beta_1 M_1 M_2 t]\,\varphi_1(x)$$
$$+ \alpha_1 t M_3 \int_0^r (r-s)\ {}_H I^\gamma g(s,{}^c D^\beta y(s), x(s))(\zeta) ds - t M_3 \varphi_2(y)$$
$$+ \beta_1 M_1 t \int_0^r (r-s) I_\eta^{\epsilon,\lambda} f(s,{}^c D^\alpha x(s), y(s))(\zeta) ds$$
$$- t\ M_3 \int_0^T (T-s) g(s,{}^c D^\beta y(s), x(s)) ds - t\ M_1 \int_0^T (T-s) f(s,{}^c D^\alpha x(s), y(s)) ds,$$

and

$$T_2(x,y)(t) = \int_0^t (t-s) g(s,{}^c D^\beta y(s), x(s)) ds + \alpha_2 N_3 t \int_0^r (r-s)\ {}_H I^\gamma g(s,{}^c D^\beta y(s), x(s))(\omega) ds$$
$$+ [1+tN_3]\varphi_2(y) - tN_3 \int_0^T (T-s) g(s,{}^c D^\beta y(s), x(s)) ds$$
$$+ \beta_2 N_1 t \int_0^r (r-s) I_\eta^{\epsilon,\lambda} f(s,{}^c D^\alpha x(s), y(s))(\omega) ds + \beta_2 N_2 t \varphi_1(x)$$
$$- tN_1 \int_0^T (T-s) f(s,{}^c D^\alpha x(s), y(s)) ds.$$

In these forms of T_1 ad T_2, we put

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix}.$$

Clearly, the fixed points of the operator T are solutions of the problem (1).

Now, we deal with the existence of a solution of (1) via the technique that relies on the concept of measures of noncompactness and Mönch's fixed point theorem. We shall prove that T satisfies all assumptions of Theorem 20.

Step 1. T is continuous.

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence such that $(x_n, y_n) \to (x, y)$ in $\Delta \times \bar{\Delta}$. Then, for each $t \in J$,

$$\begin{aligned}
&\|T_1(x_n, y_n)(t) - T_1(x, y)(t)\| \\
&\leq \int_0^t |t-s|\,|f(s,^c D^\alpha x_n(s), y_n(s)) - f(s,^c D^\alpha x(s), y(s))|\,ds \\
&+ |\alpha_1 T M_3| \int_0^r |r-s|_H I^\gamma \left(\left|g(s,^c D^\beta y_n(s), x_n(s)) - g(s,^c D^\beta y(s), x(s))\right|\right)(\zeta) ds \\
&+ |T M_3|\,|\varphi_2(y_n) - \varphi_2(y)| + |1 + \beta_1 M_1 M_2 T|\,|\varphi_1(x_n) - \varphi_1(x)| \\
&+ |T M_3| \int_0^T |T-s|\,\left|g(s,^c D^\beta y_n(s), x_n(s)) - g(s,^c D^\beta y(s), x(s))\right| ds \\
&+ |\beta_1 M_1 T| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left(|f(s,^c D^\alpha x_n(s), y_n(s)) - f(s,^c D^\alpha x(s), y(s))|\right)(\zeta) ds \\
&+ |T M_1| \int_0^T |T-s|\,|f(s,^c D^\alpha x_n(s), y_n(s)) - f(s,^c D^\alpha x(s), y(s))|\,ds \\
&\leq l_1 \int_0^t |t-s| \left(|^c D^\alpha x_n(s) -^c D^\alpha x(s)| + |y_n(s) - y(s)|\right) ds + |1 + \beta_1 M_1 M_2 T|\, k_1\, |x_n - x| \\
&+ |\alpha_1 T M_3 l_2| \int_0^r |r-s|_H I^\gamma \left(\left|^c D^\beta y_n(s) -^c D^\beta y(s)\right| + |x_n(s) - x(s)|\right)(\zeta) ds \\
&+ |T M_3 k_2|\,|y_n - y| + |T M_3 l_2| \int_0^T |T-s| \left(\left|^c D^\beta y_n(s) -^c D^\beta y(s)\right| + |x_n(s) - x(s)|\right) ds \\
&+ |\beta_1 M_1 T l_1| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left(|^c D^\alpha x_n(s) -^c D^\alpha x(s)| + |y_n(s) - y(s)|\right)(\zeta) ds \\
&+ |T M_1 l_1| \int_0^T |T-s| \left(|^c D^\alpha x_n(s) -^c D^\alpha x(s)| + |y_n(s) - y(s)|\right) ds.
\end{aligned}$$

Then we have

$$\|T_1(x_n, y_n) - T_1(x, y)\| \to 0 \qquad as \quad n \to \infty.$$

Similarly, we get

$$\begin{aligned}
&\|T_2(x_n, y_n)(t) - T_2(x, y)(t)\| \\
&\leq \int_0^t |t-s| \left|g(s,^c D^\beta y_n(s), x_n(s)) - g(s,^c D^\beta y(s), x(s))\right| ds \\
&+ |\alpha_2 N_3 T| \int_0^r |r-s|\ _H I^\gamma \left(\left|g(s,^c D^\beta y_n(s), x_n(s)) - g(s,^c D^\beta y(s), x(s))\right|\right)(\omega) ds
\end{aligned}$$

$$+ |1+TN_3|\, |\varphi_2(y_n)-\varphi_2(y)| + |\beta_2 N_2 T|\, |\varphi_1(x_n)-\varphi_1(x)|$$

$$+ |TN_3| \int_0^T |T-s| \left| g(s, {}^c D^\beta y_n(s), x_n(s)) - g(s, {}^c D^\beta y(s), x(s)) \right| ds$$

$$+ |\beta_2 N_1 T| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left(|f(s, {}^c D^\alpha x_n(s), y_n(s)) - f(s, {}^c D^\alpha x(s), y(s))| \right)(\omega) ds$$

$$+ |TN_1| \int_0^T |T-s|\, |f(s, {}^c D^\alpha x_n(s), y_n(s)) - f(s, {}^c D^\alpha x(s), y(s))|\, ds$$

$$\leq l_2 \int_0^t |t-s| \left(\left| {}^c D^\beta y_n(s) - {}^c D^\beta y(s) \right| + |x_n(s)-x(s)| \right) ds + |1+TN_3| k_2\, |y_n - y|$$

$$+ |\alpha_2 N_3 T l_2| \int_0^r |r-s|_H I^\gamma \left(\left| {}^c D^\beta y_n(s) - {}^c D^\beta y(s) \right| + |x_n(s)-x(s)| \right)(\omega) ds$$

$$+ |TN_3 l_2| \int_0^T |T-s| \left(\left| {}^c D^\beta y_n(s) - {}^c D^\beta y(s) \right| + |x_n(s)-x(s)| \right) ds$$

$$+ |\beta_2 N_1 T l_1| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left(|{}^c D^\alpha x_n(s) - {}^c D^\alpha x(s)| + |y_n(s)-y(s)| \right)(\omega) ds$$

$$+ |TN_1 l_1| \int_0^T |T-s| \left(|{}^c D^\alpha x_n(s) - {}^c D^\alpha x(s)| + |y_n(s)-y(s)| \right) ds + |\beta_2 N_2 T k_1|\, |x_n - x|\,.$$

Then we have

$$\|T_2(x_n,y_n) - T_2(x,y)\| \to 0 \quad as \;\; n \to \infty.$$

Therefore,

$$\|T(x_n,y_n) - T(x,y)\| \to 0 \quad as \;\; n \to \infty.$$

Step 2. T **maps** D_R **into itself.**
For each $(x,y) \in D_R$ and from (H1),(H2) and (H4), we have for each $t \in J$,

$$|T_1(x,y)(t)| = \sup_{t\in J} \left\{ \int_0^t (t-s) f(s, {}^c D^\alpha x(s), y(s)) ds + [1 + \beta_1 M_1 M_2 t]\, \varphi_1(x) \right.$$

$$+\alpha_1 t M_3 \int_0^r (r-s)\ {}_H I^\gamma g(s, {}^c D^\beta y(s), x(s))(\zeta) ds - t M_3 \varphi_2(y)$$

$$-\ t M_3 \int_0^T (T-s) g(s, {}^c D^\beta y(s), x(s)) ds$$

$$+\beta_1 M_1 t \int_0^r (r-s) I_\eta^{\epsilon,\lambda} f(s, {}^c D^\alpha x(s), y(s))(\zeta) ds$$

$$\left. -t M_1 \int_0^T (T-s) f(s, {}^c D^\alpha x(s), y(s)) ds \right\}$$

$$
\begin{aligned}
&\leq \int_0^t |t-s| \left(|f(s,{}^c D^\alpha x(s), y(s)) - f(s,0,0)| + |f(s,0,0)|\right) ds \\
&+ |1+\beta_1 M_1 M_2 T| \left(|\varphi_1(x) - \varphi_1(0)| + |\varphi_1(0)|\right) + |TM_3| \left(|\varphi_2(y) - \varphi_2(0)| + |\varphi_2(0)|\right) \\
&+ |\alpha_1 T M_3| \int_0^r |r-s| \ {}_H I^\gamma \left(\left|g(s,{}^c D^\beta y(s), x(s)) - g(s,0,0)\right| + |g(s,0,0)|\right)(\zeta) ds \\
&+ |TM_3| \int_0^T |T-s| \left(\left|g(s,{}^c D^\beta y(s), x(s)) - g(s,0,0)\right| + |g(s,0,0)|\right) ds \\
&+ |\beta_1 M_1 T| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left(|f(s,{}^c D^\alpha x(s), y(s)) - f(s,0,0)| + |f(s,0,0)|\right)(\zeta) ds \\
&+ |TM_1| \int_0^T |T-s| \left(|f(s,{}^c D^\alpha x(s), y(s)) - f(s,0,0) + |f(s,0,0)|\right) ds \\
&\leq \int_0^t |t-s| \left[l_1 \left(\|{}^c D^\alpha x(s)\| + \|y(s)\|\right) + f^*\right] ds + |1+\beta_1 M_1 M_2 T| \left(k_1 \|x\| + \varphi_1(0)\right) \\
&+ |\alpha_1 T M_3| \int_0^r |r-s| \ {}_H I^\gamma \left[l_2 \left(\|{}^c D^\beta y(s)\| + \|x(s)\|\right) + g^*\right](\zeta) ds + |TM_3| \left(k_2 \|y\| + \varphi_2(0)\right) \\
&+ |TM_3| \int_0^T |T-s| \left[l_2 \left(\|{}^c D^\beta y(s)\| + \|x(s)\|\right) + g^*\right] ds \\
&+ |\beta_1 M_1 T| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left[l_1 \left(\|{}^c D^\alpha x(s)\| + \|y(s)\|\right) + f^*\right](\zeta) ds \\
&+ |TM_1| \int_0^T |T-s| \left[l_1 \left(\|{}^c D^\alpha x(s)\| + \|y(s)\|\right) + f^*\right] ds \\
&\leq \frac{T^2}{2} l_1 \|{}^c D^\alpha x\| + \frac{T^2}{2} l_1 \|y\| + \frac{T^2}{2} f^* + |k_1 + k_1 \beta_1 M_1 M_2 T| \|x\| + |\varphi_1(0) + \varphi_1(0) \beta_1 M_1 M_2 T| \\
&+ |\alpha_1 T M_3| \left| l_2 \left(\|{}^c D^\beta y\| + \|x\|\right) + g^* \right| \int_0^r |r-s| \ {}_H I^\gamma (1)(\zeta) ds + |TM_3 k_2| \|y\| + |TM_3 \varphi_2(0)| \\
&+ \left|\frac{T^3}{2} M_3 l_2\right| \|{}^c D^\beta y\| + \left|\frac{T^3}{2} M_3 l_2\right| \|x\| + \left|\frac{T^3}{2} M_3 g^*\right| \\
&+ |\beta_1 M_1 T| \left| l_1 \left(\|{}^c D^\alpha x(s)\| \|y(s) + f^*\right| \int_0^r |r-s| I_\eta^{\epsilon,\lambda}(1)(\zeta) ds \| \right) \\
&+ \left|\frac{T^3}{2} M_1 l_1\right| \|{}^c D^\alpha x\| + \left|\frac{T^3}{2} M_1 l_1\right| \|y\| + \left|\frac{T^3}{2} M_1 f^*\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq \|(x,y)\| \Bigg| & T^2 l_1 + \beta_1 M_1 T^3 l_1 \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} + T^3 M_1 l_1 \\
& + k_1 + k_1 \beta_1 M_1 M_2 T + \alpha_1 T^3 M_3 l_2 \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + T^3 M_3 l_2 + T M_3 k_2 \Bigg| \\
& + \Bigg| \frac{T^2}{2} f^* \left[1 + \beta_1 M_1 T \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)} + TM_1\right] + \varphi_1(0) \left[1 + \beta_1 M_1 M_2 T\right] \\
& + \frac{T^3}{2} M_3 g^* \left[\alpha_1 \frac{(\log T)^\gamma}{\Gamma(\gamma+1)} + 1\right] + TM_3 \varphi_2(0) \Bigg|
\end{aligned}
$$

$$\leq R\left|T^2l_1+\beta_1M_1T^3l_1\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)}+T^3M_1l_1+T^3M_3l_2+k_1+k_1\beta_1M_1M_2T\right.$$
$$\left.+\alpha_1T^3M_3l_2\frac{(\log T)^\gamma}{\Gamma(\gamma+1)}+TM_3k_2\right|+\left|\frac{T^2}{2}f^*\left[1+\beta_1M_1T\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)}+TM_1\right]\right.$$
$$\left.+\varphi_1(0)\left[1+\beta_1M_1M_2T\right]+\frac{T^3}{2}M_3g^*\left[\alpha_1\frac{(\log T)^\gamma}{\Gamma(\gamma+1)}+1\right]+TM_3\varphi_2(0)\right|.$$

Making use of condition (H4) yields

$$\|T_1(x,y)\|_\Delta\leq R.$$

In the same way, we can obtain that

$$|T_2(x,y)(t)|\leq R\left(\left|T^2l_2+\alpha_2N_3T^3l_2\frac{(\log T)^\gamma}{\Gamma(\gamma+1)}+T^3N_3l_2+k_2+k_2TN_3\right.\right.$$
$$\left.+\beta_2N_1T^3l_1\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)}+T^3N_1l_1+\beta_2N_2Tk_1\right|$$
$$+\left|\frac{T^2}{2}g^*\left[1+\alpha_2N_3T\frac{(\log T)^\gamma}{\Gamma(\gamma+1)}+TN_3\right]+\varphi_2(0)\left[1+TN_3\right]\right.$$
$$\left.\left.+\frac{T^3}{2}N_1f^*\left[1+\beta_2\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1+\lambda)}\right]+\beta_2N_2T\varphi_1(0)\right|\right).$$

Again, using condition $(H4)$, one has

$$\|T_2(x,y)\|_{\overline{\Delta}}\leq R.$$

Finally

$$\|T(x,y)(t)\|\leq R.$$

Step 3. T **maps bounded sets into bounded sets in** D_R**.**

Indeed, from Step 2, it is enough to remark that for any $\eta^*>0$, and for each $(x,y)\in B_{\eta^*}=\{(x,y)\in\Delta\times\overline{\Delta}:\|(x,y)\|\leq\eta^*\}$, we have $\|T(x,y)\|_\infty\leq R$. which means that $T(B_{\eta^*})$ is bounded.

Step 4. T **maps bounded sets into an equicontinuous sets in** D_R**.**

Let $t_1,t_2\in J$, $t_1<t_2$, B_{η^*} be a bounded set of D_R as in Step 2, and let $(x,y)\in B_{\eta^*}$. Then

$$\|T_1(x,y)(t_2)-T_1(x,y)(t_1)\|\leq\int_0^{t_1}|t_2-t_1||f(s,{}^cD^\alpha x(s),y(s))|ds$$
$$+\int_{t_1}^{t_2}|t_2-s||f(s,{}^cD^\alpha x(s),y(s))|ds+|t_2-t_1|\left|\beta_1M_1M_2\varphi_1(x)\right|$$
$$+|t_2-t_1||\alpha_1M_3|\int_0^r|r-s|\;{}_HI^\gamma\left(\left|g(s,{}^cD^\beta y(s),x(s))\right|\right)(\zeta)ds$$
$$+|t_2-t_1||M_3\varphi_2(y)|+|t_2-t_1||M_3|\int_0^T|T-s|\left|g(s,{}^cD^\beta y(s),x(s))\right|ds$$
$$+|t_2-t_1||\beta_1M_1|\int_0^r|r-s|I_\eta^{\epsilon,\lambda}\left|f(s,{}^cD^\alpha x(s),y(s))\right|(\zeta)ds$$
$$+|t_2-t_1||M_1|\int_0^T|T-s|\left|f(s,{}^cD^\alpha x(s),y(s))\right|ds.$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.

Analogously, we obtain

$$\|T_2(x,y)(t_2) - T_2(x,y)(t_1)\| \leq \int_0^{t_1} |t_2 - t_1| \left|g(s,{}^c D^\beta y(s), x(s))\right| ds$$

$$+ \int_{t_1}^{t_2} |t_2 - s| \left|g(s,{}^c D^\beta y(s), x(s))\right| ds$$

$$+ |t_2 - t_1||\alpha_2 N_3| \int_0^r |r-s| \ {}_H I^\gamma \left|g(s,{}^c D^\beta y(s), x(s))\right| (\omega) ds$$

$$+ |t_2 - t_1||N_3 \varphi_2(y)| + |t_2 - t_1||N_3| \int_0^T |T-s| \left|g(s,{}^c D^\beta y(s), x(s))\right| ds$$

$$+ |t_2 - t_1||\beta_2 N_1| \int_0^r |r-s| I_\eta^{\epsilon,\lambda} \left|f(s,{}^c D^\alpha x(s), y(s))\right| (\omega) ds$$

$$+ |t_2 - t_1||\beta_2 N_2 \varphi_1(x)| + |t_2 - t_1||N_1| \int_0^T |T-s| \left|f(s,{}^c D^\alpha x(s), y(s))\right| ds.$$

Therefore, the operator T maps the bounded set into an equicontinuous one.

Now let V be a subset of D_R such that $V \subset conv(T(V) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $t \to v(t) = (\widetilde{\nu}(V(t))$ is continuous and bounded on J.
From Example 12 and properties of the measure of noncompactness $\widetilde{\nu}$, we have for each $t \in J$,

$$\begin{aligned} v(t) &\leq \widetilde{\nu}(T(V)(t)) \cup \{0\}) \\ &\leq \widetilde{\nu}(T(V)(t)) \\ &= \nu(T_1(V)(t)) + \nu(T_2(V)(t)). \end{aligned}$$

First we will estimate $\nu(T_1(V)(t))$ and $\nu(T_2(V)(t))$.
Denote by V_1, V_2 the natural projection of $V \subset D_R$ over Δ, $\bar{\Delta}$ respectively, Using Lemma 10, Remark 19, (H_3) and the properties of the measure of noncompactness $\widetilde{\nu}$ one has,

$$\nu(T_1 V(t)) \leq l_1 \int_0^t (t-s) \left(\nu(V_1(s)) + \nu(V_2(s))\right) ds$$

$$+ C_{\varphi_1} \nu(V_1(t)) \left[1 + \frac{\beta_1 \Gamma(\epsilon+1) - \Gamma(\epsilon+1+\lambda)}{\Gamma(\epsilon+1+\lambda)\left(T - \frac{\beta_1 \zeta \Gamma(\epsilon + \frac{1}{\eta} + 1)}{\Gamma(\epsilon + \frac{1}{\eta} + \lambda + 1)}\right)} t \right]$$

$$+ \frac{l_2 \alpha_1 t}{(T - \alpha_1 \zeta)\Gamma(\gamma)} \int_0^\zeta \int_0^r \left(\log \frac{\zeta}{r}\right)^{\gamma - 1} (r-s) \frac{(\nu(V_1(s)) + \nu(V_2(s)))}{r} ds dr$$

$$+ \frac{t}{|T - \alpha_1 \zeta|} C_{\varphi_2} V_2(t) + \frac{l_2 t}{|T - \alpha_1 \zeta|} \int_0^T (T-s) \left(\nu(V_1(s)) + \nu(V_2(s))\right) ds$$

$$+\frac{l_1\beta_1\eta\zeta^{-\eta(\lambda+\epsilon)}t}{\Gamma(\lambda)\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)}\int_0^{\zeta}\int_0^{r}\frac{r^{\eta\epsilon+\eta-1}(r-s)}{(\zeta^{\eta}-r^{\eta})^{1-\lambda}}\left(\nu(V_1(s))+\nu(V_2(s))\right)dsdr$$

$$+\frac{l_1t}{|\left(T-\frac{\beta_1\zeta\Gamma(\epsilon+\frac{1}{\eta}+1)|}{\Gamma(\epsilon+\frac{1}{\eta}+\lambda+1)}\right)|}\int_0^{T}(T-s)\left(\nu(V_1(s))+\nu(V_2(s))\right)ds.$$

Thus,

$$\begin{aligned}\nu(T_1V(t)) \quad &\leq \|v\|_{\infty}\bigg(2l_1T+C_{\varphi_1}(1+\beta_1\ M_1\ M_2\ T)+\frac{2l_2\alpha_1M_3T\zeta^2}{2^{\gamma}}\\ &+TM_3C_{\varphi_2}+2l_2M_3T^2+2l_1T^2M_1+\frac{2l_1M_1\beta_1\eta T^3\Gamma(\epsilon+\frac{2}{\eta}+1)}{2\Gamma(\epsilon+\frac{2}{\eta}+\lambda+1)}\bigg)\\ &\leq \|v\|_{\infty}K_{\eta}^{\epsilon,\lambda}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\nu\left(T_2V(t)\right)=&\|v\|_{\infty}\bigg(\frac{2l_2\alpha_2N_3T\omega^2}{2^{\gamma}}+2l_2T+(1+T\ N_3)\,C_{\varphi_2}+\beta_2N_2T\ C_{\varphi_1}\\ &+2l_2N_3T^2+2l_1T^2N_1+\frac{2l_1\ N_1\beta_2\eta\ T^3\Gamma(\epsilon+\frac{2}{\omega}+1)}{2\Gamma(\epsilon+\frac{2}{\omega}+\lambda+1)}\bigg)\\ &\leq \|v\|_{\infty}L_{\eta}^{\epsilon,\lambda}.\end{aligned}$$

It follows then that

$$v(t)\leq \|v\|_{\infty}\left(K_{\eta}^{\epsilon,\lambda}+L_{\eta}^{\epsilon,\lambda}\right),$$

which means that

$$\|v\|_{\infty}\left(1-K_{\eta}^{\epsilon,\lambda}-L_{\eta}^{\epsilon,\lambda}\right)\leq 0.$$

By (9) it follows that $\|v\|_{\infty}=0$; that is, $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzela theorem, V is relatively compact in D_R. Applying now Theorem 9, we conclude that T has a fixed point which is a solution of the problem (1). □

4. Example

As an application of our results, we consider the following fractional coupled system of differential equations.

$$\begin{cases} x_n''(t)=\left(\dfrac{7+|{}^cD^{\frac{3}{2}}x_n(t)|+|y_n(t)|}{(20+e^t)(1+|y_n(t)|+|{}^cD^{\frac{3}{2}}x_n(t)|)}\right)_n, & t\in(0,1),\ \alpha=\dfrac{3}{2},\\ y_n''(t)=\left(\dfrac{|x_n(t)|+|{}^cD^{\frac{3}{2}}y_n(t)|}{10(1+|x_n(t)|+|{}^cD^{\frac{3}{2}}y_n(t)|)}\right)_n, & t\in(0,1),\ \beta=\dfrac{3}{2},\\ x_n(0)=\sum_{i=1}^{n}c_ix_n(t_i),\qquad y_n(0)=\sum_{i=1}^{n}d_iy_n(t_i),\\ x_n(1)={}_HI^{\frac{1}{2}}y_n(2)-I_{\frac{1}{2}}^{\frac{1}{2},1}x_n(2),\\ y_n(1)={}_HI^{\frac{1}{2}}y_n(\frac{1}{2})+I_{\frac{1}{2}}^{\frac{1}{2},1}x_n(\frac{1}{2}).\end{cases}\tag{10}$$

Let E be the set of real sequences such that

$$E = \{x = \{x_i\}_{i\in\mathbb{N}},\ \sup |x_n| < \infty\},$$

with the norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|.$$

Then E is a Banach space, and we consider the product space $(\Delta \times \bar{\Delta}, \|(\cdot,\cdot)\|)$ as defined in Section 2. Then $\Delta \times \bar{\Delta}$ is a Banach space with norm,

$$\|(x, y)\| = \|x\|_{\Delta} + \|y\|_{\overline{\Delta}}.$$

The problem (10) can be regarded as a problem of the form (1), where $0 < t_1 < t_2 < \cdots < t_n < 1$, $c_i,\ i = 1, \ldots, n,$ and $d_i,\ i = 1, \ldots, n,$ are given positive constants with

$$\sum_{i=1}^{+\infty} c_i < \frac{2}{25},\ \ \sum_{i=1}^{+\infty} d_i < \frac{2}{25},$$

$$\alpha_1 = \alpha_2 = 1,\ \beta_1 = -1,\ \beta_2 = 1,\ \eta = \epsilon = \gamma = \omega = \frac{1}{2},\ \zeta = 2,\ \lambda = 1,$$

$$f(t, x, y) = (f_1(t, x, y), f_2(t, x, y), \ldots, f_n(t, x, y))\ \ with,$$

$$f_n(t, x, y) = \frac{7 + |x_n(t)| + |y_n(t)|}{(20 + e^{4t})(1 + |x_n(t)| + |y_n(t)|)},$$

$$g(t, x, y) = (g_1(t, x, y), \ldots, g_n(t, x, y))\ \ with,$$

$$g_n(t, x, y) = \frac{|x_n(t)| + |y_n(t)|}{10(1 + |x_n(t)| + |y_n(t)|)},$$

$$\varphi_1(x_n) = \sum_{i=1}^{n} c_i x_n(t_i),\ \ \varphi_2(y_n) = \sum_{i=1}^{n} d_i y_n(t_i).$$

It clear that $f \in C(J \times E, E)$ and $g \in C(J \times E, E)$. With the aid of straightforward computations, we find, for every $x, y \in \mathbb{R}$ and $t \in J$,

$$\begin{aligned}
|f(t, x, y) - f(t, \overline{x}, \overline{y})| &= \frac{1}{(20 + e^{4t})}\left|\frac{7 + |x| + |y|}{1 + |x| + |y|} - \frac{7 + |\overline{x}| + |\overline{y}|}{1 + |\overline{x}| + |\overline{y}|}\right| \\
&= \frac{6}{20 + e^{4t}}\left|\frac{1}{1 + |\overline{x}| + |\overline{y}|} - \frac{1}{1 + |x| + |y|}\right| \\
&\leq \frac{6}{20 + e^{4t}}\Big||x| + |y| - |\overline{x}| - |\overline{y}|\Big| \\
&\leq \frac{1}{10}\left(\Big|x - \overline{x}\Big| + \Big|y - \overline{y}\Big|\right).
\end{aligned}$$

Similarly, we obtain

$$|g(t, x, y) - g(t, \overline{x}, \overline{y})| \leq \frac{1}{10}\left(\Big|x - \overline{x}\Big| + \Big|y - \overline{y}\Big|\right),$$

and hence condition $(H1)$ holds with $l_1 = l_2 = \frac{1}{10}$.

From the definitions of φ_1 and φ_2, we have

$$\|\varphi_1(x) - \varphi_1(y)\| \leq \frac{2}{25}\|x-y\|, \text{ for all, } x,\ y\ \in V_1 \subset \Delta,$$

$$\|\varphi_2(\overline{x}) - \varphi_2(\overline{y})\| \leq \frac{2}{25}\|\overline{x}-\overline{y}\|, \text{for all, } \overline{x},\ \overline{y}\ \in V_2 \subset \overline{\Delta},$$

which implies that the condition (H2) is satisfied. On the another hand, we have

$$\nu(\varphi_1(V_1)) \leq \frac{2}{25}\nu_c(V_1) \text{ for all subset } V_1 \subset \Delta,$$

$$\nu(\varphi_2(V_2)) \leq \frac{2}{25}\nu_c(V_2) \text{ for all subsets } V_2 \subset \overline{\Delta},$$

hence condition $(H3)$ is satisfied with $C_{\varphi_1} = \frac{2}{25}$, $C_{\varphi_2} = \frac{2}{25}$.

An easy computation gives that, with

$$M_1 = \frac{7}{5},\ M_2 = \frac{5}{3},\ M_3 = -1,\ N_1 = \frac{7}{5},\ N_2 = \frac{-1}{3},\ N_3 = -1,$$

$$K_\eta^{\epsilon,\lambda} = 0.158,\ L_\eta^{\epsilon,\lambda} = 0.57,$$

and then condition (H4) is satisfied.

We shall verify the condition (9). From the calculus, we obtain

$$\mathcal{K}_\eta^{\epsilon,\lambda} = 0.38,\ \mathcal{L}_\eta^{\epsilon,\lambda} = 0.12,$$

which implies that

$$1 - \mathcal{K}_\eta^{\epsilon,\lambda} - \mathcal{L}_\eta^{\epsilon,\lambda} > 0,$$

and hence condition (9) is realized.

Therefore, Theorem 20 ensures that problem (10) has at least one solution.

5. Conclusion

We deal, in this paper, with the existence of solution for a coupled system of implicit differential equations with Hadamard and Erdélyi–Kober type of fractional integral boundary value problems and nonlocal conditions at the boundaries in Banach space. We make use of the Mönch fixed point theorem combined with the Kuratowski measure of noncompactness to give our existence result.

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