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Lie group analysis for initial and boundary value problem of time-fractional nonlinear generalized KdV partial differential equation

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Abstract: The Lie group analysis or in other word the symmetry analysis method is extended to deal with a time-fractional order derivative nonlinear generalized KdV equation. Our research in this work aims to use transformation methods and their analysis to search for exact solutions to the nonlinear generalized KdV differential equation. It is shown that this equation can be reduced to an equation with Erdelyi–Kober fractional derivative. In this study, we research the initial and boundary conditions, considering them invariant, and so we get two ordinary initial value problems, i.e. two Cauchy problems. Conservation laws for the given equation are also investigated. In this work, we introduce symmetry analysis and find conservation laws for the nonlinear generalized time-fractional KdV equation by the Lie groups method using fractional derivatives in the Riemann–Liouville sense.

Key words: Lie groups method, conservation laws, generalized KdV equation, Riemann–Liouville derivative

1. Introduction

The search for exact solutions is motivated by the aspiration to understand the mathematical structure of solutions and hence a deeper understanding of the physical phenomena described by them. As is known, finding a general solution of the nonlinear partial differential equation with fractional derivatives is quite complicated. Thus, many methods of mathematical physics have been developed to solve differential equations, including the Lie group method, which is an efficient approach to derive the exact solution of nonlinear partial differential equations (PDEs). The Lie group methods are fundamental to the development of systematic procedures that lead to invariant transformations. These transformations may be used to generate new solutions from known ones [8, 12, 16, 23].

Some researchers have applied the Lie group method to fractional differential equations. More recently, in 2014, the fractional Lie group method was proposed in [8] to solve the fractional KdV-type equation and for time-fractional generalized Burgers and KdV equations in [22]. It is an effective tool in solving fractional differential equations.

In this work we will discuss new solutions by using the Lie symmetry group method for the time-fractional nonlinear generalized KdV equation:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + P(u^n)_x = Qg(t)(u^n)_{xxx}, & \\ u(t,0) = \Psi(t), & t \in [0,T], \\ u(0,x) = \Phi(x), & x \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

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where $u := u(t,x)$ is a real function for $(t,x) \in [0,T] \times \mathbb{R}^+$, $T > 0$, $n \neq 1$, and $P, Q \in \mathbb{R}$ are constants. $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the fractional derivative of $u(x,t)$ in the Riemann–Liouville sense. This equation generates propagation of nonlinear water-waves in the long-wavelength region [1]. We would like to note that we study the initial and boundary value problem with different initial and boundary conditions than Abd-el-Malek and Amin in their work [1].

Our goal in this work is to discuss the time-fractional generalized KdV equation by using the symmetry method. During the investigation we show that the time-fractional generalized KdV equation can be transformed into a nonlinear ordinary differential equation (ODE) of fractional order by using the obtained corresponding infinitesimal operators. We also find conservation laws for the time-fractional generalized KdV equation.

The paper is organized as follows: Section 2 introduces fractional derivatives and integrals with some properties, Section 3 discusses the Lie symmetry analysis of the fractional partial differential equation (FPDE). In Section 4, the symmetry method is applied to reduce the time-fractional generalized KdV equation into an ordinary differential equation. In Section 5, we study the initial condition for our problem, and in Section 6 we show conservation laws for the generalized fractional KdV equation. Finally, we present conclusions in the last section.

2. Fractional calculus and some properties

The left-side time-fractional Riemann–Liouville derivative of order α differentiates with respect to t, which is denoted by $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$, as in the following formula [7]:

$$D_t^\alpha f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau,x)}{(t-\tau)^{n-\alpha-1}} d\tau, & if \quad \alpha \notin \mathbb{N}, \quad n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t,x), & if \quad \alpha \in \mathbb{N}. \end{cases}$$

Here $\Gamma(n-\alpha)$ is a gamma function, which is defined as:

$$\Gamma(n) = \int_0^{+\infty} t^{n-1} e^{-t} dt.$$

The Riemann–Liouville time-fractional derivative $D_t^\alpha f(t)$ can be written as:

$$D_t^\alpha f(t,x) = D_t^n ({}_0I_t^{n-\alpha} f(t,x)),$$

where ${}_0I_t^{n-\alpha} f(t,x)$ is the left-sided time-fractional Riemann–Liouville integral of order $n-\alpha$ as

$${}_0I_t^{n-\alpha} f(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(\tau,x)}{(t-\tau)^{n-\alpha-1}} d\tau.$$

Also, we need to denote that ${}_tI_T^{n-\alpha} f(t,x)$ is the right-sided time-fractional Riemann–Liouville integral of order $n-\alpha$ defined by

$${}_tI_T^{n-\alpha} f(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{f(\tau,x)}{(\tau-t)^{n-\alpha-1}} d\tau.$$

The following are useful formulas of the Leibnitz rule and chain rule, respectively [3, 15, 19, 21]:

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \qquad \binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}, \tag{2.1}$$

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!} (-g(t))^r \frac{d^m}{dt^m} \left(g(t)^{k-r}\right) \frac{d^k f(g)}{dg^k}, \tag{2.2}$$

and these will be used in the following sections.

3. Applying the Lie method to the time-fractional nonlinear KdV equation

Now we will determine a Lie group of scaling transformations acting on a space of two independent variables (x,t) and dependent variable u as

$$\begin{aligned} \bar{t} &= t + \varepsilon\tau(t,x,u) + O(\varepsilon^2), \\ \bar{x} &= x + \varepsilon\xi(t,x,u) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon\eta(t,x,u) + O(\varepsilon^2), \end{aligned} \tag{3.1}$$

where $\varepsilon > 0$ is an infinitesimals parameter and τ, ξ, and η are to be determined. To find τ, ξ, η, we should find each derivative of $u(t,x)$ for (1.1). Thus,

$$\begin{aligned} \tfrac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \tfrac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^0 + O(\varepsilon^2), \\ \tfrac{\partial \bar{u}}{\partial \bar{x}} &= \tfrac{\partial u}{\partial x} + \varepsilon\eta_1^1 + O(\varepsilon^2), \\ \tfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \tfrac{\partial^2 u}{\partial x^2} + \varepsilon\eta_2^1 + O(\varepsilon^2), \\ \tfrac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \tfrac{\partial^3 u}{\partial x^3} + \varepsilon\eta_3^1 + O(\varepsilon^2). \end{aligned} \tag{3.2}$$

Here $\eta_\alpha^0, \eta_1^1, \eta_2^1$, and η_3^1 are extended infinitesimals of orders $\alpha, 1, 2$, and 3, respectively [5, 9, 20]. The explicit expressions of them are:

$$\begin{aligned} \eta_\alpha^0 &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \\ \eta_1^1 &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta_2^1 &= D_x(\eta_1^1) - u_{xx} D_x(\xi) - u_{tx} D_x(\tau), \\ \eta_3^1 &= D_x(\eta_2^1) - u_{xxx} D_x(\xi) - u_{txx} D_x(\tau). \end{aligned} \tag{3.3}$$

Here we focus our attention on η_α^0. Using the generalized Leibnitz rule (2.1), we get

$$\xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) = -\sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi),$$

and

$$D_t^\alpha(u D_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u) = -\alpha D_t(\tau) D_t^\alpha(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau).$$

Thus, we obtain the expression

$$\eta_\alpha^0 = D_t^\alpha(\eta) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi) - \alpha D_t(\tau) D_t^\alpha(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau).$$

Using the compound function of the chain rule (2.2), our η_α^0 takes the form

$$\begin{aligned}\eta_\alpha^0 &= \tfrac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha(\tau_t + u_t\tau_u))\tfrac{\partial^\alpha u}{\partial t^\alpha} - u\tfrac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\tfrac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau\right] D_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}(D_t^n\xi)(D_t^{\alpha-n}u_x),\end{aligned} \tag{3.4}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)}\frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

The infinitesimal generator associated with the above transformations can be written as

$$X = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}$$

with $\xi = \frac{d\bar{x}}{d\varepsilon}\mid_{\varepsilon=0}$, $\tau = \frac{d\bar{t}}{d\varepsilon}\mid_{\varepsilon=0}$, and $\eta = \frac{d\bar{u}}{d\varepsilon}\mid_{\varepsilon=0}$.

Thus, by following transformation (3.1), prolongation $pr^{(\alpha,3)}X$ to equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(t,x,u,u_x,u_{xx},u_{xxx})$$

has the form

$$pr^{(\alpha,3)}X(\Delta)\mid_{\Delta=0} = 0, \qquad \Delta = \frac{\partial^\alpha u}{\partial t^\alpha} - F(t,x,u,u_x,u_{xx},u_{xxx}) = 0,$$

or

$$pr^{(\alpha,3)}X = X + \eta_\alpha^0\partial_{\partial_t^\alpha u} + \eta_1^1\partial_{u_x} + \eta_2^1\partial_{u_{xx}} + \eta_3^1\partial_{u_{xxx}}.$$

Our equation (1.1) can be expanded as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}u_x =& n(n-1)(n-2)Qg(t)u^{n-3}u_x^3 \\ &+ 3n(n-1)Qg(t)u^{n-2}u_xu_{xx} + nQg(t)u^{n-1}u_{xxx}.\end{aligned} \tag{3.5}$$

After putting transformations (3.1) and (3.3) into (3.5) and equating factor ε to zero, we get the following invariance criterion:

$$\begin{aligned}&\eta_\alpha^0 + nPu^{n-1}\eta_1^1 + n(n-1)Pu^{n-2}u_x\eta - n(n-1)(n-2)Qg'(t)u^{n-3}u_x^3\tau \\ &- n(n-1)(n-2)(n-3)Qg(t)u^{n-4}u_x^3\eta - 3n(n-1)(n-2)Qg(t)u^{n-3}u_x^2\eta_1^1 \\ &- 3n(n-1)Qg'(t)u^{n-2}u_xu_{xx}\tau - 3n(n-1)Qg(t)u^{n-2}u_x\eta_2^1 \\ &- 3n(n-1)(n-2)Qg(t)u^{n-3}u_xu_{xx}\eta - 3n(n-1)Qg(t)u^{n-2}u_{xx}\eta_1^1 \\ &- nQg'(t)u^{n-1}u_{xxx}\tau - nQg(t)u^{n-1}\eta_3^1 - n(n-1)Qg(t)u^{n-2}u_{xxx}\eta = 0.\end{aligned} \tag{3.6}$$

Here, by using the infinitesimals (3.3), we get following system:

$$
\begin{aligned}
&\xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0,\\
&Pu(\alpha\tau_t - \xi_x) + (n-1)P\eta - 3(n-1)Qg(t)\eta_{xx} - Qg(t)u(3\eta_{xxu} - \xi_{xxx}) = 0,\\
&g(t)u(\eta_u - \alpha\tau_t) - (n-3)g(t)\eta - 3g(t)u(\eta_u - \xi_x) - g'(t)u\tau = 0,\\
&g(t)u(\eta_u - \alpha\tau_t) - (n-2)g(t)\eta - g(t)u(2\eta_u - 3\xi_x) - g'(t)u\tau = 0,\\
&g(t)u(\eta_u - \alpha\tau_t) - (n-1)g(t)\eta - g(t)u(\eta_u - 3\xi_x) - g'(t)u\tau = 0,\\
&-(n-2)\eta_x - u(2\eta_{xu} - \xi_{xx}) = 0,\\
&-(n-1)\eta_x - u(\eta_{xu} - \xi_{xx}) = 0,\\
&\eta_{ut} - \tfrac{\alpha-1}{2}\tau_{tt} = 0,\\
&\tfrac{\partial^\alpha \eta}{\partial t^\alpha} - u\tfrac{\partial^\alpha \eta_u}{\partial t^\alpha} + nPu^{n-1}\eta_x - nQg(t)u^{n-1}\eta_{xxx} = 0.
\end{aligned}
\tag{3.7}
$$

According to the Lie group theory, we can classify the solutions of Eq. (1.1) as the following cases:

Case 1: For arbitrary $g(t)$ and $0 < \alpha \leq 1$ we obtain the following infinitesimals:

$$\xi = c, \qquad \tau = 0, \qquad \eta = 0,$$

where c is an arbitrary constant. We can obtain the corresponding infinitesimal operator:

$$X_1 = \frac{\partial}{\partial x}.$$

Case 2: For $g(t) = 1$ we obtain the following infinitesimals:

$$\xi = c_1, \qquad \tau = c_2 t + c_3, \qquad \eta = \frac{\alpha}{1-n}c_2 u,$$

where c_1, c_2, and c_3 are arbitrary constants and there are two additional infinitesimal operators:

$$X_2 = \frac{\partial}{\partial t}, \qquad X_3 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-n}u\frac{\partial}{\partial u}.$$

Case 3: For $g(t) = t^\lambda$ with $\lambda \neq 0$ we obtain the following infinitesimals:

$$\xi = c_1\frac{\lambda x}{2} + c_2, \qquad \tau = c_1 t, \qquad \eta = \frac{2\alpha - \lambda}{2-2n}c_1 u,$$

where c_1 and c_2 are arbitrary constants and there is an additional infinitesimal operator:

$$X_2 = \frac{\lambda}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2\alpha-\lambda}{2-2n}u\frac{\partial}{\partial u}.$$

Case 4: For $g(t) = e^t$ we obtain the following infinitesimals:

$$\xi = c_1 x + c_2, \qquad \tau = 2c_1, \qquad \eta = \frac{c_1}{n-1}u,$$

where c_1 and c_2 are arbitrary constants and there is one additional infinitesimal operator:

$$X_2 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial t} + \frac{1}{n-1}u\frac{\partial}{\partial u}.$$

4. Search for some exact solutions of the time-fractional nonlinear generalized KdV differential equation

In this section, we classify corresponding invariant solutions of the fractional nonlinear generalized KdV equation.

Case 1: For arbitrary $g(t)$ and $0 < \alpha \leq 1$ we have one infinitesimal operator:

$$X_1 = \frac{\partial}{\partial x}.$$

For this infinitesimal operator, the corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Solving it, we obtain the similarity transformation $u = \varphi(t)$, and so we have

$$D_t^\alpha \varphi(t) = 0.$$

The exact solution of the time-fractional nonlinear generalized KdV differential equation (1.1) is

$$u(t,x) = ct^{\alpha-1},$$

where c is arbitrary constant.

Case 2: For $g(t) = 1$ we have three infinitesimal operators:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-n}u\frac{\partial}{\partial u}.$$

Here, for infinitesimal operator X_2, by solving the corresponding characteristic equation

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0},$$

we can obtain the similarity transformation $u = \varphi(x)$, and by substituting it into (1.1) we have

$$u(t,x) = 0.$$

In the same way we get the similarity reduction for infinitesimal operator X_3 as

$$u = t^{\frac{\alpha}{1-n}}\varphi(x).$$

After substituting it into (1.1) we get

$$\frac{\varphi(x)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\varsigma)^{-\alpha}\varsigma^{\frac{\alpha}{1-n}}d\varsigma + nP(t^{\frac{\alpha}{1-n}}\varphi(x))^{n-1}t^{\frac{\alpha}{1-n}}\varphi'(x) = -nQ(t^{\frac{\alpha}{1-n}}\varphi(x))^{n-1}t^{\frac{\alpha}{1-n}}\varphi'''(x)$$
$$-n(n-1)(n-2)Q(t^{\frac{\alpha}{1-n}}\varphi(x))^{n-3}(t^{\frac{\alpha}{1-n}}\varphi'(x))^3 - 3n(n-1)Q(t^{\frac{\alpha}{1-n}}\varphi(x))^{n-2}t^{\frac{\alpha}{1-n}}\varphi'(x)t^{\frac{\alpha}{1-n}}\varphi''(x).$$

After some easy transformations we obtain the following nonlinear ODE:

$$\begin{aligned} &G(\alpha,n)\varphi(x) + nP(\varphi(x))^{n-1}\varphi'(x) - n(n-1)(n-2)Q(\varphi(x))^{n-3}(\varphi'(x))^3 \\ &-3n(n-1)Q(\varphi(x))^{n-2}\varphi'(x)\varphi''(x) - nQ(\varphi(x))^{n-1}\varphi'''(x) = 0, \end{aligned} \tag{4.1}$$

where $G(\alpha,n) = \frac{\Gamma\left(1+\frac{\alpha}{(1-n)}\right)}{\Gamma\left(1-\alpha+\frac{\alpha}{(1-n)}\right)}$.

Case 3: For $g(t)=t^{\lambda}$ with $\lambda\neq 0$ we have obtained the following two infinitesimal operators:

$$X_1=\frac{\partial}{\partial x},\qquad X_2=\frac{\lambda}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+\frac{2\alpha-\lambda}{2-2n}u\frac{\partial}{\partial u}.$$

Here the infinitesimal operator X_2, by solving the corresponding characteristic equations

$$\frac{dx}{\frac{\lambda}{2}x}=\frac{dt}{t}=\frac{du}{\frac{2\alpha-\lambda}{2-2n}u},$$

gives the similarity transformation with new variable p:

$$u(t,x)=t^{\frac{2\alpha-\lambda}{2-2n}}\varphi(p),\quad where\quad p=xt^{-\frac{\lambda}{2}}. \tag{4.2}$$

According to the above transformation, we have the following result.

Theorem: Upon the similarity transformation $u(t,x)=t^{\frac{2\alpha-\lambda}{2-2n}}\varphi(p)$ with $p=xt^{-\frac{\lambda}{2}}$, the time-fractional nonlinear generalized KdV differential equation with $g(t)=t^{\lambda}$ can be reduced to the following nonlinear fractional ODE [13]:

$$\begin{aligned}&\left(\mathrm{P}_{2/\lambda}^{1-\alpha+\frac{2\alpha-\lambda}{2-2n},\alpha}\varphi\right)(p)+nP(\varphi(p))^{n-1}\varphi'(p)=Qn(n-1)(n-2)(\varphi'(p))^3(\varphi(p))^{n-3}\\&+3Qn(n-1)\varphi'(p)\varphi''(p)(\varphi(p))^{n-2}+Qn\varphi'''(p)(\varphi(p))^{n-1},\end{aligned} \tag{4.3}$$

where $\mathrm{P}_{\gamma}^{\alpha,\beta}$ is the Erdelyi–Kober fractional differential operator:

$$\left(\mathrm{P}_{\gamma}^{\alpha,\beta}f\right)(x):=\prod_{j=0}^{n-1}\left(\alpha+j-\frac{1}{\gamma}x\frac{d}{dx}\right)(\mathrm{K}_{\gamma}^{\alpha+\beta,n-\beta}f)(x), \tag{4.4}$$

with

$$n=\begin{cases}[|\beta|]+1, & \beta\notin\mathbb{N},\\ \beta, & \beta\in\mathbb{N},\end{cases}$$

where $\mathrm{K}_{\gamma}^{\alpha,\beta}$ is Erdelyi–Kober fractional integral operator,

$$(\mathrm{K}_{\gamma}^{\alpha,\beta}f)(x):=\begin{cases}\frac{1}{\Gamma(\beta)}\int\limits_1^{\infty}(u-1)^{\beta-1}u^{-(\alpha+\beta)}f(xu^{1/\gamma})du, & \beta>0,\\ f(x), & \beta=0.\end{cases} \tag{4.5}$$

Proof: Let $m-1<\alpha<m$ and $m\in\mathbb{N}$ according to the time-fractional Riemann–Liouville derivative and the transformation (4.2), and we obtain

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int\limits_0^t(t-\varsigma)^{m-\alpha-1}\varsigma^{\frac{2\alpha-\lambda}{2-2n}}\varphi(x\varsigma^{-\frac{\lambda}{2}})d\varsigma\right].$$

Let $s=\frac{t}{\varsigma}$; then we get

$$\begin{aligned}&\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int\limits_1^{\infty}t^{m-\alpha-1+\frac{2\alpha-\lambda}{2-2n}+1}(s-1)^{m-\alpha-1}s^{-\frac{2\alpha-\lambda}{2-2n}1-m+\alpha-1}\varphi(ps^{\frac{\lambda}{2}})ds\right]\\&=\frac{d^m}{dt^m}\left[\frac{t^{m-\alpha+\frac{2\alpha-\lambda}{2-2n}}}{\Gamma(m-\alpha)}\int\limits_1^{\infty}(s-1)^{m-\alpha-1}s^{-(m-\alpha-1+\frac{2\alpha-\lambda}{2-2n})}\varphi(ps^{\frac{\lambda}{2}})ds\right]\end{aligned}$$

$$= \frac{d^m}{dt^m}\left[t^{m-\alpha+\frac{2\alpha-\lambda}{2-2n}}\left(\mathrm{K}_{\frac{2}{\lambda}}^{1+\frac{2\alpha-\lambda}{2-2n},m-\alpha}\varphi\right)(p)\right].$$

By considering that $p = xt^{-\frac{\lambda}{2}}$ and finding the derivative m times, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{d^{m-1}}{dt^{m-1}}\left[\frac{d}{dt}t^{m-\alpha+\frac{2\alpha-\lambda}{2-2n}}\left(\mathrm{K}_{\frac{2}{\lambda}}^{1+\frac{2\alpha-\lambda}{2-2n},m-\alpha}\varphi\right)(p)\right]$$

$$= \frac{d^{m-1}}{dt^{m-1}}\left[t^{m-\alpha+\frac{2\alpha-\lambda}{2-2n}-1}\left(m-\alpha+\frac{2\alpha-\lambda}{2-2n}-\frac{2}{\lambda}p\frac{d}{dp}\right)\left(\mathrm{K}_{\frac{2}{\lambda}}^{1+\frac{2\alpha-\lambda}{2-2n},m-\alpha}\varphi\right)(p)\right] =$$

$$\vdots$$

$$= t^{-\alpha+\frac{2\alpha-\lambda}{2-2n}}\prod_{j=0}^{m-1}\left(1+\frac{2\alpha-\lambda}{2-2n}-\alpha+j-\frac{2}{\lambda}p\frac{d}{dp}\right)(\mathrm{K}_{\frac{2}{\lambda}}^{1+\frac{2\alpha-\lambda}{2-2n},m-\alpha}\varphi)(p)$$

$$= t^{-\alpha+\frac{2\alpha-\lambda}{2-2n}}\left(\mathrm{P}_{\frac{2}{\lambda}}^{1+\frac{2\alpha-\lambda}{2-2n}-\alpha,\alpha}\varphi\right)(p).$$

Thus, we obtain the following nonlinear ODE, which is invariant to our nonlinear generalized KdV differential equation:

$$\left(\mathrm{P}_{2/\lambda}^{1-\alpha+\frac{2\alpha-\lambda}{2-2n},\alpha}\varphi\right)(p) + nP(\varphi(p))^{n-1}\varphi'(p) = Qn(n-1)(n-2)(\varphi'(p))^3(\varphi(p))^{n-3}$$
$$+ 3Qn(n-1)\varphi'(p)\varphi''(p)(\varphi(p))^{n-2} + Qn\varphi'''(p)(\varphi(p))^{n-1}.$$

This completes the proof.

5. Symmetry analysis of initial and boundary conditions

As we know, the PDE can describe real physical, biological, and socioeconomic processes if there are given initial and boundary conditions for the PDE [3].

Although Lie symmetry analysis is one of the most widely applicable methods of finding exact solutions of differential equations, it was not widely used for solving initial and boundary value problems [2]. The reason is that the corresponding initial and boundary conditions are usually not invariant under any transformations. That is, not all symmetries of the PDE leave the initial and boundary conditions invariant as well. For the PDE, an invariant solution resulting from applying symmetry transformation solves a given boundary value problem while the symmetry transformation leaves invariant all boundary conditions and the domain of the boundary value problem. In [2], Bluman gave a definition of Lie's invariance for initial and boundary conditions by means of which there are some classes of initial and boundary value problems that can be solved.

In this work we study the initial and boundary value problem for the generalized KdV equation, so we consider $\alpha = 1$ and our problem (1.1) takes the following form:

$$\begin{cases} u_t + P(u^n)_x = Qg(t)(u^n)_{xxx}, & \\ u(t,0) = \Psi(t), & t \in [0,T], \\ u(0,x) = \Phi(x), & x \in \mathbb{R}^+. \end{cases} \tag{5.1}$$

By using Bluman's definition of the boundary value problem below, we obtain symmetry analysis for problem (5.1).

Let the Lie symmetry X be as follows:

$$X = \xi(t,x)\frac{\partial}{\partial x} + \tau(t,x)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{5.2}$$

admitted by the boundary value problem defined on a domain Ω:

$$u_t = F\left(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^k u}{\partial x^k}\right), \qquad (t,x) \in \Omega \subset \mathbb{R}^2, \tag{5.3}$$

$$d_a(t,x) = 0 : B_a\left(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^{k-1} u}{\partial x^{k-1}}\right) = 0, \qquad a = 1,...,p. \tag{5.4}$$

Here $B_a(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^{k-1}u}{\partial x^{k-1}})$ is the boundary condition on $d_a(t,x)$. Suppose that the above boundary value problem has a unique solution.

Definition 5.1 *Symmetry X, which has the form (5.2), is allowed by the boundary value problem (5.3, 5.4) if:*

- $X^{(k)}(u_t - F(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^k u}{\partial x^k})) = 0$ *for* $u_t = F(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^k u}{\partial x^k})$;
- $Xd_a(t,x) = 0$ *for* $d_a(t,x) = 0$, $a = 1,...,p$;
- $X^{(k-1)}B_a(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^{k-1}u}{\partial x^{k-1}}) = 0$ *for* $B_a(t,x,u,\frac{\partial u}{\partial x},...,\frac{\partial^{k-1}u}{\partial x^{k-1}}) = 0$ *on* $d_a(t,x) = 0$, $a = 1,..,p$.

According to Bluman's definition, the invariance of $t = 0$ leads to $\tau(0) = 0$ and the invariance of $x = 0$ leads to $\xi(0) = 0$. Thus, for each $g(t)$, we have:

Case 1: For arbitrary $g(t)$ we have only one infinitesimal operator, $X = c_1\frac{\partial}{\partial x}$, and as the invariance of $x = 0$ leads to $\xi(0) = 0$, which gives $c_1 = 0$, it means that for arbitrary $g(t)$ the initial boundary value problem (5.1) has no symmetries.

Case 2: For $g(t) = 1$ we get $c_1 = 0$ and $c_3 = 0$ with $t\Psi'(t) = \frac{c_2}{1-n}\Psi(t)$ and $\frac{c_2}{1-n}\Phi(x) = 0$, which gives us $\Psi(t) = t^{\frac{1}{1-n}}M$ and $\Phi(x) = 0$, respectively, where M is an arbitrary constant. We have one infinitesimal operator,

$$X = t\frac{\partial}{\partial t} + \frac{1}{1-n}u\frac{\partial}{\partial u},$$

which leaves the initial boundary value problem (5.1) invariant with initial and boundary conditions

$$\begin{cases} u(t,0) = t^{\frac{1}{1-n}}M, & t \in [0,T], \\ u(0,x) = 0, & x \in \mathbb{R}^+. \end{cases}$$

By using this operator we get transformation $u(t,x) = t^{\frac{1}{1-n}}\psi(x)$ and can obtain the invariant to problem (5.1), an ODE with initial condition as below:

$$\begin{cases} \frac{1}{1-n}\psi(x) + nP\psi(x)^{n-1}\psi'(x) - Q(n(n-1)(n-2)\psi(x)^{n-3}\psi'(x)^3 + \\ 3n(n-1)\psi(x)^{n-2}\psi'(x)\psi''(x) + n\psi(x)^{n-1}\psi'''(x)) = 0, \\ \psi(0) = M. \end{cases} \tag{5.5}$$

Case 3: For $g(t) = t^\lambda$ we get $c_2 = 0$ with two equations, $\frac{x\lambda}{2}\Phi'(x) - \frac{2-\lambda}{2-2n}\Phi(x) = 0$ and $t\Psi'(t) = \frac{2-\lambda}{2-2n}\Psi(t)$, which give us $\Phi(x) = x^{\frac{2-\lambda}{\lambda(1-n)}}N_1$ and $\Psi(t) = t^{\frac{2-\lambda}{2(1-n)}}N_2$, where N_1, N_2 are arbitrary constants. Thus, the infinitesimal operator,

$$X = \frac{\lambda}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2-\lambda}{2-2n}u\frac{\partial}{\partial u},$$

leaves problem (5.1) invariant with initial and boundary conditions $u(t,0) = t^{\frac{2-\lambda}{2(1-n)}}N_2$, $u(0,x) = x^{\frac{2-\lambda}{\lambda(1-n)}}N_1$, and gives us transformation $u(t,x) = t^{\frac{2-\lambda}{2(1-n)}}\varphi(z)$ with $z = xt^{-\lambda/2}$. By using this transformation we can obtain the following invariant to problem (5.1), an ODE with initial condition

$$\begin{cases} \frac{2-\lambda}{2(1-n)}\varphi(z) - \frac{1}{2}\lambda z\varphi'(z) + nP\varphi(z)^{n-1}\varphi'(z) - Q(n(n-1)(n-2)\varphi(z)^{n-3}\varphi'(z)^3 + \\ 3n(n-1)\varphi(z)^{n-2}\varphi'(z)\varphi''(z) + n\varphi(z)^{n-1}\varphi'''(z)) = 0, \\ \varphi(0) = N_2. \end{cases} \tag{5.6}$$

Here we want to note that the infinitesimal operator $X = t\frac{\partial}{\partial t} + \frac{1}{1-n}u\frac{\partial}{\partial u}$ gives one more transformation, $u(t,x) = x^{\frac{2-\lambda}{\lambda(1-n)}}\vartheta(\iota)$ with $\iota = x^{-2/\lambda}t$. This transformation gives us an invariant ODE with initial condition as below:

$$\begin{cases} \vartheta'(\iota) + Pn\vartheta(\iota)^{n-1}(\frac{2-\lambda}{a(1-n)}\vartheta(\iota) - \frac{2\iota}{\lambda}\vartheta'(\iota)) - Q\iota^\lambda(n(n-1)(n-2)\vartheta(\iota)^{n-3}(\frac{2-\lambda}{a(1-n)}\vartheta(\iota) - \frac{2\iota}{\lambda}\vartheta'(\iota))^3 + \\ 3n(n-1)\vartheta(\iota)^{n-2}(\frac{2-\lambda}{a(1-n)}\vartheta(\iota) - \frac{2\iota}{\lambda}\vartheta'(\iota))(\frac{2-\lambda}{a(1-n)}(\frac{2-\lambda}{a(1-n)} - 1)\vartheta(\iota) - \frac{4n+4-6\lambda+2\lambda n}{\lambda(1-n)}\iota\vartheta'(\iota) + \frac{4}{\lambda^2}\iota^2\vartheta''\iota) + \\ n\vartheta(\iota)^{n-1}(\frac{2-\lambda}{a(1-n)}(\frac{2-\lambda}{a(1-n)} - 1)(\frac{2-\lambda}{a(1-n)} - 2)\vartheta(\iota) - \frac{2(2-\lambda)}{\lambda^2(1-n)}(\frac{3\lambda-3\lambda n-2n+2n}{\lambda(1-n)} - \frac{2(\lambda+1)(1-n)}{2-\lambda})\iota\vartheta'(\iota) + \\ \frac{12n(2-\lambda)}{\lambda^3(1-n)}\iota^2\vartheta''(\iota) - \frac{8}{\lambda^3}\iota^3\vartheta'''(\iota))) = 0, \\ \varphi(0) = N_1. \end{cases} \tag{5.7}$$

Case 4: For $g(t) = e^t$ we have the same situation as for arbitrary $g(t)$, and because of $c_1 = 0$ and $c_2 = 0$ there are no symmetries for (5.1).

Ordinary boundary value problems (5.5), (5.6), and (5.7) obtained above can be solved by different methods like the power series method, the reduction method introduced by Nucci, or by numerical methods [6, 17, 18].

The work of Bluman was based on the PDE of the following form:

$$u_t = F\left(t, x, u, \frac{\partial u}{\partial x}, ..., \frac{\partial^k u}{\partial x^k}\right),$$

with finite and infinite x, and t values of initial and boundary conditions for the above equation, respectively. Our future study will be based on the PDE of the form

$$D_t^\alpha u = F\left(t, x, u, \frac{\partial u}{\partial x}, ..., \frac{\partial^k u}{\partial x^k}\right),$$

with $\alpha \neq 1$ and finite and infinite x, and t values of initial and boundary conditions for the above equation.

6. Conservation laws

Now we will construct the conservation laws of time-fractional nonlinear generalized KdV differential equation (1.1) using Ibragimov's theorem [10]. This theorem was given for differential equations with integer order, but it can be applied to fractional differential equations [14, 23]. A vector field $C = (C^t, C^x)$ where $C^t = C^t(t, x, u, u_x, ...)$ and $C^x = C^x(t, x, u, u_x, ...)$ is called a conserved vector for (1.1) on all its solutions if it satisfies the following conservation equation:

$$D_t(C^t) + D_x(C^x) = 0, \tag{6.1}$$

where C^t and C^x is the conservation law for Eq. (1.1).

It should be noted that our Eq. (1.1) can be written in the form of conservation law (6.1) as follows:

$$C^t = I_t^{1-\alpha} u, \qquad C^x = Pu^n - Qg(t)(u^n)_{xx}.$$

A formal Lagrangian function for (1.1) is given by

$$L = v(t, x)E. \tag{6.2}$$

Here $v(t, x)$ is a new dependent variable and

$$E = \frac{\partial^\alpha u}{\partial t^\alpha} + nPu^{n-1}u_x - n(n-1)(n-2)Qg(t)u^{n-3}u_x^3 - 3n(n-1)Qg(t)u^{n-2}u_x u_{xx} - nQg(t)u^{n-1}u_{xxx}.$$

The Euler–Lagrange operator with respect to u is defined by [4, 14]

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}}, \tag{6.3}$$

where $(D_t^\alpha)^*$ is an adjoint operator of D_t^α, which has the form

$$(D_t^\alpha)^* = (-1)_t^n I_T^{n-\alpha}(D_t^n). \tag{6.4}$$

Then the adjoint equation of Eq. (1.1) is defined by

$$\frac{\delta L}{\delta u} = 0. \tag{6.5}$$

After calculations, Eq. (6.5) takes the following form:

$$\frac{\delta L}{\delta u} = (D_t^\alpha)^* v - nv_x Pu^{n-1} + nv_{xxx}Qg(t)u^{n-1}.$$

The definition of the nonlinear self-adjointness of the differential equation that is integer-order [10, 11] can be extended to the time-fractional nonlinear generalized KdV differential equation. We thus say that the generalized KdV equation is nonlinearly self-adjoint if the adjoint equation (6.5) is satisfied for all solutions u of equation (1.1) upon a substitution $v=\varphi(t,x,u)$ and $\varphi(t,x,u)\neq 0$. We find the factor $\varphi(t,x,u)=-\frac{u^n}{n}$ by substitution $v=\varphi(t,x,u)$ and its necessary derivatives into (6.5) considering the self-adjoint condition

$$\frac{\delta L}{\delta u}|_{v=\varphi(t,x,u)}=\Lambda E,$$

where Λ is a certain function.

Further, as our equation (1.1) does not involve the space-fractional derivative with respect to x, the conservation law for the x-component has the form [4]

$$\begin{aligned} C^x =&\xi L+W_i\left(\frac{\partial L}{\partial u_x}-D_x\frac{\partial L}{\partial u_{xx}}+D_x^2\frac{\partial L}{\partial u_{xxx}}\right)\\ &+D_x(W_i)\left(\frac{\partial L}{\partial u_{xx}}-D_x\frac{\partial L}{\partial u_{xxx}}\right)+D_x^2(W_i)\frac{\partial L}{\partial u_{xxx}}, \end{aligned} \tag{6.6}$$

where $W_i=\eta_i-\xi_i u_x-\tau_i u_t$.

As our equation (1.1) involves the time-fractional derivative with respect to t, the conservation law for the t-component has the form [14]

$$C^t=\tau L+\sum_{k=0}^{m-1}(-1)^k D_t^{\alpha-1-k}(W_i)D_t^k(\frac{\partial L}{\partial D_t^\alpha u})-(-1)^m J(W_i,D_t^m\frac{\partial L}{\partial D_t^\alpha u}) \tag{6.7}$$

for $m-1<\alpha<m$ and J is a integral:

$$J(f,g)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^T\frac{f(x,s)g(x,p)}{(p-s)^{\alpha+1-m}}dpds.$$

Thus, by using (6.2) and $\varphi(t,x,u)=-\frac{u^n}{n}$ C^x with C^t, our problem takes the form

$$\begin{aligned} C^x =&\xi(-\frac{u^n}{n}(D_t^\alpha u+nPu^{n-1}u_x-Qg(t)(n(n-1)(n-2)u^{n-3}u_x^3-3n(n-1)u^{n-2}u_xu_{xx}-nu^{n-1}u_xu_{xxx})))+\\ &W_i\left(-Pu^{2n-1}+(n-1)(n-2)Qg(t)u^{2n-3}u_x^2+(2n-1)Qg(t)u^{2n-2}u_{xx}\right)\\ &+D_x(W_i)(n-2)Qg(t)u^{2n-2}u_x+D_x^2(W_i)Qg(t)u^{2n-1}, \end{aligned} \tag{6.8}$$

$$C^t=-\frac{u^n}{n}D_t^{\alpha-1}(W_i)+J(W_i,-u^{n-1}u_t). \tag{6.9}$$

At this rate, for general $g(t)$ we have $X_1=\frac{\partial}{\partial x}$ infinitesimal operator, which gives $W_1=-u_x$ and

$$\begin{aligned} C^x =&-\frac{u^n}{n}(D_t^\alpha u+nPu^{n-1}u_x-Qg(t)(n(n-1)(n-2)u^{n-3}u_x^3-3n(n-1)u^{n-2}u_xu_{xx}-nu^{n-1}u_xu_{xxx}))+\\ &Pu^{2n-1}u_x-(n-1)(n-2)Qg(t)u^{2n-3}u_x^3-3(n-1)Qg(t)u^{2n-2}u_xu_{xx}-Qg(t)u^{2n-1}u_xu_{xxx}, \end{aligned} \tag{6.10}$$

$$C^t=-\frac{u^n}{n}I_t^{1-\alpha}(-u_x)+J(-u_x,-u^{n-1}u_t). \tag{6.11}$$

For $g(t)=t^\lambda$, we have $X_2=\frac{2\alpha-\lambda}{2-2n}\frac{\partial}{\partial u}+t\frac{\partial}{\partial t}+\frac{\lambda}{2}x\frac{\partial}{\partial x}$ infinitesimal operator, which gives $W_2=\frac{2\alpha-\lambda}{2-2n}u-tu_t-\frac{\lambda}{2}xu_x$ and

$$\begin{aligned} C^x =&-\frac{\lambda x u^n}{2n}(D_t^\alpha u+nPu^{n-1}u_x-Qg(t)(n(n-1)(n-2)u^{n-3}u_x^3-3n(n-1)u^{n-2}u_xu_{xx}-nu^{n-1}u_xu_{xxx}))+\\ &\left(\frac{2\alpha-\lambda}{2-2n}u-tu_t-\frac{\lambda}{2}xu_x\right)\left(-Pu^{2n-1}+(n-1)(n-2)Qt^\lambda u^{2n-3}u_x^2+(2n-1)Qt^\lambda u^{2n-2}u_{xx}\right)\\ &+\left(\frac{2\alpha-\lambda}{2-2n}u_x-tu_{xt}-\frac{\lambda}{2}xu_{xx}-\frac{\lambda}{2}u_x\right)(n-2)Qt^\lambda u^{2n-2}u_x\\ &+\left(\frac{2\alpha-\lambda}{2-2n}u_{xx}-tu_{xxt}-\lambda u_{xx}-\frac{\lambda}{2}xu_{xxx}\right)Qt^\lambda u^{2n-1}, \end{aligned} \tag{6.12}$$

$$C^t = -\frac{u^n}{n} I_t^{1-\alpha}\left(\frac{2\alpha-\lambda}{2-2n}u - tu_t - \frac{\lambda}{2}xu_x\right) + J\left(\frac{2\alpha-\lambda}{2-2n}u - tu_t - \frac{\lambda}{2}xu_x, -u^{n-1}u_t\right). \tag{6.13}$$

For $g(t)=1$, we have $X_3 = \frac{\partial}{\partial t}$ and $X_4 = \frac{\alpha}{1-n}u\frac{\partial}{\partial u} + t\frac{\partial}{\partial t}$ infinitesimal operators, which give $W_3 = u_t$, $W_4 = \frac{\alpha}{1-n}u - tu_t$, and

$$\begin{aligned} C^x =& Pu^{2n-1}u_t - (n-1)(n-2)Qg(t)u^{2n-3}u_x^2u_t - (2n-1)Qg(t)u^{2n-2}u_tu_{xx} \\ &- (n-1)Qg(t)u^{2n-2}u_xu_{tx} - Qg(t)u^{2n-1}u_xu_{xxt}, \end{aligned} \tag{6.14}$$

$$C^t = -\frac{u^n}{n}I_t^{1-\alpha}(-u_t) + J(-u_t, -u^{n-1}u_t). \tag{6.15}$$

$$\begin{aligned} C^x =& \left(\frac{\alpha}{1-n}u - tu_t\right)\left(-Pu^{2n-1} + (n-1)(n-2)Qt^\lambda u^{2n-3}u_x^2 + (2n-1)Qt^\lambda u^{2n-2}u_{xx}\right) \\ &+ \left(\frac{\alpha}{1-n}u_x - tu_{xt}\right)(n-2)Qt^\lambda u^{2n-2}u_x + \left(\frac{\alpha}{1-n}u_{xx} - tu_{xxt}\right)Qt^\lambda u^{2n-1}, \end{aligned} \tag{6.16}$$

$$C^t = -\frac{u^n}{n}I_t^{1-\alpha}\left(\frac{\alpha}{1-n}u - tu_t\right) + J\left(\frac{\alpha}{1-n}u - tu_t, -u^{n-1}u_t\right). \tag{6.17}$$

7. Conclusion

In this work, we illustrate the application of Lie group analysis to study time-fractional nonlinear partial differential equations. We introduce a new solution for the nonlinear generalized time-fractional KdV equation. Using the Lie point symmetries, we showed that the equation can be reduced to an equation with the Erdelyi–Kober fractional derivative. We also investigated conservation laws for the given equation and we studied the initial and boundary conditions. For $\alpha = 1$ we found invariants to our boundary value problem. We are now working on symmetry analysis for the boundary value problem with $\alpha \neq 1$.

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