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A new solution of pair matrix equations with arbitrary triangular fuzzy numbers

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Abstract: Pair matrix equations have numerous applications in control system engineering, such as for stability analysis of linear control systems and also for reduction of nonlinear control system models. There are some situations in which the classical pair matrix equations are not well equipped to deal with the uncertainty problem during the process of stability analysis and reduction in control system engineering. Thus, this study presents a new algorithm for solving fully fuzzy pair matrix equations where the parameters of the equations are arbitrary triangular fuzzy numbers. The fuzzy Kronecker product and fuzzy Vec-operator are employed to transform the fully fuzzy pair matrix equations to a fully fuzzy pair linear system. Then a new associated linear system is developed to convert the fully fuzzy pair linear system to a crisp linear system. Finally, the solution is obtained by using a pseudoinverse method. Some related theoretical developments and examples are constructed to illustrate the proposed algorithm. The developed algorithm is also able to solve the fuzzy pair matrix equation.

Key words: Fully fuzzy pair matrix equation, fully fuzzy linear system, Kronecker product, Vec-operator, associated linear system

1. Introduction

In real world applications, matrix equations play an essential role in several situations. In the literature, a few researchers reported that the matrix equation has been used in control system engineering [30], image restoration [6], model reduction [5], signal processing [28], and stochastic control [31]. In control system engineering, for example, the matrix equation is used as a technical tool in stability analysis of linear control system and also in reduction of nonlinear control system models.

Considering that many uncertain situations may occur during system processes, such as conflicting requirements during the system process, instability of environmental conditions [3], and noise distraction [1], the classical matrix equation is sometimes not well equipped to deal with those situations. Thus, a fuzzy number has been embedded to deal with the uncertainty parameters.

To date, considerable work has been conducted on matrix equations, such as the fuzzy matrix equation (FME), $A\tilde{X}_m = \tilde{B}_m$ [17], and fuzzy Sylvester matrix equation (FSE), $A\tilde{X} + \tilde{X}B = \tilde{C}$ [2, 15, 16, 18, 19, 27]. In these studies, some of the parameters were considered in the form of fuzzy numbers. On the other hand, there are also a number of studies in which all the parameters of the matrix equations are in the form of fuzzy

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numbers. As examples there are the fully fuzzy matrix equation (FFME) of $\tilde{A}\tilde{X}_m = \tilde{B}_m$ [25], fully fuzzy continuous-time Sylvester matrix equation of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ [23, 29] and $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ [8, 13], and also the fully fuzzy discrete-time Sylvester matrix equation of $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$ [9].

Meanwhile, there are some applications where two matrix equations are required to be solved simultaneously. In this case, the combination of two matrix equations is called a pair matrix equation (PME). The PME is also important in various applications, such in control systems. According to [30], the PME is used to make the computational process less complicated, especially in analyzing the stability of control systems so that the control system always performs well according to its specifications.

In the classical case of the PME, where the parameters of the PME are in crisp form, many studies have been conducted, such as [7, 11, 32]. However, far fewer studies have been conducted for solving the PME with a fuzzy environment, such as [26] and also [10]. In [26], the PME consists of $A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1$ and $A_2\tilde{X}B_2 = \tilde{C}_2$, where only some of the parameters of the equation are in the form of arbitrary fuzzy numbers. A numerical iterative method was used to obtain the approximate solutions. In [10], all the parameters of the PME are in fuzzy form, whereby the fully fuzzy matrix equation (FFME) involved consists of $\tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1$ and $\tilde{A}_2\tilde{X}\tilde{B}_2 - \tilde{X} = \tilde{C}_2$. In that study, a direct method was proposed to solve the positive PFFME. Due to the limitations of these two studies, we aim to provide an algorithm for solving an arbitrary PFFME of

$$\begin{cases} \tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1 \\ \tilde{A}_2\tilde{X}\tilde{B}_2 = \tilde{C}_2, \end{cases} \tag{1}$$

where $\tilde{A}_i$ and $\tilde{B}_i$ $(i = 1, 2)$ are arbitrary fuzzy matrices with some common sizes, whereas $\tilde{C}_i$ $(i = 1, 2)$ are arbitrary common size fuzzy matrices, where the fuzzy matrix $\tilde{X}$ is to be determined.

The main contribution of this study would be the improvement of the associated linear system that was originally constructed in [22]. Besides that, the fuzzy Kronecker product and fuzzy Vec-operator are also utilized in this algorithm to convert the PFFME into a simpler form of equations. The development of the algorithm presented in this study provides the first investigation on how to solve the PFFME in Eq. (1). With that, the algorithm will contribute to real-world applications, such as the process of analyzing the stability of linear control systems involved with the uncertainty problem.

The remaining part of the paper proceeds as follows. In Section 2, the fundamental concepts of fuzzy set theory and Kronecker operation are provided. In Section 3, new definitions, theorems, and corollaries are defined and then a new algorithm for solving the PFFME is constructed. Next, two numerical examples are illustrated in Section 5 and the solution of the PFME is shown in Section 6. Finally, the conclusion is drawn in Section 7.

2. Preliminaries

This section will recall some definitions and theorems that will be used in this study.

Definition 1 *[33] A fuzzy number is a function such as $u : R \to [0, 1]$ satisfying the following properties:*

- u is normal; that is, there exists an $x_0 \in R$ such that $u(x_0) = 1$.
- u is fuzzy convex; that is, $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$ for any $x, y \in R$, $\lambda \in [0, 1]$.

- u is upper semicontinuous.
- supp $u = \{x \in R | u(x) > 0\}$ is the support of u, and its closure cl(supp u) is compact.

Definition 2 *[33] A fuzzy number* $\tilde{M} = (m, \alpha, \beta)$ *is said to be a triangular fuzzy number (TFN) if its membership function is given by:*

$$\mu_{\tilde{M}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq m + \beta, \beta > 0, \\ \quad 0, & \textit{otherwise}. \end{cases} \tag{2}$$

In this case, m is the mean value of $\tilde{M}$, whereas α and β are right and left spreads, respectively.

Definition 3 *[33] A fuzzy number* $\tilde{M} = (m, \alpha, \beta)$ *is called an arbitrary fuzzy number where it may be positive, negative, or near zero, which can be classified as follows:*

- $\tilde{M}$ *is a positive (negative) fuzzy number iff* $m - \alpha \geq 0$ $(\beta + m \leq 0)$.
- $\tilde{M}$ *is a zero fuzzy number if* $(m = 0,\ \alpha,\ \beta = 0)$.
- $\tilde{M}$ *is a near zero fuzzy number iff* $m - \alpha \leq 0 \leq \beta + m$.

Definition 4 *[14] The arithmetic operations of two positive fuzzy numbers* $\tilde{M} = (m, \alpha, \beta)$ *and* $\tilde{N} = (n, \gamma, \delta)$ *are as follows:*

- Addition:

$$\tilde{M} \oplus \tilde{N} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta). \tag{3}$$

- Opposite:

$$-\tilde{M} = -(m, \alpha, \beta) = (-m, \beta, \alpha). \tag{4}$$

- Subtraction:

$$\tilde{M} \ominus \tilde{N} = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma). \tag{5}$$

- Multiplication:

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta). \tag{6}$$

Definition 5 *[20] Let* $\tilde{M} = (m, \alpha, \beta)$ *and* $\tilde{N} = (n, \gamma, \delta)$ *be two arbitrary triangular fuzzy numbers. Then Kaufmann's approximation for multiplication of arbitrary triangular fuzzy numbers is defined as:*

$$\tilde{M} \otimes \tilde{N} = (f, p, q), \tag{7}$$

where $f = mn$, $p = f - r$, $q = s - f$,

$$r = min((m - \alpha)(n - \gamma), (m - \alpha)(n + \delta)) = \begin{cases} (m - \alpha)(n - \gamma) & \text{if} \quad m - \alpha \geq 0 \\ (m - \alpha)(n + \delta) & \text{if} \quad m - \alpha < 0, \end{cases} \tag{8}$$

$$s = max((m + \beta)(n - \gamma), (m + \beta)(n + \delta)) = \begin{cases} (m + \beta)(n - \gamma) & \text{if} \quad m + \beta < 0 \\ (m + \beta)(n + \delta) & \text{if} \quad m + \beta \geq 0. \end{cases} \tag{9}$$

Theorem 1 *[24] Consider an arbitrary fuzzy number* $\tilde{M} = (m^a, \alpha^a, \beta^a)$ *and positive fuzzy solution* $\tilde{X} = (m^x, \alpha^x, \beta^x)$.

- *If* $\tilde{M}$ *is positive, then the following inequalities are satisfied:*

$$0 \leq (m^x - \alpha^x)(m^a - \alpha^a) \leq (m^x + \beta^x)(m^a - \alpha^a), \tag{10}$$

$$0 \leq (m^x - \alpha^x)(m^a + \beta^a) \leq (m^x + \beta^x)(m^a + \beta^a). \tag{11}$$

- *If* $\tilde{M}$ *is negative, then the following inequalities are satisfied:*

$$0 \geq (m^x - \alpha^x)(m^a - \alpha^a) \geq (m^x + \beta^x)(m^a - \alpha^a), \tag{12}$$

$$0 \geq (m^x - \alpha^x)(m^a + \beta^a) \geq (m^x + \beta^x)(m^a + \beta^a). \tag{13}$$

- *If* $\tilde{M}$ *is near zero, then the inequalities in Eqs. (11) and (12) are satisfied for all* $\tilde{X}$.

Definition 6 *[12] An* $n \times n$ *fully fuzzy linear system (FFLS) is defined as follows:*

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + ... + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + ... + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \vdots \\ \tilde{a}_{m1}\tilde{x}_1 + \tilde{a}_{m2}\tilde{x}_2 + ... + \tilde{a}_{mn}\tilde{x}_n = \tilde{b}_m, \end{cases} \tag{14}$$

which can also be written in a matrix form of

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_m \end{pmatrix}, \tag{15}$$

and it is usually denoted in a form of

$$\tilde{M}\tilde{X} = \tilde{B}, \tag{16}$$

where all the entries $\tilde{M}, \tilde{B}$, *and* $\tilde{X}$ *are arbitrary triangular fuzzy numbers.*

Definition 7 *[4] Let* $(A)_{ij}$ *be any real* $m \times n$ *matrix. The pseudoinverse of* A *is an* $n \times m$ *matrix* X *satisfying the following conditions:*

- $AXA = A$,
- $XAX = X$,
- $(AX)^T = AX$,
- $(XA)^T = XA$.

Definition 8 *[23] Let $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{p\times q}$ be fuzzy matrices. The fuzzy Kronecker product is represented as $\tilde{A} \otimes_k \tilde{B}$ with the operation*

$$\tilde{A} \otimes_k \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{B} & \tilde{a}_{m2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix} \tag{17}$$
$$= [\tilde{a}_{ij}\tilde{B}]_{(mp)\times(nq)}.$$

Definition 9 *[23] The Vec-operator of a fuzzy matrix is a linear transformation that converts the fuzzy matrix of $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$ into a column vector as*

$$Vec(\tilde{C}) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix}. \tag{18}$$

Theorem 2 *[23] If $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$ is a fuzzy matrix and $\tilde{U} = (\tilde{u}_{ij})_{p\times p}$ is a unitary fuzzy matrix defined as*

$$\tilde{U} = \begin{pmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{pmatrix}, \tag{19}$$

then

- $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$,
- $\tilde{U}^T = \tilde{U}$.

Theorem 3 *[23] Let $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$, and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$. Then:*

- $Vec[\tilde{A}\tilde{X}] = [\tilde{U}_n \otimes_k \tilde{A}]Vec(\tilde{X})$,
- $Vec[\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{U}_m]Vec(\tilde{X})$,

where $\tilde{U}_m$ and $\tilde{U}_n$ denote the fuzzy unitary matrices with orders m and n, respectively.

Theorem 4 *[23] Let $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$, and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$. Then the FFME $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ can be transformed as an FFLS:*

$$[(\tilde{U}_n \otimes_k \tilde{A}) + (\tilde{B}^T \otimes_k \tilde{U}_m)]Vec(\tilde{X}) = Vec(\tilde{C}), \tag{20}$$

where $\tilde{U}_m$ and $\tilde{U}_n$ denote the fuzzy unitary matrices with orders m and n, respectively.

3. Theoretical development for solving the PFFME

This section contains definitions, theorems, and a corollary that have been developed for solving the PFFME.

Theorem 5 *Let* $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$ *and* $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$. *Then*

$$\tilde{A} \otimes \tilde{B} = [(\tilde{A} \otimes \tilde{U})(\tilde{U} \otimes \tilde{B})]. \tag{21}$$

Proof According to Definition 8 and also the concept of matrix multiplication,

$$\tilde{A} \otimes \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{21}\tilde{B} & \dots & \tilde{a}_{m1}\tilde{B} \\ \tilde{a}_{12}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{m2}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1m}\tilde{B} & \tilde{a}_{2m}\tilde{B} & \dots & \tilde{a}_{mm}\tilde{B} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}(\tilde{A} \otimes \tilde{U})(\tilde{U} \otimes \tilde{B}) &= \begin{pmatrix} \tilde{a}_{11}\tilde{U} & \tilde{a}_{21}\tilde{U} & \dots & \tilde{a}_{m1}\tilde{U} \\ \tilde{a}_{12}\tilde{U} & \tilde{a}_{22}\tilde{U} & \dots & \tilde{a}_{m2}\tilde{U} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1m}\tilde{U} & \tilde{a}_{2m}\tilde{U} & \dots & \tilde{a}_{mm}\tilde{U} \end{pmatrix} \begin{pmatrix} \tilde{B} & 0 & \dots & 0 \\ 0 & \tilde{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{B} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{21}\tilde{B} & \dots & \tilde{a}_{m1}\tilde{B} \\ \tilde{a}_{12}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{m2}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1m}\tilde{B} & \tilde{a}_{2m}\tilde{B} & \dots & \tilde{a}_{mm}\tilde{B} \end{pmatrix}.\end{aligned}$$

Hence, the theorem is proved. □

Theorem 6 *Let* $\tilde{A} = (\tilde{a}_{ij})_{p\times q}$, $\tilde{B} = (\tilde{b}_{ij})_{r\times s}$, *and* $\tilde{X} = (\tilde{x}_{ij})_{q\times r}$. *Then*

$$Vec[\tilde{A}\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{A}]Vec(\tilde{X}). \tag{22}$$

Proof Use Theorem 2(1) to write a matrix $\tilde{X}$ as

$$\tilde{X} = \tilde{X}\tilde{U}.$$

Then

$$\begin{aligned} Vec[\tilde{A}\tilde{X}\tilde{B}] &= Vec[\tilde{A}\tilde{X}\tilde{U}^T\tilde{B}] && \text{by Theorem 2(2)} \\ &= Vec[(\tilde{A}\tilde{X})(\tilde{B}^T\tilde{U})^T] && \text{by matrix transpose} \\ &= (\tilde{B}^T\tilde{U} \otimes_k \tilde{U})Vec(\tilde{A}\tilde{X}) && \text{by Theorem 3(2)} \\ &= (\tilde{B}^T \otimes_k \tilde{U})Vec(\tilde{A}\tilde{X}) && \text{by Theorem 2(1)} \\ &= (\tilde{B}^T \otimes_k \tilde{U})[(\tilde{U} \otimes \tilde{A})Vec(\tilde{X})] && \text{by Theorem 3(1)} \\ &= [(\tilde{B}^T \otimes_k \tilde{U})(\tilde{U} \otimes_k \tilde{A})]Vec(\tilde{X}) && \text{by associative matrix} \\ &= (\tilde{B}^T \otimes_k \tilde{A})Vec(\tilde{X}) && \text{by Theorem 5.} \end{aligned}$$

Therefore, the theorem is obviously proved. □

Corollary 1 *Let* $\tilde{A}_1 = (\tilde{a}_{1ij})_{p\times p}$, $\tilde{A}_2 = (\tilde{a}_{2ij})_{r\times p}$, $\tilde{B}_1 = (\tilde{b}_{1ij})_{q\times q}$, *and* $\tilde{B}_2 = (\tilde{b}_{2ij})_{q\times s}$ *be parameters of the PFFME. Then a PFFME is equivalent to a pair of fully fuzzy linear systems (PFFLS):*

$$\begin{cases} \left[\left(\tilde{U}_{q\times q} \otimes_k (\tilde{A}_1)_{p\times p}\right) + \left((\tilde{B}_1^T)_{q\times q} \otimes_k \tilde{U}_{p\times p}\right)\right] \tilde{X}v = \tilde{C}v_1 \\ \left((\tilde{B}_2^T)_{s\times q} \otimes_k (\tilde{A}_2)_{r\times p}\right) \tilde{X}v = \tilde{C}v_2, \end{cases} \tag{23}$$

where

$$\tilde{C}v_1 = Vec(\tilde{C}_1); \quad \tilde{C}v_2 = Vec(\tilde{C}_2); \quad \tilde{X}v = Vec(\tilde{X}).$$

Proof According to Theorem 4,

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$$

can be converted to be

$$[(\tilde{U}_n \otimes_k \tilde{A}) + (\tilde{B}^T \otimes_k \tilde{U}_m)]Vec(\tilde{X}) = Vec(\tilde{C}),$$

similarly to the equation

$$\tilde{A}\tilde{X}\tilde{B} = \tilde{C}, \tag{24}$$

which can be transformed to

$$(\tilde{B}^T \otimes_k \tilde{A})Vec(\tilde{X}) = \tilde{C} \tag{25}$$

based on Theorem 6. Thus, it is obvious that the PFFME in Eq. (1) can be converted to a PFFLS:

$$\begin{cases} \left[\left(\tilde{U}_{q\times q} \otimes_k (\tilde{A}_1)_{p\times p}\right) + \left((\tilde{B}_1^T)_{q\times q} \otimes_k \tilde{U}_{p\times p}\right)\right] \tilde{X}v = \tilde{C}v_1 \\ \left((\tilde{B}_2^T)_{s\times q} \otimes_k (\tilde{A}_2)_{r\times p}\right) \tilde{X}v = \tilde{C}v_2. \end{cases}$$

□

Theorem 7 *Let* $\tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1$ *and* $\tilde{A}_2\tilde{X}\tilde{B}_2 = \tilde{C}_2$ *be a PFFME, where* $\tilde{A}_1$ *and* $\tilde{B}_1$ *are square matrices whereas* $\tilde{A}_2$ *and* $\tilde{B}_2$ *are rectangle matrices. Then the number of columns of the Kronecker product for these two equations is always the same.*

Proof According to Corollary 1, the Kronecker product of

$$(\tilde{A}_1)_{p\times p}\tilde{X}_{p\times q} + \tilde{X}_{p\times q}(\tilde{B}_1)_{q\times q} = (\tilde{C}_1)_{p\times q} \tag{26}$$

is

$$\left[\left(\tilde{U}_{q\times q} \otimes_k (\tilde{A}_1)_{p\times p}\right) + \left((\tilde{B}_1^T)_{q\times q} \otimes_k \tilde{U}_{p\times p}\right)\right] (\tilde{X})_{p\times q} = (\tilde{C}_1)_{p\times q}. \tag{27}$$

Applying Definition 8, the equation will yield a FFLS with the size of the coefficient matrix being $qp \times qp$.

On the other hand, the Kronecker product for

$$(\tilde{A}_2)_{r\times p}\tilde{X}_{p\times q}(\tilde{B}_2)_{q\times s} = (\tilde{C}_2)_{r\times s} \tag{28}$$

is

$$\left((\tilde{B}_2^T)_{s\times q} \otimes_k (\tilde{A}_2)_{r\times p}\right) (\tilde{X})_{p\times q} = (\tilde{C}_2)_{r\times s}. \tag{29}$$

Then, by Definition 8, the coefficient of the FFLS for this equation is a matrix with size of $sr \times qp$.

Thus, this is proof that the PFFME will yield the same number of columns since they have same solution of $\tilde{X}$. □

Remark 1 *The size of coefficient matrix* $\left[\left(\tilde{U}_{q\times q} \otimes_k (\tilde{A}_1)_{p\times p}\right) + \left((\tilde{B}_1^T)_{q\times q} \otimes_k \tilde{U}_{p\times p}\right)\right]$ *in Eq. (27) is always a square matrix since the fuzzy matrices* $\tilde{A}_1$ *and* $\tilde{B}_1$ *in Eq. (26) are always square matrices.*

Theorem 8 *The fuzzy matrices* $\tilde{A}_1$ *and* $\tilde{B}_1$ *for the fully fuzzy matrix equation in Eq. (26) must be square matrices.*

Proof Let

$$\begin{aligned}(\tilde{A}_1)_{p\times p}\tilde{X}_{p\times q} + \tilde{X}_{p\times q}(\tilde{B}_1)_{q\times q} &= \tilde{C}_{p\times q},\\ (\tilde{A}_1\tilde{X})_{p\times q} + (\tilde{X}\tilde{B}_1)_{p\times q} &= \tilde{C}_{p\times q},\end{aligned}$$

be the fully fuzzy matrix equation as shown in Eq. (26), where $\tilde{A}_1$ and $\tilde{B}_1$ are fuzzy coefficients and $\tilde{X}_{p\times q}$ is the fuzzy solution. If the fuzzy coefficients $\tilde{A}_1$ and $\tilde{B}_1$ are nonsquare with order $(\tilde{A}_1)_{q\times p}$ and $(\tilde{B}_1)_{q\times p}$, and the solution is $\tilde{X}_{p\times q}$, then

$$(\tilde{A}_1)_{q\times p}\tilde{X}_{p\times q} + \tilde{X}_{p\times q}(\tilde{B}_1)_{q\times p}$$

will yield

$$(\tilde{A}_1\tilde{X})_{q\times q} + (\tilde{X}\tilde{B}_1)_{p\times p}.$$

However, the addition of $(\tilde{A}_1\tilde{X})_{q\times q}$ and $(\tilde{X}\tilde{B}_1)_{p\times p}$ is invalid due to the difference in sizes. Thus, in all cases, $\tilde{A}_1$ and $\tilde{B}_1$ must be square matrices. □

Remark 2 *The size of coefficient matrix* $\left((\tilde{B}_2^T)_{s\times q} \otimes_k (\tilde{A}_2)_{r\times p}\right)$ *in Eq. (29) will only be a square matrix if*

- *both matrices* $\tilde{A}_2$ *and* $\tilde{B}_2$ *are square;*
- *both matrices* $\tilde{A}_2$ *and* $\tilde{B}_2$ *are nonsquare, but the sizes of both matrices are the same.*

Definition 10 *A pair of fully fuzzy linear systems (PFFLS) as stated in Eq. (23) can also be represented as follows:*

$$\begin{cases}(F_1, M_1, N_1) \otimes (m^x, \alpha^x, \beta^x) = (m^{c_1}, \alpha^{c_1}, \beta^{c_1})\\ (F_2, M_2, N_2) \otimes (m^x, \alpha^x, \beta^x) = (m^{c_2}, \alpha^{c_2}, \beta^{c_2}).\end{cases} \tag{30}$$

Remark 3 *Based on Corollary 1, the following terms as in Definition 10 can be given:*

$$(F_1, M_1, N_1) = \left[\left(\tilde{U}_{q\times q} \otimes_k (\tilde{A}_1)_{p\times p}\right) + \left((\tilde{B}_1^T)_{q\times q} \otimes_k \tilde{U}_{p\times p}\right)\right],$$

$$(F_2, M_2, N_2) = \left((\tilde{B}_2^T)_{s\times q} \otimes_k (\tilde{A}_2)_{r\times p}\right),$$

$$(m^{c_1}, \alpha^{c_1}, \beta^{c_1}) = Vec(\tilde{C}_1),$$

$$(m^{c_2}, \alpha^{c_2}, \beta^{c_2}) = Vec(\tilde{C}_2),$$

$$(m^x, \alpha^x, \beta^x) = \tilde{X}.$$

Definition 11 *An associated linear system for the PFFLS is defined as*

$$FX = B, \tag{31}$$

where

$$F = \left(\begin{array}{c|c|c} F_1 & 0 & 0 \\ F_2 & 0 & 0 \\ \hline M_1 & (F_1 - M_1)^+ & -(F_1 - M_1)^- \\ M_2 & (F_2 - M_2)^+ & -(F_2 - M_2)^- \\ \hline N_1 & -(F_1 + N_1)^- & (F_1 + N_1)^+ \\ N_2 & -(F_2 + N_2)^- & (F_2 + N_2)^+ \end{array}\right), \quad X = \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix}, \quad B = \left(\begin{array}{c} m^{c_1} \\ m^{c_2} \\ \hline \alpha^{c_1} \\ \alpha^{c_2} \\ \hline \beta^{c_1} \\ \beta^{c_2} \end{array}\right). \tag{32}$$

Theorem 9 *The solution of the PFFLS is equivalent to the solution of the associated linear system.*

Proof In order to show the equivalence of the solution, the PFFLS needs to be converted to the associated linear system.

Considering that the PFFLS in Definition 10 is an arbitrary PFFLS, the arbitrary fuzzy multiplication arithmetic formulas as stated in Definition 5 are used in obtaining the solution. Hence,

$$\begin{cases} F_1 m^x = m^{c_1} \\ F_2 m^x = m^{c_2}, \end{cases} \tag{33}$$

$$\begin{cases} F_1 m^x - min((F_1 - M_1)(m^x - \alpha^x), (F_1 - M_1)(m^x + \beta^x)) = \alpha^{c_1} \\ F_2 m^x - min((F_2 - M_2)(m^x - \alpha^x), (F_2 - M_2)(m^x + \beta^x)) = \alpha^{c_2}, \end{cases} \tag{34}$$

$$\begin{cases} max((F_1 + N_1)(m^x + \beta^x), (F_1 + N_1)(m^x - \alpha^x)) - F_1 m^x = \beta^{c_1} \\ max((F_2 + N_2)(m^x + \beta^x), (F_2 + N_2)(m^x - \alpha^x)) - F_2 m^x = \beta^{c_2}, \end{cases} \tag{35}$$

where $(F_i - M_i)$ and $(F_i + N_i)$ for $i = 1, 2$ would have two cases that both need to be considered. These cases are as follows:

$$\begin{cases} \text{if} \quad F_i - M_i \geq 0 \quad \text{then} \quad (F_i - M_i)^+ \\ \text{if} \quad F_i - M_i < 0 \quad \text{then} \quad (F_i - M_i)^-, \end{cases} \tag{36}$$

$$\begin{cases} \text{if} \quad F_i + N_i \geq 0 \quad \text{then} \quad (F_i + N_i)^+ \\ \text{if} \quad F_i + N_i < 0 \quad \text{then} \quad (F_i + N_i)^-, \end{cases} \tag{37}$$

for $i = 1, 2$.

$$\begin{cases} F_1 m^x - ((F_1 - M_1)^+ (m^x - \alpha^x) + (F_1 - M_1)^- (m^x + \beta^x)) = \alpha^{c_1} \\ F_2 m^x - ((F_2 - M_2)^+ (m^x - \alpha^x) + (F_2 - M_2)^- (m^x + \beta^x)) = \alpha^{c_2}, \end{cases} \tag{38}$$

$$\begin{cases} ((F_1 + N_1)^+ (m^x + \beta^x) + (F_1 + N_1)^- (m^x - \alpha^x)) - F_1 m^x = \beta^{c_1} \\ ((F_2 + N_2)^+ (m^x + \beta^x) + (F_2 + N_2)^- (m^x - \alpha^x)) - F_2 m^x = \beta^{c_2}. \end{cases} \tag{39}$$

Subsequently, Eq. (38) can be simplified as follows:

$$\begin{cases} F_1 m^x - (F_1 - M_1)^+ m^x + (F_1 - M_1)^+ \alpha^x - (F_1 - M_1)^- m^x - (F_1 - M_1)^- \beta^x = \alpha^{c_1} \\ F_2 m^x - (F_2 - M_2)^+ m^x + (F_2 - M_2)^+ \alpha^x - (F_2 - M_2)^- m^x - (F_2 - M_2)^- \beta^x = \alpha^{c_2}, \end{cases}$$

$$\begin{cases} F_1 m^x - ((F_1 - M_1)^+ + (F_1 - M_1)^-) m^x + (F_1 - M_1)^+ \alpha^x - (F_1 - M_1)^- \beta^x = \alpha^{c_1} \\ F_2 m^x - ((F_2 - M_2)^+ + (F_2 - M_2)^-) m^x + (F_2 - M_2)^+ \alpha^x - (F_2 - M_2)^- \beta^x = \alpha^{c_2}. \end{cases} \tag{40}$$

Similarly to Eq. (39),

$$\begin{cases} (F_1 + N_1)^+ m^x + (F_1 + N_1)^+ \beta^x + (F_1 + N_1)^- m^x - (F_1 + N_1)^- \alpha^x - F_1 m^x = \beta^{c_1} \\ (F_2 + N_2)^+ m^x + (F_2 + N_2)^+ \beta^x + (F_2 + N_2)^- m^x - (F_2 + N_2)^- \alpha^x - F_2 m^x = \beta^{c_2}, \end{cases}$$

$$\begin{cases} ((F_1 + N_1)^+ + (F_1 + N_1)^-) m^x + (F_1 + N_1)^+ \beta^x - (F_1 + N_1)^- \alpha^x - F_1 m^x = \beta^{c_1} \\ ((F_2 + N_2)^+ + (F_2 + N_2)^-) m^x + (F_2 + N_2)^+ \beta^x - (F_2 + N_2)^- \alpha^x - F_2 m^x = \beta^{c_2}. \end{cases} \tag{41}$$

By assuming that $((F_i - M_i)^+ + (F_i - M_i)^-) = (F_i - M_i)$ for $i = 1, 2$, then Eq. (40) will be as follows:

$$\begin{cases} F_1 m^x - (F_1 - M_1) m^x + (F_1 - M_1)^+ \alpha^x - (F_1 - M_1)^- \beta^x = \alpha^{c_1} \\ F_2 m^x - (F_2 - M_2) m^x + (F_2 - M_2)^+ \alpha^x - (F_2 - M_2)^- \beta^x = \alpha^{c_2}, \end{cases}$$

which can be reduced to

$$\begin{cases} F_1 m^x - F_1 m^x - M_1 m^x + (F_1 - M_1)^+ \alpha^x - (F_1 - M_1)^- \beta^x = \alpha^{c_1} \\ F_2 m^x - F_2 m^x - M_2 m^x + (F_2 - M_2)^+ \alpha^x - (F_2 - M_2)^- \beta^x = \alpha^{c_2}. \end{cases} \tag{42}$$

For Eq. (41), where $((F_i + N_i)^+ + (F_i + N_i)^-) = (F_i + N_i)$ for $i = 1, 2$, then

$$\begin{cases} (F_1 + N_1) m^x + (F_1 + N_1)^+ \beta^x - (F_1 + N_1)^- \alpha^x - F_1 m^x = \beta^{c_1} \\ (F_2 + N_2) m^x + (F_2 + N_2)^+ \beta^x - (F_2 + N_2)^- \alpha^x - F_2 m^x = \beta^{c_2}, \end{cases}$$

and this can be reduced to

$$\begin{cases} F_1 m^x + N_1 m^x + (F_1+N_1)^+\beta^x - (F_1+N_1)^-\alpha^x - F_1 m^x = \beta^{c_1} \\ F_2 m^x + N_2 m^x + (F_2+N_2)^+\beta^x - (F_2+N_2)^-\alpha^x - F_2 m^x = \beta^{c_2}. \end{cases} \tag{43}$$

Therefore, by taking Eq. (33) and simplifying Eq.s (42) and (43), we obtain

$$\begin{aligned} &F_1 m^x = m^{c_1}, \\ &F_2 m^x = m^{c_2}, \\ &M_1 m^x + (F_1 - M_1)^+\alpha^x - (F_1 - M_1)^-\beta^x = \alpha^{c_1}, \\ &M_2 m^x + (F_2 - M_2)^+\alpha^x - (F_2 - M_2)^-\beta^x = \alpha^{c_2}, \\ &N_1 m^x + (F_1 + N_1)^+\beta^x - (F_1 + N_1)^-\alpha^x = \beta^{c_1}, \\ &N_2 m^x + (F_2 + N_2)^+\beta^x - (F_2 + N_2)^-\alpha^x = \beta^{c_2}, \end{aligned} \tag{44}$$

which can be written in a matrix form of

$$FX = B, \tag{45}$$

where

$$F = \left(\begin{array}{c|c|c} F_1 & 0 & 0 \\ F_2 & 0 & 0 \\ \hline M_1 & (F_1 - M_1)^+ & -(F_1 - M_1)^- \\ M_2 & (F_2 - M_2)^+ & -(F_2 - M_2)^- \\ \hline N_1 & -(F_1 + N_1)^- & (F_1 + N_1)^+ \\ N_2 & -(F_2 + N_2)^- & (F_2 + N_2)^+ \end{array}\right), \quad X = \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix}, \quad B = \left(\begin{array}{c} m^{c_1} \\ m^{c_2} \\ \hline \alpha^{c_1} \\ \alpha^{c_2} \\ \hline \beta^{c_1} \\ \beta^{c_2} \end{array}\right). \tag{46}$$

Thus, the solution of the associated linear system is the same as the solution of the PFFLS. □

4. Solution for arbitrary PFFME

In this section, an algorithm for finding the solution of the PFFME is presented by considering that the parameters are arbitrary fuzzy numbers.

Step 1: Transforming the PFFME to PFFLS.

Eq. (1) is transformed accordingly as shown in Corollary 1.

Step 2: Converting the PFFLS to an associated linear system.

The conversion of the associated linear system is based on Theorem 9.

Step 3: Obtaining the final solution.

The final solution $\tilde{X}$ can be obtained by computing the inverse of the coefficient for Eq. (46). In this case, coefficient F would be a nonsquare matrix, and thus a generalized inverse method, such as the pseudoinverse method, should be applied. Thus,

$$X = F^{\dagger}B, \tag{47}$$

where $F^{\dagger}$ is a pseudoinverse [4].

5. Numerical examples

Example 1 *Consider the PFFME,*

$$\begin{cases} \tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1 \\ \tilde{A}_2\tilde{X}\tilde{B}_2 = \tilde{C}_2, \end{cases}$$

where

$$\tilde{A}_1 = \begin{pmatrix} (12,8,5) & (7,3,3) & (11,8,5) \\ (9,5,6) & (8,5,7) & (9,3,7) \\ (10,1,8) & (12,8,6) & (8,3,6) \end{pmatrix}, \tilde{B}_1 = \begin{pmatrix} (8,3,2) & (10,8,2) \\ (7,4,7) & (6,4,3) \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} (268,212,233) & (296,265,217) \\ (248,161,240) & (270,198,228) \\ (316,218,342) & (318,236,324) \end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix} (9,2,3) & (8,4,3) & (9,2,3) \\ (8,3,2) & (7,3,8) & (11,3,3) \end{pmatrix}, \tilde{B}_2 = \begin{pmatrix} (7,2,2) & (10,4,2) & (11,3,5) \\ (9,2,1) & (7,2,3) & (10,2,9) \end{pmatrix},$$

$$\tilde{C}_2 = \begin{pmatrix} (2732,2045,2442) & (2971,2466,2910) & (3564,2641,4944) \\ (2836,2271,2914) & (2962,2683,3487) & (3686,2914,5685) \end{pmatrix},$$

and

$$\tilde{X} = \begin{pmatrix} (m_{11}^x,\alpha_{11}^x,\beta_{11}^x) & (m_{12}^x,\alpha_{12}^x,\beta_{12}^x) \\ (m_{21}^x,\alpha_{21}^x,\beta_{21}^x) & (m_{22}^x,\alpha_{22}^x,\beta_{22}^x) \\ (m_{31}^x,\alpha_{31}^x,\beta_{31}^x) & (m_{32}^x,\alpha_{32}^x,\beta_{32}^x) \end{pmatrix} \geq 0.$$

Solution:

Step 1: Transform the PFFME into Eq. (30) according to its Kronecker product.

$$(F_1, M_1, N_1) = \begin{pmatrix} (20,11,7) & (7,3,3) & (11,8,5) & (7,4,7) & (0,0,0) & (0,0,0) \\ (9,5,6) & (16,8,9) & (9,3,7) & (0,0,0) & (7,4,7) & (0,0,0) \\ (10,1,8) & (12,8,6) & (16,6,8) & (0,0,0) & (0,0,0) & (7,4,7) \\ (10,8,2) & (0,0,0) & (0,0,0) & (18,12,8) & (7,3,3) & (11,8,5) \\ (0,0,0) & (10,8,2) & (0,0,0) & (9,5,6) & (14,9,10) & (9,3,7) \\ (0,0,0) & (0,0,0) & (10,8,2) & (10,1,8) & (12,8,6) & (14,7,9) \end{pmatrix},$$

$$(m^{c_1}, \alpha^{c_1}, \beta^{c_1}) = \begin{pmatrix} (268,212,233) \\ (248,161,240) \\ (316,218,342) \\ (296,265,217) \\ (270,198,228) \\ (318,236,324) \end{pmatrix},$$

and

$$(F_2, M_2, N_2) = \begin{pmatrix} (63,32,39) & (56,44,37) & (63,32,39) & (81,36,36) & (72,52,35) & (81,36,36) \\ (56,37,30) & (49,35,70) & (77,43,43) & (72,43,26) & (63,41,79) & (99,49,38) \\ (90,56,48) & (80,72,46) & (90,56,48) & (63,32,48) & (56,44,45) & (63,32,48) \\ (80,62,36) & (70,58,94) & (110,74,52) & (56,37,38) & (49,35,77) & (77,43,54) \\ (99,49,78) & (88,68,73) & (99,49,78) & (90,38,111) & (80,56,102) & (90,38,111) \\ (88,57,62) & (77,54,123) & (121,66,88) & (80,46,92) & (70,44,143) & (110,52,129) \end{pmatrix},$$

$$(m^{c_2}, \alpha^{c_2}, \beta^{c_2}) = \begin{pmatrix} (2732,2045,2442) \\ (2836,2271,2914) \\ (2872,2466,2910) \\ (2962,2683,3487) \\ (3564,2641,4944) \\ (3686,2914,5685) \end{pmatrix}.$$

Step 2: Considering all the fuzzy matrices in Step 1, the crisp matrices are obtained as follows:

$$F_1 = \begin{pmatrix} 20 & 7 & 11 & 7 & 0 & 0 \\ 9 & 16 & 9 & 0 & 7 & 0 \\ 10 & 12 & 16 & 0 & 0 & 7 \\ 10 & 0 & 0 & 18 & 7 & 11 \\ 0 & 10 & 0 & 9 & 14 & 9 \\ 0 & 0 & 10 & 10 & 12 & 14 \end{pmatrix}, M_1 = \begin{pmatrix} 11 & 3 & 8 & 4 & 0 & 0 \\ 5 & 8 & 3 & 0 & 4 & 0 \\ 1 & 8 & 6 & 0 & 0 & 4 \\ 8 & 0 & 0 & 12 & 3 & 8 \\ 0 & 8 & 0 & 5 & 9 & 3 \\ 0 & 0 & 8 & 1 & 8 & 7 \end{pmatrix}, N_1 = \begin{pmatrix} 7 & 3 & 5 & 7 & 0 & 0 \\ 6 & 9 & 7 & 0 & 7 & 0 \\ 8 & 6 & 8 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 & 3 & 5 \\ 0 & 2 & 0 & 6 & 10 & 7 \\ 0 & 0 & 2 & 8 & 6 & 9 \end{pmatrix},$$

and from $(m^{c_1}, \alpha^{c_1}, \beta^{c_1})$,

$$m^{c_1} = \begin{pmatrix} 268 \\ 248 \\ 316 \\ 296 \\ 270 \\ 318 \end{pmatrix}, \quad \alpha^{c_1} = \begin{pmatrix} 212 \\ 161 \\ 218 \\ 265 \\ 198 \\ 236 \end{pmatrix}, \quad \beta^{c_1} = \begin{pmatrix} 233 \\ 240 \\ 342 \\ 217 \\ 228 \\ 324 \end{pmatrix}.$$

On the other hand,

$$F_2 = \begin{pmatrix} 63 & 56 & 63 & 81 & 72 & 81 \\ 56 & 49 & 77 & 72 & 63 & 99 \\ 90 & 80 & 90 & 63 & 56 & 63 \\ 80 & 70 & 110 & 56 & 49 & 77 \\ 99 & 88 & 99 & 90 & 80 & 90 \\ 88 & 77 & 121 & 80 & 70 & 110 \end{pmatrix}, M_2 = \begin{pmatrix} 32 & 44 & 32 & 36 & 52 & 36 \\ 37 & 35 & 43 & 43 & 41 & 49 \\ 56 & 72 & 56 & 32 & 44 & 32 \\ 62 & 58 & 74 & 37 & 35 & 43 \\ 49 & 68 & 49 & 38 & 56 & 38 \\ 57 & 54 & 66 & 46 & 44 & 52 \end{pmatrix}, N_2 = \begin{pmatrix} 39 & 37 & 39 & 36 & 35 & 36 \\ 30 & 70 & 43 & 26 & 79 & 38 \\ 48 & 46 & 48 & 48 & 45 & 48 \\ 36 & 94 & 52 & 38 & 77 & 54 \\ 78 & 73 & 78 & 111 & 102 & 111 \\ 62 & 123 & 88 & 92 & 143 & 129 \end{pmatrix},$$

and

$$m^{c_2} = \begin{pmatrix} 2732 \\ 2836 \\ 2872 \\ 2962 \\ 3564 \\ 3686 \end{pmatrix}, \quad \alpha^{c_2} = \begin{pmatrix} 2045 \\ 2271 \\ 2466 \\ 2683 \\ 2641 \\ 2914 \end{pmatrix}, \quad \beta^{c_2} = \begin{pmatrix} 2442 \\ 2914 \\ 2910 \\ 3487 \\ 4944 \\ 5685 \end{pmatrix}.$$

Then an associated linear systems, $FX = B$, is obtained as follows:

$$\left(\begin{array}{cccccc|cccccc|cccccc}
20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 16 & 9 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
63 & 56 & 63 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
11 & 3 & 8 & 4 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 8 & 3 & 0 & 4 & 0 & 9 & 16 & 9 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 8 & 6 & 0 & 0 & 4 & 10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 12 & 3 & 8 & 10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 5 & 9 & 3 & 0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 1 & 8 & 7 & 0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 44 & 32 & 36 & 52 & 36 & 63 & 56 & 63 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 & 0 \\
37 & 35 & 43 & 43 & 41 & 49 & 56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 & 0 \\
56 & 72 & 56 & 32 & 44 & 32 & 90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 & 0 \\
62 & 58 & 74 & 37 & 35 & 43 & 80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 & 0 \\
49 & 68 & 49 & 38 & 56 & 38 & 99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 & 0 \\
57 & 54 & 66 & 46 & 44 & 52 & 88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
7 & 3 & 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 \\
6 & 9 & 7 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 16 & 9 & 0 & 7 & 0 \\
8 & 6 & 8 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 12 & 16 & 0 & 0 & 7 \\
2 & 0 & 0 & 8 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 18 & 7 & 11 \\
0 & 2 & 0 & 6 & 10 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 9 & 14 & 9 \\
0 & 0 & 2 & 8 & 6 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 12 & 14 \\
39 & 37 & 39 & 36 & 35 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 63 & 56 & 63 & 81 & 72 & 81 \\
30 & 70 & 43 & 26 & 79 & 38 & 0 & 0 & 0 & 0 & 0 & 0 & 56 & 49 & 77 & 72 & 63 & 99 \\
48 & 46 & 48 & 48 & 45 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 90 & 80 & 90 & 63 & 56 & 63 \\
36 & 94 & 52 & 38 & 77 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 70 & 110 & 56 & 49 & 77 \\
78 & 73 & 78 & 111 & 102 & 111 & 0 & 0 & 0 & 0 & 0 & 0 & 99 & 88 & 99 & 90 & 80 & 90 \\
62 & 123 & 88 & 92 & 143 & 129 & 0 & 0 & 0 & 0 & 0 & 0 & 88 & 77 & 121 & 80 & 70 & 110
\end{array}\right)
\begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix}
=
\begin{pmatrix} 268 \\ 248 \\ 316 \\ 296 \\ 270 \\ 318 \\ 2732 \\ 2836 \\ 2872 \\ 2962 \\ 3564 \\ 3686 \\ 212 \\ 161 \\ 218 \\ 265 \\ 198 \\ 236 \\ 2045 \\ 2271 \\ 2466 \\ 2683 \\ 2641 \\ 2914 \\ 233 \\ 240 \\ 342 \\ 217 \\ 228 \\ 324 \\ 2442 \\ 2914 \\ 2910 \\ 3487 \\ 4944 \\ 5685 \end{pmatrix}.$$

Step 3: Therefore, the final solution is obtained as follows by using pseudoinverse method:

$$\begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix}
=
\begin{pmatrix} 5 \\ 7 \\ 7 \\ 6 \\ 4 \\ 10 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 6 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

Hence,

$$\tilde{X}=\begin{pmatrix}(m^x_{11},\alpha^x_{11},\beta^x_{11}) & (m^x_{12},\alpha^x_{12},\beta^x_{12})\\(m^x_{21},\alpha^x_{21},\beta^x_{21}) & (m^x_{22},\alpha^x_{22},\beta^x_{22})\\(m^x_{31},\alpha^x_{31},\beta^x_{31}) & (m^x_{32},\alpha^x_{32},\beta^x_{32})\end{pmatrix}$$
$$=\begin{pmatrix}(5,1,1) & (6,1,2)\\(7,1,0) & (4,0,1)\\(7,2,6) & (10,3,4)\end{pmatrix}.$$

Example 2 *Consider the PFFME,*

$$\begin{cases}\tilde{A}_1\tilde{X}+\tilde{X}\tilde{B}_1=\tilde{C}_1\\ \tilde{A}_2\tilde{X}\tilde{B}_2=\tilde{C}_2,\end{cases}$$

where

$$\tilde{A}_1=\begin{pmatrix}(-8,3,2) & (10,8,2)\\(7,4,7) & (-6,4,3)\end{pmatrix},\tilde{B}_1=\begin{pmatrix}(-12,8,5) & (7,3,3) & (11,8,5)\\(9,5,6) & (-8,5,7) & (9,3,7)\\(10,1,8) & (12,8,6) & (-8,3,6)\end{pmatrix},$$

$$\tilde{C}_1=\begin{pmatrix}(122,313,329) & (151,379,312) & (59,268,349)\\(56,264,420) & (49,354,367) & (170,293,389)\end{pmatrix},$$

$$\tilde{A}_2=\begin{pmatrix}(5,1,1) & (6,1,2)\\(7,1,0) & (4,0,1)\\(9,2,6) & (8,2,6)\end{pmatrix},\tilde{B}_2=\begin{pmatrix}(-9,2,3) & (8,4,3) & (7,4,3)\\(8,3,2) & (-7,3,8) & (6,3,1)\\(11,3,3) & (9,2,3) & (-10,2,8)\end{pmatrix},$$

$$\tilde{C}_2=\begin{pmatrix}(1059,1841,2739) & (798,2235,2700) & (339,1818,2303)\\(1113,1598,2055) & (862,1878,2153) & (215,1463,1951)\\(1671,3705,6387) & (1274,4502,5995) & (445,3805,4989)\end{pmatrix},$$

and

$$\tilde{X}=\begin{pmatrix}(m^x_{11},\alpha^x_{11},\beta^x_{11}) & (m^x_{12},\alpha^x_{12},\beta^x_{12}) & (m^x_{13},\alpha^x_{13},\beta^x_{13})\\(m^x_{21},\alpha^x_{21},\beta^x_{21}) & (m^x_{22},\alpha^x_{22},\beta^x_{22}) & (m^x_{23},\alpha^x_{23},\beta^x_{23})\end{pmatrix}\geq 0.$$

Solution:

Step 1: Transform the PFFME to the form of Eq. (30) according to its Kronecker product.

$$(F_2,M_2,N_2)=\begin{pmatrix}(-20,11,7) & (10,8,2) & (9,5,6) & (0,0,0) & (10,1,8) & (0,0,0)\\(7,4,7) & (-18,12,8) & (0,0,0) & (9,5,6,) & (0,0,0) & (10,1,8)\\(7,3,3) & (0,0,0) & (-16,8,9) & (10,8,2) & (12,8,6) & (0,0,0)\\(0,0,0) & (7,3,3) & (7,4,7) & (-14,9,10) & (0,0,0) & (12,8,6)\\(11,8,5) & (0,0,0) & (9,3,7) & (0,0,0) & (-16,6,8) & (10,8,2)\\(0,0,0) & (11,8,5) & (0,0,0) & (9,3,7) & (7,4,7) & (-14,7,9)\end{pmatrix},$$

$$(m^{c_1},\alpha^{c_1},\beta^{c_1})=\begin{pmatrix}(122,313,329)\\(56,264,420)\\(151,379,312)\\(49,354,367)\\(59,268,349)\\(170,293,389)\end{pmatrix},$$

and

$$(F_2, M_2, N_2) = \begin{pmatrix} (-45,21,21) & (-54,34,24) & (40,20,20) & (48,23,32) & (55,23,29) & (66,26,46) \\ (-63,14,27) & (-36,19,12) & (56,26,14) & (32,12,18) & (77,29,21) & (44,12,26) \\ (-81,84,39) & (-72,82,36) & (72,37,78) & (64,34,76) & (99,43,111) & (88,40,108) \\ (40,24,26) & (48,28,40) & (-35,25,41) & (-42,38,50) & (45,17,27) & (54,19,42) \\ (56,32,21) & (32,16,23) & (-49,21,56) & (-28,22,33) & (63,21,21) & (36,8,24) \\ (72,44,93) & (64,40,90) & (-63,87,78) & (-56,84,70) & (81,32,99) & (72,30,96) \\ (35,23,25) & (42,27,38) & (30,18,12) & (36,21,20) & (-50,22,42) & (-60,36,50) \\ (49,31,21) & (28,16,22) & (42,24,7) & (24,12,11) & (-70,14,58) & (-40,20,32) \\ (63,42,87) & (56,38,84) & (54,33,51) & (48,30,50) & (-90,90,76) & (-80,88,68) \end{pmatrix},$$

$$(m^{c_2}, \alpha^{c_2}, \beta^{c_2}) = \begin{pmatrix} (1059,1841,2739) \\ (1113,1598,2055) \\ (1671,3705,6387) \\ (798,2235,2700) \\ (862,1878,2153) \\ (1274,4502,5995) \\ (339,1818,2303) \\ (215,1463,1951) \\ (445,3805,4989) \end{pmatrix}.$$

Step 2: The equation is converted to the associated linear system.

$$F_1 = \begin{pmatrix} -20 & 10 & 9 & 0 & 10 & 0 \\ 7 & -18 & 0 & 9 & 0 & 10 \\ 7 & 0 & -16 & 10 & 12 & 0 \\ 0 & 7 & 7 & -14 & 0 & 12 \\ 11 & 0 & 9 & 0 & -16 & 10 \\ 0 & 11 & 0 & 9 & 7 & -14 \end{pmatrix}, M_1 = \begin{pmatrix} 11 & 8 & 5 & 0 & 1 & 0 \\ 4 & 12 & 0 & 5 & 0 & 1 \\ 3 & 0 & 8 & 8 & 8 & 0 \\ 0 & 3 & 4 & 9 & 0 & 8 \\ 8 & 0 & 3 & 0 & 6 & 8 \\ 0 & 8 & 0 & 3 & 4 & 7 \end{pmatrix}, N_1 = \begin{pmatrix} 7 & 2 & 6 & 0 & 8 & 0 \\ 7 & 8 & 0 & 6 & 0 & 8 \\ 3 & 0 & 9 & 2 & 6 & 0 \\ 0 & 3 & 7 & 10 & 0 & 6 \\ 5 & 0 & 7 & 0 & 8 & 2 \\ 0 & 5 & 0 & 7 & 7 & 9 \end{pmatrix},$$

$$m^{c_1} = \begin{pmatrix} 122 \\ 56 \\ 151 \\ 49 \\ 59 \\ 170 \end{pmatrix}, \quad \alpha^{c_1} = \begin{pmatrix} 313 \\ 264 \\ 379 \\ 354 \\ 268 \\ 293 \end{pmatrix}, \quad \beta^{c_1} = \begin{pmatrix} 329 \\ 420 \\ 312 \\ 367 \\ 349 \\ 389 \end{pmatrix}.$$

On the other hand,

$$F_2 = \begin{pmatrix} -45 & -54 & 40 & 48 & 55 & 66 \\ -63 & -36 & 56 & 32 & 77 & 44 \\ -81 & -72 & 72 & 64 & 99 & 88 \\ 40 & 48 & -35 & -42 & 45 & 54 \\ 56 & 32 & -49 & -28 & 63 & 36 \\ 72 & 64 & -63 & -56 & 81 & 72 \\ 35 & 42 & 30 & 36 & -50 & -60 \\ 49 & 28 & 42 & 24 & -70 & -40 \\ 63 & 56 & 54 & 48 & -90 & -80 \end{pmatrix}, M_2 = \begin{pmatrix} 21 & 34 & 20 & 23 & 23 & 26 \\ 14 & 19 & 26 & 12 & 29 & 12 \\ 84 & 82 & 37 & 34 & 43 & 40 \\ 24 & 28 & 25 & 38 & 17 & 19 \\ 32 & 16 & 21 & 22 & 21 & 8 \\ 44 & 40 & 87 & 84 & 32 & 30 \\ 23 & 27 & 18 & 21 & 22 & 36 \\ 31 & 16 & 24 & 12 & 14 & 20 \\ 42 & 38 & 33 & 30 & 90 & 88 \end{pmatrix}, N_2 = \begin{pmatrix} 21 & 24 & 20 & 32 & 29 & 46 \\ 27 & 12 & 14 & 18 & 21 & 26 \\ 39 & 36 & 78 & 76 & 111 & 108 \\ 26 & 40 & 41 & 50 & 27 & 42 \\ 21 & 23 & 56 & 33 & 21 & 24 \\ 93 & 90 & 78 & 70 & 99 & 96 \\ 25 & 38 & 12 & 20 & 42 & 50 \\ 21 & 22 & 7 & 11 & 58 & 32 \\ 87 & 84 & 51 & 50 & 76 & 68 \end{pmatrix},$$

$$m^{c_2} = \begin{pmatrix} 1059 \\ 1113 \\ 1671 \\ 798 \\ 862 \\ 1274 \\ 339 \\ 215 \\ 445 \end{pmatrix}, \quad \alpha^{c_2} = \begin{pmatrix} 1841 \\ 1598 \\ 3705 \\ 2235 \\ 1878 \\ 4502 \\ 1818 \\ 1463 \\ 3805 \end{pmatrix}, \quad \beta^{c_2} = \begin{pmatrix} 2739 \\ 2055 \\ 6387 \\ 2700 \\ 2153 \\ 5995 \\ 2303 \\ 1951 \\ 4989 \end{pmatrix}.$$

Thus, based on Eq. (46), we obtain

$$\left(\begin{array}{rrrrrr|rrrrrr|rrrrrr}
-20 & 10 & 9 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & -18 & 0 & 9 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & -16 & 10 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 7 & -14 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 9 & 0 & -16 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11 & 0 & 9 & 7 & -14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-45 & -54 & 40 & 48 & 55 & 66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-63 & -36 & 56 & 32 & 77 & 44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-81 & -72 & 72 & 64 & 99 & 88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & 48 & -35 & -42 & 45 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56 & 32 & -49 & -28 & 63 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
72 & 64 & -63 & -56 & 81 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 42 & 30 & 36 & -50 & -60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
49 & 28 & 42 & 24 & -70 & -40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
63 & 56 & 54 & 48 & -90 & -80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
11 & 8 & 5 & 0 & 1 & 0 & 0 & 2 & 4 & 0 & 9 & 0 & 31 & 0 & 0 & 0 & 0 & 0 \\
4 & 12 & 0 & 5 & 0 & 1 & 3 & 0 & 0 & 4 & 0 & 9 & 0 & 30 & 0 & 0 & 0 & 0 \\
3 & 0 & 8 & 8 & 8 & 0 & 4 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \\
0 & 3 & 4 & 9 & 0 & 8 & 0 & 4 & 3 & 0 & 0 & 4 & 0 & 0 & 0 & 23 & 0 & 0 \\
8 & 0 & 3 & 0 & 6 & 8 & 3 & 0 & 6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 22 & 0 \\
0 & 8 & 0 & 3 & 4 & 7 & 0 & 3 & 0 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
21 & 34 & 20 & 23 & 23 & 26 & 0 & 0 & 20 & 25 & 32 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 19 & 26 & 12 & 29 & 12 & 0 & 0 & 30 & 20 & 48 & 32 & 0 & 0 & 0 & 0 & 0 & 0 \\
84 & 82 & 37 & 34 & 43 & 40 & 0 & 0 & 35 & 30 & 56 & 48 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 28 & 25 & 38 & 17 & 19 & 16 & 20 & 0 & 0 & 28 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 16 & 21 & 22 & 21 & 8 & 24 & 16 & 0 & 0 & 42 & 28 & 0 & 0 & 0 & 0 & 0 & 0 \\
44 & 40 & 87 & 84 & 32 & 30 & 28 & 24 & 0 & 0 & 49 & 42 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 27 & 18 & 21 & 22 & 36 & 12 & 15 & 12 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
31 & 16 & 24 & 12 & 14 & 20 & 18 & 12 & 18 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & 38 & 33 & 30 & 90 & 88 & 21 & 18 & 21 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
7 & 2 & 6 & 0 & 8 & 0 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 15 & 0 & 18 & 0 \\
7 & 8 & 0 & 6 & 0 & 8 & 0 & 10 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & 15 & 0 & 18 \\
3 & 0 & 9 & 2 & 6 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 10 & 0 & 0 & 12 & 18 & 0 \\
0 & 3 & 7 & 10 & 0 & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 10 & 14 & 0 & 0 & 18 \\
5 & 0 & 7 & 0 & 8 & 2 & 0 & 0 & 0 & 0 & 8 & 0 & 16 & 0 & 16 & 0 & 0 & 12 \\
0 & 5 & 0 & 7 & 7 & 9 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 16 & 0 & 16 & 14 & 0 \\
21 & 24 & 20 & 32 & 29 & 46 & 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 60 & 80 & 84 & 112 \\
27 & 12 & 14 & 18 & 21 & 26 & 36 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 70 & 50 & 98 & 70 \\
39 & 36 & 78 & 76 & 111 & 108 & 42 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 150 & 140 & 210 & 196 \\
26 & 40 & 41 & 50 & 27 & 42 & 0 & 0 & 0 & 0 & 0 & 0 & 66 & 88 & 6 & 8 & 72 & 96 \\
21 & 23 & 56 & 33 & 21 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 77 & 55 & 7 & 5 & 84 & 60 \\
93 & 90 & 78 & 70 & 99 & 96 & 0 & 0 & 0 & 0 & 0 & 0 & 165 & 154 & 15 & 14 & 180 & 168 \\
25 & 38 & 12 & 20 & 42 & 50 & 0 & 0 & 0 & 0 & 8 & 10 & 60 & 80 & 42 & 56 & 0 & 0 \\
21 & 22 & 7 & 11 & 58 & 32 & 0 & 0 & 0 & 0 & 12 & 8 & 70 & 50 & 49 & 35 & 0 & 0 \\
87 & 84 & 51 & 50 & 76 & 68 & 0 & 0 & 0 & 0 & 14 & 12 & 150 & 140 & 105 & 98 & 0 & 0
\end{array}\right)
\begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix}
=
\begin{pmatrix} 122 \\ 56 \\ 151 \\ 49 \\ 59 \\ 170 \\ 1059 \\ 1113 \\ 1671 \\ 798 \\ 862 \\ 1274 \\ 339 \\ 215 \\ 445 \\ 313 \\ 264 \\ 379 \\ 354 \\ 268 \\ 293 \\ 1841 \\ 1598 \\ 3705 \\ 2235 \\ 1878 \\ 4502 \\ 1818 \\ 1463 \\ 3805 \\ 329 \\ 420 \\ 312 \\ 367 \\ 349 \\ 389 \\ 2739 \\ 2055 \\ 6387 \\ 2700 \\ 2153 \\ 5995 \\ 2303 \\ 1951 \\ 4989 \end{pmatrix}.$$

Step 3: By pseudoinverse method, the following solution is obtained:

$$X = \begin{pmatrix} 7 \\ 9 \\ 8 \\ 11 \\ 10 \\ 7 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 2 \\ 2 \\ 1 \\ 4 \\ 5 \\ 2 \\ 3 \end{pmatrix} \text{ or } X = \begin{pmatrix} \begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \end{pmatrix} \\ \begin{pmatrix} \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \end{pmatrix} \\ \begin{pmatrix} \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 7 \\ 9 \\ 8 \\ 11 \\ 10 \\ 7 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 2 \\ 3 \end{pmatrix} \end{pmatrix}.$$

In other words, the solution can be written as

$$\begin{aligned} \tilde{X} &= \begin{pmatrix} (m^x_{11}, \alpha^x_{11}, \beta^x_{11}) & (m^x_{12}, \alpha^x_{12}, \beta^x_{12}) & (m^x_{13}, \alpha^x_{13}, \beta^x_{13}) \\ (m^x_{21}, \alpha^x_{21}, \beta^x_{21}) & (m^x_{22}, \alpha^x_{22}, \beta^x_{22}) & (m^x_{23}, \alpha^x_{23}, \beta^x_{23}) \end{pmatrix} \\ &= \begin{pmatrix} (7,2,2) & (8,3,4) & (10,4,2) \\ (9,2,1) & (11,3,5) & (7,2,3) \end{pmatrix}. \end{aligned}$$

6. Solving the PFME using the proposed method

As mentioned earlier, the study in [26] solved the pair of fuzzy matrix equations (PFME). In that study, the PFME was written as follows:

$$\begin{cases} A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1 \\ A_2\tilde{X}B_2 = \tilde{C}_2, \end{cases} \tag{48}$$

where the parameters A_1, A_2, B_1, and B_2 are in the form of crisp numbers, whereas $\tilde{C}_1$ and $\tilde{C}_2$ are the fuzzy matrices and the solution $\tilde{X}$ is also in fuzzy form. An example is taken from that study and will be solved using our proposed algorithm.

Example 3 *According to the PFME as stated in Eq. (48), the parameters A_1, A_2, B_1, and B_2 are as follows:*

$$A_1 = \begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}, B_1 = \begin{pmatrix} 4 & -2 & -1 \\ -2 & 5 & 1 \\ 0 & -1 & 2 \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} (-41+26r, 26-41r) & (-44+23r, 23-44r) & (-25+22r, 22-25r) \\ (-50+50r, 50-50r) & (-34+67r, 67-34r) & (-19+43r, 43-19r) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 \\ -1 & 3 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{pmatrix},$$

$$\tilde{C}_2 = \begin{pmatrix} (-5+8r, 8-5r) & (-17+20r, 20-17r) \\ (-29+23r, 23-29r) & (-65+86r, 86-65r) \end{pmatrix},$$

and the exact solution is given by

$$\tilde{X} = \begin{pmatrix} (-5+2r, 2-5r) & (-4+r, 1-4r) & (-3+3r, 3-3r) \\ (-4+4r, 4-4r) & (-2+5r, 5-2r) & (-1+4r, 4-r) \end{pmatrix}.$$

Solution:
First, the PFME is converted to the following PFFME based on the triangular fuzzy numbers (m, α, β):

$$\begin{pmatrix} (3,0,0) & (-1,0,0) \\ (-2,0,0) & (5,0,0) \end{pmatrix} \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} \end{pmatrix} + \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} \end{pmatrix} \begin{pmatrix} (4,0,0) & (-2,0,0) & (-1,0,0) \\ (-2,0,0) & (5,0,0) & (1,0,0) \\ (0,0,0) & (-1,0,0) & (2,0,0) \end{pmatrix}$$
$$= \begin{pmatrix} (-15,26,41) & (-21,23,44) & (-3,22,25) \\ (0,50,50) & (33,67,34) & (24,43,19) \end{pmatrix},$$

$$\begin{pmatrix} (-1,0,0) & (0,0,0) \\ (-1,0,0) & (3,0,0) \end{pmatrix} \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} \end{pmatrix} \begin{pmatrix} (1,0,0) & (2,0,0) \\ (0,0,0) & (-1,0,0) \\ (-1,0,0) & (3,0,0) \end{pmatrix} = \begin{pmatrix} (3,8,5) & (3,20,17) \\ (-6,23,29) & (21,86,65) \end{pmatrix}.$$

By the Kronecker product, the following equation is obtained:

$$\begin{pmatrix} (7,0,0) & (-1,0,0) & (-2,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (-2,0,0) & (9,0,0) & (0,0,0) & (-2,0,0) & (0,0,0) & (0,0,0) \\ (-2,0,0) & (0,0,0) & (8,0,0) & (-1,0,0) & (-1,0,0) & (0,0,0) \\ (0,0,0) & (-2,0,0) & (-2,0,0) & (10,0,0) & (0,0,0) & (-1,0,0) \\ (-1,0,0) & (0,0,0) & (1,0,0) & (0,0,0) & (5,0,0) & (-1,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (1,0,0) & (-2,0,0) & (7,0,0) \end{pmatrix} \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{13}^x, \alpha_{13}^x, \beta_{13}^x) \\ (m_{23}^x, \alpha_{23}^x, \beta_{23}^x) \end{pmatrix} = \begin{pmatrix} (-15,26,41) \\ (0,50,50) \\ (-21,23,44) \\ (33,67,34) \\ (-3,22,25) \\ (24,43,19) \end{pmatrix},$$

$$\begin{pmatrix} (-1,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (1,0,0) & (0,0,0) \\ (-1,0,0) & (3,0,0) & (0,0,0) & (0,0,0) & (1,0,0) & (-3,0,0) \\ (-2,0,0) & (0,0,0) & (1,0,0) & (0,0,0) & (-3,0,0) & (0,0,0) \\ (-2,0,0) & (6,0,0) & (1,0,0) & (-3,0,0) & (-3,0,0) & (9,0,0) \end{pmatrix} \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{13}^x, \alpha_{13}^x, \beta_{13}^x) \\ (m_{23}^x, \alpha_{23}^x, \beta_{23}^x) \end{pmatrix} = \begin{pmatrix} (3,8,5) \\ (-6,23,29) \\ (3,20,17) \\ (21,86,65) \end{pmatrix}.$$

Then we obtain

$$
\left(\begin{array}{cccccc|cccccc|cccccc}
7 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 9 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 8 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & -2 & 10 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & -2 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 6 & 1 & -3 & -3 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 7 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 1 & 0 & 0 & 9 & 2 & 0 & 0 & 3 & 3 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 10 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 6 & 1 & 0 & 0 & 0 & 9
\end{array}\right)
\begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix}
=
\begin{pmatrix} -15 \\ 0 \\ -21 \\ 33 \\ -3 \\ 24 \\ 3 \\ -6 \\ 3 \\ 21 \\ 26 \\ 50 \\ 23 \\ 67 \\ 22 \\ 43 \\ 8 \\ 23 \\ 20 \\ 86 \\ 41 \\ 50 \\ 44 \\ 34 \\ 25 \\ 19 \\ 5 \\ 29 \\ 17 \\ 65 \end{pmatrix}.
$$

Therefore, the final solution is obtained as follows:

$$
\begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix}
=
\begin{pmatrix} -3 \\ 0 \\ -3 \\ 3 \\ 0 \\ 3 \\ 2 \\ 4 \\ 1 \\ 5 \\ 3 \\ 4 \\ 5 \\ 4 \\ 4 \\ 2 \\ 3 \\ 1 \end{pmatrix},
$$

which is

$$\tilde{X} = \begin{pmatrix} (m^x_{11}, \alpha^x_{11}, \beta^x_{11}) & (m^x_{12}, \alpha^x_{12}, \beta^x_{12}) & (m^x_{13}, \alpha^x_{13}, \beta^x_{13}) \\ (m^x_{21}, \alpha^x_{21}, \beta^x_{21}) & (m^x_{22}, \alpha^x_{22}, \beta^x_{22}) & (m^x_{23}, \alpha^x_{23}, \beta^x_{23}) \end{pmatrix} = \begin{pmatrix} (-3,2,5) & (-3,1,4) & (0,3,3) \\ (0,4,4) & (3,5,2) & (3,4,1) \end{pmatrix}.$$

It is obvious that the solution obtained is also similar to the exact solution stated in [26].

Based on Example 3, we find that:

1. Our constructed algorithm is able to solve the PFFME and PFME since the FME is a subset of FFME and the FLS is also a subset of FFLS [21].
2. The solutions obtained for the example is exactly the same as the actual solutions in [26]; however, the numerical approach proposed in [26] only obtained the approximation solutions.
3. The computational time for the method in [26] is longer compared to our proposed method because the method in [26] involved many iterations to converge to the solution.

7. Conclusion

This study proposed a new algorithm for solving the PFFME, where the coefficients and the solution $\tilde{X}$ are in the form of arbitrary triangular fuzzy numbers. The proposed algorithm utilized the fuzzy Kronecker product and fuzzy Vec-operator and developed a new associated linear system. The solution is then obtained by using the pseudoinverse method. Moreover, the constructed algorithm is not only able to solve the PFFME; it can also solve the PFME. In conclusion, any real application involving the PFFME and PFME can employ the constructed algorithm in obtaining an exact solution with less computational time.

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