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The impact of hyperfine interaction on the charge radius of Protons

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Abstract

In this paper, the wave function of a Proton has been calculated via solving the Dirac equation for a three-particle system. Then, the charge radius of the proton was measured. Isospin-Isospin interaction was added to the system as a perturbation factor, and the variation of the charge radius of Proton was considered. Finally, the spin-spin and spin-isospin interactions on the charge radius of the Proton were calculated based on this method.

Key Words: Wave function of Protons, isospin, Dirac equation, Jacobean coordinate, charge radius

1. Introduction

Protons fall under the category of baryons and contain two up quarks and a down quark, hence a baryon of type uud. To calculate the wave function of Protons we employ the Dirac equation for a three-particle system, since a Proton is formed of three particles almost of the same order of size. In our scheme, we consider these particles alike in terms of their mass and energy.

Under a bag model, these three quarks thought to be influenced by those interactions whose origins are the potentials with which each quark affects the others. The potential is modeled having three components. The first observed potential is that of a harmonic oscillator (ar^2), whose quarks exhibit an oscillating movement around their mass center. The second is linear potential, which arises from the quarks not able to leave their bag. This potential is a limiting type. The third part of the potential, relating to inverse of the distance, $-\frac{c}{r}$, is based on the existence of a quark property known as color. The total potential then exhibits the form [1]

$$U(r) = ar^2 + br - \frac{c}{r}. \qquad (1)$$

2. Calculation of the wave function of the proton and its charge radius

To obtain the wave function of the proton, we used the Dirac equation, writing it separately for each quark. The wave function of the Proton then results from multiplying the tensors of these three wave functions. Wave function of the first particle is:

$$\psi_1 = \begin{pmatrix} \phi_1 \\ \chi_1 \end{pmatrix}.$$

So, the Dirac equation will be in this form:

First particle: $[\vec{\alpha}_1.\vec{p}_1 + \beta(m_1 + U_0)]\psi_1(x) = (E - V_0)\psi_1(x)$;

Second particle: $[\vec{\alpha}_2.\vec{p}_2 + \beta(m_2 + U_0)]\psi_2(x) = (E - V_0)\psi_2(x)$;

Third particle: $[\vec{\alpha}_3.\vec{p}_3 + \beta(m_3 + U_0)]\psi_3(x) = (E - V_0)\psi_3(x)$.

Hence, the total wave function is $\psi = \psi_1 \otimes \psi_2 \otimes \psi_3$.

In the wave function of the Proton, ψ contains eight components. The following equations would be achieved by applying components of the wave function in the Dirac equation, regarding the consideration of the scalar and pseudo-scalar potentials U_0, V_0 as the same:

$$\begin{aligned} (\vec{\sigma}_i.\vec{p}_i)\ \chi + (m_i + 2V_0)\,\phi &= \varepsilon_i\,\phi \\ (\vec{\sigma}_i.\vec{p}_i)\ \phi - (m_i)\,\chi &= \varepsilon_i\,\chi, \end{aligned} \tag{2}$$

where ϕ, χ are $\phi = g_\gamma(x)Y^{j_3}_{jl}$ and $\chi = if_\gamma(x)Y^{j_3}_{jl}$, respectively. From equation (2) and the following equation,

$$(\vec{\sigma}_i.\vec{p}_i)^2 = p_i^2 = p_1^2 + p_2^2 + p_3^2,$$

we get this differential equation:

$$(p_1^2 + p_2^2 + p_3^2)\phi = 9(\varepsilon^2 - m^2)\phi - 6V_0(\varepsilon + m)\phi. \tag{3}$$

But, to obtain the equation above, we use Jacobean coordinate system as the form: [1]

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3}\ ,\ \ \vec{\lambda} = \frac{\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3}{\sqrt{6}}\ \ ,\ \ \ \vec{\rho} = \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}}. \tag{4}$$

In this coordinate system, by omitting the sentence which is related to central mass, we'll have these relations:

$$p_\rho^2 = -\nabla_\rho^2\ ,\ \ p_\lambda^2 = -\nabla_\lambda^2\ \ ,\ P^2 = P_\rho^2 + P_\lambda^2.$$

With the relation between the hyper spherical radius and space angles Ω_ρ , Ω_λ, $x = \sqrt{\rho^2 + \lambda^2}$, we obtain the following relation:

$$(p_\rho^2 + p_\lambda^2)\phi = 3[3(\varepsilon^2 - m^2)\phi - 2V_0(\varepsilon + m)\phi]. \tag{5}$$

But,

$$(\nabla_\rho^2 + \nabla_\lambda^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{5}{x}\frac{\partial}{\partial x} + \frac{L^2(\Omega)}{x^2}\right)\ \ ,\ \ \ \ L^2(\Omega) = -\gamma(\gamma + 4). \tag{6}$$

were $n = 1, 2, 3, .., \; \gamma = 2n + l_\rho + l_\lambda$ [2, 3]; and the ground state, which is related to our calculations, is$\gamma = 0$. In lieu of equation (6) is equation (5), and considering the ground state, we reach the differential equation

$$a_1 = (\varepsilon + m)a \;\;, b_1 = (\varepsilon + m)\, b\,, \quad c_1 = (\varepsilon + m)c \;\;, \;\; \zeta = \varepsilon^2 - m^2,$$

$$\left[\frac{d^2}{dx^2} + \frac{5d}{x\,dx} + \frac{L^2(\Omega)}{x^2}\right]\phi_0(x) = -3\left[3\zeta - a_1x^2 - b_1x + \frac{c_1}{x}\right]\phi_0(x). \tag{7}$$

Now, suggesting $\phi\,(x) = x^{\frac{-5}{2}}\xi(x)$, we obtain the following differential equation [4]:

$$\frac{d^2\xi}{dx^2} - \{\frac{15}{4} - L^2(\Omega)\}\frac{1}{x^2}\xi = -3\left[3\zeta - a_1x^2 - b_1x + \frac{c_1}{x}\right]\xi. \tag{8}$$

We now have a solution, $\xi(x) = \exp(h(x))$ [5]; and in this relation we consider $h(x) = -\frac{1}{2}\alpha x^2 + \beta x + \delta \ln x$. Putting this relation into equation (8), we get these solutions:

$$\delta = \frac{1}{2} \pm \sqrt{4 - L^2(\Omega)}, \quad \alpha = \sqrt{3a_1},$$

$$\beta = \frac{-3c_1}{1 \pm 2\sqrt{4 - L^2(\Omega)}}, \quad b_1 = \frac{2c_1\sqrt{3a_1}}{1 \pm 2\sqrt{4 - L^2(\Omega)}}.$$

This equation for the ground state $L^2(\Omega) = 0$ is solved thusly:

$$\alpha = \sqrt{3a_1}\;, \quad b_1 = \frac{2c_1}{5}\sqrt{3a_1}\;, \quad \beta = -\frac{3}{5}c_1\;,\; \zeta = \frac{14}{9}\sqrt{3a_1} - \frac{1}{25}c_1^2\;,\; \delta = \frac{5}{2}, \frac{3}{2}. \tag{9}$$

So the total wave function related to Proton is calculated in this form:

$$\psi = \frac{N}{4\pi}\begin{pmatrix} 1 \\ \frac{i\vec{\sigma}.\hat{r}}{\varepsilon + m}\left(\sqrt{3a_1}\,x + \frac{3}{5}c_1\right)\end{pmatrix} \times \exp\left(\frac{-\sqrt{3a_1}}{2}\,x^2 - \frac{3}{5}\,c_1\,x\right). \tag{10}$$

In equation (10), N is normalization constant and is defined via the integral

$$N^2\int_0^\infty \left[|\phi|^2 + |\chi|^2\right]\,x^5\,dx = 1. \tag{11}$$

To calculate potential coefficient, we use definite quantity of $\frac{g_A}{g_V} = 1.26$, which is known as

$$\frac{g_A}{g_V} = \frac{5}{3}\left[1 - \frac{4}{3}\frac{\int_0^\infty |\phi|^2\;x^5\,dx}{\int_0^\infty \left[|\phi|^2 + |\chi|^2\right]x^5\,dx}\right] = 1.26. \tag{12}$$

So, to calculate the potential coefficient, we use these three equations:

$$\zeta = \frac{14}{9}\sqrt{3a_1} - \frac{1}{25}c_1^2,$$

$$b_1 = \frac{2c_1}{5}\sqrt{3a_1}, \tag{13}$$

$$\frac{g_A}{g_V} = 1.26.$$

By applying the amount of the wave function from equation (10) to equation (13), and choosing the potential coefficient quantity $a_1 = 1$ (since nothing non-physical occurs in the problem by choosing the same arbitrary amount), the other coefficients calculated from (10) and (13) equations are $a_1 = 1$. The following equations were reached:

$$\begin{aligned} a_1 &= 1 \ \text{fm}^{-4}, \\ c_1 &= 0.535 \ \text{fm}^{-1}, \\ m_q &= 1.251 \ \text{fm}^{-1}, \\ b_1 &= 0.463 \ \text{fm}^{-3}_{,} \end{aligned} \tag{14}$$

So the wave function is developed to this form:

$$\psi = \frac{N}{4\pi}\begin{pmatrix} 1 \\ \frac{i\vec{\sigma}.\hat{x}}{\varepsilon + m}(1.732\,x + 0.401) \end{pmatrix} \times \exp\left(-0.866\,x^2 - 0.401\,x\right). \tag{15}$$

Considering equation (11), the constant normalization will be $N = 2.788$. With this value and the wave function (15), we can obtain the charge radius of the Proton from the following equation:

$$\left\langle x^2 \right\rangle^{\frac{1}{2}} = N^2 \int_0^\infty \left[|\phi|^2 + |\chi|^2\right] x^5\, dx. \tag{16}$$

According to the equation (15), which

$$\phi = \frac{N}{4\pi}\exp\left(-0.866x^2 - 0.401x\right)$$

$$\chi = \frac{N}{4\pi}\frac{i\vec{\sigma}\cdot\hat{x}}{(\varepsilon + m)}(1.732x + 0.401)\exp(-0.866x^2 - 0.401x).$$

Placing these relations in equation (16), the charge radius of the Proton is found to be

$$\left\langle x^2 \right\rangle^{\frac{1}{2}} = 0.897\ \text{fm}. \tag{17}$$

3. Isospin-Isospin potential dependence on wave function and charge radius

Here, we consider isospin-isospin potential in the form [5]

$$H_I = A_I \sum_{i<j}^{\frac{1}{(\sqrt{\pi}\sigma_I)^3}\vec{T}_i\cdot} \cdot\vec{T}_j e^{-\frac{r_{ij}^2}{\alpha}}. \tag{18}$$

Following the above, we can calculate perturbation effect of this potential with this equation:

$$E' = \langle\psi| H_I |\psi\rangle\,,\ \text{that}\ \langle E'\rangle = \int \psi^{+} H_I\, \psi\, x^5\, dx. \tag{19}$$

So the value of perturbed energy is equal to $E' = 0.197\ \text{fm}^{-1}$. Therefore, after the effect of isospin potential, the value of system's energy will be

$$E = 1.584 + 0.197 = 1.781\ \text{fm}^{-1}.$$

With this equation the new potential coefficient and the system's new wave function are

$$\begin{aligned} a'_1 &= 1\ \text{fm}^{-4}, \\ c'_1 &= 0.445\ \text{fm}^{-1}, \\ m'_q &= 1.490\ \text{fm}^{-1}, \\ b'_1 &= 0.385\ \text{fm}^{-3}. \end{aligned} \tag{20}$$

The wave function will then be

$$\psi' = \frac{N'}{4\pi}\begin{pmatrix} 1 \\ \dfrac{i\vec{\sigma}.\hat{x}}{\varepsilon + m}(1.732\ x + 0.334) \end{pmatrix} \times \exp\left(-0.866\ x^2 - 0.334\, x\right), \tag{21}$$

and the constant normalization for this wave function is $N' = 2.535$. Considering (16), and using the isospin perturbation equation, the value of Proton's charge radius is found to be

$$\left\langle x^2 \right\rangle^{\frac{1}{2}} = 0.883\ \text{fm}. \tag{22}$$

4. The spin-spin potential dependence on wave function and charge radius

In fact, all three quarks are placed under the spin-spin potential in the Proton, regarding their spins interactions, which naturally give rise to their magnetic moment. The above is expressed in the equation [3]

$$H_s = A_s \sum_{i<j} \frac{1}{(\sqrt{\pi}\sigma_s)^3} e^{-(r_{ij}^2/\sigma_s^2)} (\vec{S}_i.\vec{S}_j). \tag{23}$$

Here, $A_s = 67.4\ \text{fm}^2$ and $\sigma_s = 2.87$ fm. But, since the spin-spin potential, as does the isospin-isospin potential, has an (but not a considerable) impact on the potential of the proton's quarks, and because analytic calculation of differential equation—which we have calculated above with these potentials—is difficult or impossible to calculate, we consider the effect of this potential as a perturbation. Within this approach, the perturbed amount of this potential would be

$$E'' = \langle\psi| H_s |\psi\rangle\,, \quad \langle E''\rangle = \int \psi^{+} H_s\, \psi\, x^5\, dx.$$

This value, with attention to wave function (15), will be

$$\langle E'' \rangle = 0.0019\,\mathrm{fm}^{-1}. \tag{24}$$

So the value of system's energy after the effect of spin-spin potential is

$$E = 1.584 + 0.0019 = 1.586\ \mathrm{fm}^{-1}. \tag{25}$$

With this works the new potential coefficients; and the new wave function is

$$a_1'' = 1\ \ \mathrm{fm}^{-4},\ c_1'' = 0.534\,\mathrm{fm}^{-1}\ ,\ m_q'' =\ 1.253\ \mathrm{fm}^{-1}\ ,\ b_1'' = 0.462\ \mathrm{fm}^{-3}, \tag{26}$$

Therefore the wave function will be calculated with this equation:

$$\psi'' = \frac{N''}{4\pi}\begin{pmatrix} 1 \\ \dfrac{i\vec{\sigma}.\hat{r}}{\varepsilon + m}\,(1.732\ x + 0.4\,) \end{pmatrix} \times \exp\left(-0.866\ x^2 - 0.4\,x\right), \tag{27}$$

where $N'' = 2.785$. According to equation (16), the value of charge radius after using the perturbation of spin-spin potential is

$$\left\langle x^2 \right\rangle^{\frac{1}{2}} = 0.896 fm. \tag{28}$$

5. The spin-isospin potential dependence on wave function and charge radius

The spin-isospin potential is of the form [3]

$$H_{sI} = A_{sI}\sum_{i<j}\frac{1}{(\sqrt{\pi}\sigma_{sI})^3}e^{-(r_{ij}^2/\sigma_{sI}^2)}(\vec{S}_i.\vec{S}_j)(\vec{T}_i.\vec{T}_j). \tag{29}$$

With equation (15), value of the perturbed energy is

$$\langle E''' \rangle = \int \psi^+ H_{sI}\,\psi\,x^5\,dx, \quad \langle E''' \rangle = -0.004\ \mathrm{fm}^{-1}. \tag{30}$$

So, the whole energy of system is calculated $E = 1.584 - 0.004 = 1.58\ \mathrm{fm}^{-1}$.

Therefore, the new potential coefficients, i.e., the new wave function is

$$a_1''' = 1\ \ \mathrm{fm}^{-4},\quad c_1''' = 0.537\ \mathrm{fm}^{-1},\quad m_q''' =\ 1.244\ \mathrm{fm}^{-1},\quad b_1''' = 0.465\ \mathrm{fm}^{-3}. \tag{31}$$

So the wave function will be calculated as

$$\psi''' = \frac{N'''}{4\pi}\begin{pmatrix} 1 \\ \dfrac{i\vec{\sigma}.\hat{r}}{\varepsilon + m}\,(1.732\ x + 0.403) \end{pmatrix} \times \exp\left(-0.866\ x^2 - 0.403\,x\right). \tag{32}$$

The normalization constant is $N''' = 2.788$. And with regards to the equation (15), the amount of the charge radius of the Proton would be calculated

$$\left\langle x^2 \right\rangle^{\frac{1}{2}} = 0.898\ \mathrm{fm}. \tag{33}$$

The obtained amount can be compared with the three numbers—computed from previous relations (17), (23), (29)—for the charge radius of the Proton.

6. Conclusion

As it is understood from the process of the article, we can consider the Proton as a collection of three quarks. The wave function of solving the Dirac equation for a three-particle system could be calculated through (15).

In fact, we can consider isospin-isospin, spin-spin and spin-isospin interaction of quarks as a potential. And we can enter the effect of three potentials as a perturbation to calculate the wave function.

In this paper, it was observed that the isospin-isospin dependence was more than the other perturbing potentials, and this potential is significant. The amount obtained is closer to the experimental results for the Proton's charge radius ($\left\langle x^2 \right\rangle^{1/2} = 0.88 \pm 0.03$ fm), compared to the two calculated amounts in (28) and (33).

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