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α -Associated metrics on rigged null hypersurfaces

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Abstract: Let $x : (M, g) \rightarrow (\overline{M}, \overline{g})$ be a null hypersurface isometrically immersed into a proper semi-Riemannian manifold $(\overline{M}, \overline{g})$. A rigging for M is a vector field ζ defined on some open subset of \overline{M} containing M such that $\zeta_p \notin T_p M$ for every $p \in M$. Such a vector field induces an everywhere transversal null vector field N defined over M and which induces on M the same geometrical objects as ζ . Let $\overline{\eta}$ be the 1-form that is \overline{g} -metrically equivalent to N and let $\eta = x^* \overline{\eta}$ be its pullback on M . For a given nowhere vanishing smooth function α on M , we have introduced and studied the so-called α -associated (semi-)Riemannian metric $g_\alpha = g + \alpha \eta \otimes \eta$. It turns out that this perturbation of the induced metric along a transversal null vector field is always nondegenerate, so we have established some relationships between geometrical objects of the (semi-)Riemannian manifold (M, g_α) and those of the lightlike hypersurface (M, g) . For instance, in the case where N is closed, we give a constructive method to find a α -associated metric whose Levi-Civita connection coincides with the connection ∇ induced on M through the projection of the Levi-Civita connection $\overline{\nabla}$ of \overline{M} along N . As an application, we show that given a null Monge hypersurface M in \mathbb{R}_q^{n+1} , there always exists a rigging and a α -associated metric whose Levi-Civita connection is the induced connection on M .

Key words: Null hypersurface, Monge hypersurface, screen distribution, rigging vector field, associated metric

1. Introduction

Let $(\overline{M}, \overline{g})$ be a proper semi-Riemannian manifold and $x : M \rightarrow \overline{M}$ an embedded hypersurface of \overline{M} . The pullback metric $g = x^* \overline{g}$ can be either degenerate or nondegenerate on M . When g is nondegenerate, one says that (M, g) is a semi-Riemannian hypersurface of $(\overline{M}, \overline{g})$; otherwise, (M, g) is called a null (or degenerate, or lightlike) hypersurface of $(\overline{M}, \overline{g})$. Since any semi-Riemannian hypersurface has (locally) a canonical transversal vector field that is everywhere orthogonal to the hypersurface (the Gauss map), there is a standard way to study the extrinsic geometry of such a hypersurface. In fact, geometrical objects of the ambient manifold \overline{M} are projected orthogonally on M and give new objects, which can be used to study the extrinsic geometry of the hypersurface.

For a null hypersurface, the tangent bundle contains the orthogonal bundle; hence, the orthogonal projection is no longer possible. One of the most used techniques to study a null hypersurface (M, g) immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is to arbitrarily fix on it a screen distribution $S(TM)$ (a complementary to the normal bundle TM^\perp in TM) and a (null) section $\xi \in \Gamma(TM^\perp)$. These choices locally fix a null

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transversal vector field N , which is orthogonal to the screen distribution, transversal to M , and satisfies $\bar{g}(N, \xi) = 1$. (See, for instance, [1, 5].) Instead of choosing a null section of the normal bundle and a screen distribution independently, we can make only one arbitrary choice of a transversal vector field ζ defined on an open neighborhood of M in \bar{M} . This vector field ζ is called rigging for M and induces a null section ξ (called the associated rigged vector field) of the normal bundle, a screen distribution, and a null transversal vector field N , all of them defined over M . This second technique (the rigging technique) was introduced in [8] and also used in other works such as [3, 11, 12].

A null rigging N for M induces a family (g_α) of nondegenerate metrics on M as follows. Let $\bar{\eta}$ be the 1-form that is \bar{g} -metrically equivalent to N (i.e. $\bar{\eta} = \bar{g}(N, \cdot)$) and η the pullback of $\bar{\eta}$ on M via the immersion x^* . For a given nowhere vanishing smooth function α on M we define the so-called α -associated (semi-)Riemannian metric $g_\alpha = g + \alpha\eta \otimes \eta$. When $\alpha = 1$, the metric $g_1 = g + \eta \otimes \eta$ is the so-called associated metric defined in [2]. Notice that for a function $\alpha > 0$ the associated metric corresponding to the change of rigging $N' = \sqrt{\alpha}N$ is equal to the α -associated metric g_α corresponding to N . When the ambient manifold (\bar{M}, \bar{g}) is a Lorentzian manifold, the associated metric is a Riemannian metric. This Riemannian metric was recently studied in [8].

When the null rigging N is defined over \bar{M} , it induces a perturbation $\bar{g}_\alpha = \bar{g} + \alpha\bar{\eta} \otimes \bar{\eta}$ of the metric \bar{g} whose restriction on M gives the associated metric. Such perturbations including those defined by spacelike or timelike vector fields at the place of null rigging have been considered in several works (see [9] for the α -associated type and [11] for canonical variation $g_t = g + t\eta \otimes \eta$ where t is constant). The Levi-Civita connection of the α -associated metric does not coincide in general with the connection ∇ induced on M from the Levi-Civita connection $\bar{\nabla}$ of \bar{g} through the projection along N . A necessary and sufficient condition to have this coincidence for the case $\alpha = 1$ was given in [2, 12].

In the present work, giving a null rigging N , we give a constructive method to obtain a nowhere vanishing smooth function α such that the Levi-Civita connection of the corresponding α -associated metric coincides with the induced connection. We also give relationships between curvatures of (M, ∇) and those of (M, ∇_α) . We give some applications of our formalism on null Monge hypersurfaces in \mathbb{R}_1^{n+1} .

This paper is organized as follows: the first section is the Introduction. In Section 2 we present the twisted metric or a perturbation of a semi-Riemannian metric along a null vector field. Section 3 is devoted to the general setup on null hypersurfaces and new results on the α -associated metric. Theorem 3.2 gives a necessary and sufficient condition for the α -associated connection to coincide with the induced connection provided that α is constant along the leaves of the screen distribution. Section 4 is devoted to the computation of curvatures of the induced connection and the α -associated connection. Finally, in Section 5, we apply the formalism developed in the previous sections to null Monge hypersurfaces in \mathbb{R}_1^{n+1} by showing that such hypersurfaces always admit suitable riggings and functions α such that $\nabla_\alpha = \nabla$.

2. Twisted metrics on a semi-Riemannian manifold

Throughout this work, (\bar{M}, \bar{g}) is an $(n + 1)$ -dimensional semi-Riemannian manifold of index $q > 0$, and $\bar{\nabla}$ and \bar{R} will represent respectively the Levi-Civita connection and the Riemannian curvature of \bar{g} . (Objects from the metric \bar{g} will be denoted with a line.) All manifolds are taken as smooth and connected. Let Σ be a d -dimensional manifold with $d \leq n + 1$. If there exists an immersion $x : \Sigma \rightarrow \bar{M}$, then $x(\Sigma)$ is called a

d -dimensional immersed submanifold of \bar{M} . If, moreover, x is injective, one says that $x(\Sigma)$ is a d -dimensional submanifold of \bar{M} . In addition, if the inverse map x^{-1} is a continuous map from $x(\Sigma)$ to Σ , $x(\Sigma)$ is a d -dimensional embedded submanifold of \bar{M} . When $x(\Sigma)$ is an embedded submanifold, one identifies Σ and $x(\Sigma)$. All submanifolds will be assumed embedded, and through the identification, stating that $x : M \rightarrow \bar{M}$ is a submanifold will mean that there is an embedding map $x : \Sigma \rightarrow \bar{M}$ such that $M = x(\Sigma)$. A hypersurface of \bar{M} is a submanifold of \bar{M} of dimension $d = n$. One says that $x : (M, g) \rightarrow (\bar{M}, \bar{g})$ is an isometrically immersed submanifold when, $x : M \rightarrow \bar{M}$ is a submanifold of \bar{M} and $g = x^*\bar{g}$. An isometrically immersed submanifold $x : (M, g) \rightarrow (\bar{M}, \bar{g})$ is called a nondegenerate submanifold if (M, g) is a semi-Riemannian manifold. Otherwise, one says that (M, g) is a degenerate or null or lightlike submanifold. The latter means that at any point $p \in M$ there exists a nonzero vector $u \in T_pM$ such that $g_p(u, v) = 0$ for every $v \in T_pM$.

Let N be a lightlike vector field defined over \bar{M} and α be a nowhere vanishing smooth function on \bar{M} . We set $\bar{\eta}$ to be the 1-form \bar{g} -metrically equivalent to N . Using \bar{g} , we define the α -twisted metric on \bar{M} as

$$\bar{g}_\alpha = \bar{g} + \alpha\bar{\eta} \otimes \bar{\eta}. \tag{1}$$

Lemma 2.1 *The pair $(\bar{M}, \bar{g}_\alpha)$ is a semi-Riemannian manifold.*

Proof Let $p \in \bar{M}$ and $u \in T_p\bar{M}$ such that $\bar{g}_\alpha(u, v) = 0$ for every $v \in T_p\bar{M}$. In particular, $\bar{g}_\alpha(u, N_p) = 0$, and hence $\bar{\eta}(u) = 0$, since N_p is a null vector. It follows that $\bar{g}(u, v) = 0$ for every $v \in T_p\bar{M}$, and then $u = 0$ since \bar{g} is nondegenerate. This proves that \bar{g}_α is nondegenerate on \bar{M} . \square

Let $\bar{\nabla}_\alpha$ be the Levi-Civita connection of \bar{g}_α . The metrics \bar{g} and \bar{g}_α are two semi-Riemannian metrics on \bar{M} . The following gives the relationship between their Levi-Civita connections.

Proposition 2.1 *The connections $\bar{\nabla}$ and $\bar{\nabla}_\alpha$ are related by*

$$\begin{aligned} \bar{\nabla}_\alpha U V &= \bar{\nabla} U V + \frac{1}{2} \left[\alpha\bar{\eta}(U) (i_V d\bar{\eta})^{\#\bar{g}_\alpha} + \alpha\bar{\eta}(V) (i_U d\bar{\eta})^{\#\bar{g}_\alpha} - \bar{\eta}(U)\bar{\eta}(V) d\alpha^{\#\bar{g}_\alpha} \right] \\ &\quad + \frac{1}{2} [\alpha (L_N \bar{g})(U, V) + d\alpha(U)\bar{\eta}(V) + d\alpha(V)\bar{\eta}(U)] N, \end{aligned} \tag{2}$$

where $d\alpha^{\#\bar{g}_\alpha}$ is the vector field \bar{g}_α -metrically equivalent to the 1-form $d\alpha$, and $L_N \bar{g}$ is the Lie derivative of \bar{g} along N .

Proof Let us start by recalling the Koszul equation defining $\bar{\nabla}_\alpha$. For all sections U, V, W of the tangent bundle $T\bar{M}$,

$$\begin{aligned} 2\bar{g}_\alpha(\bar{\nabla}_\alpha U V, W) &= U \cdot \bar{g}_\alpha(V, W) + V \cdot \bar{g}_\alpha(W, U) - W \cdot \bar{g}_\alpha(U, V) \\ &\quad + \bar{g}_\alpha([U, V], W) - \bar{g}_\alpha([V, W], U) + \bar{g}_\alpha([W, U], V). \end{aligned}$$

Using (1) and the fact that $\bar{\nabla}$ is torsion-free and \bar{g} -metric, the latter equation leads to

$$\begin{aligned}
 2\bar{g}_\alpha(\bar{\nabla}_U V, W) &= \bar{g}(\bar{\nabla}_U V, W) + \bar{g}(V, \bar{\nabla}_U W) + \alpha U \cdot (\bar{\eta}(V)\bar{\eta}(W)) + d\alpha(U)\bar{\eta}(V)\bar{\eta}(W) \\
 &\quad + \bar{g}(\bar{\nabla}_V W, U) + \bar{g}(W, \bar{\nabla}_V U) + \alpha V \cdot (\bar{\eta}(U)\bar{\eta}(W)) + d\alpha(V)\bar{\eta}(U)\bar{\eta}(W) \\
 &\quad - \bar{g}(\bar{\nabla}_W U, V) - \bar{g}(U, \bar{\nabla}_W V) - \alpha W \cdot (\bar{\eta}(U)\bar{\eta}(V)) - d\alpha(W)\bar{\eta}(U)\bar{\eta}(V) \\
 &\quad + \bar{g}(\bar{\nabla}_U V - \bar{\nabla}_V U, W) + \alpha\bar{\eta}([U, V])\bar{\eta}(W) - \bar{g}(\bar{\nabla}_V W - \bar{\nabla}_W V, U) \\
 &\quad - \alpha\bar{\eta}([V, W])\bar{\eta}(U) + \bar{g}(\bar{\nabla}_W U - \bar{\nabla}_U W, V) + \alpha\bar{\eta}([W, U])\bar{\eta}(V) \\
 &= 2\bar{g}(\bar{\nabla}_U V, W) + 2\alpha\bar{\eta}(\bar{\nabla}_U V)\bar{\eta}(W) + \alpha(L_N\bar{g})(U, V)\bar{\eta}(W) + d\alpha(U)\bar{\eta}(V)\bar{\eta}(W) \\
 &\quad + \alpha\bar{\eta}(U)d\bar{\eta}(V, W) + \alpha\bar{\eta}(V)d\bar{\eta}(U, W) + d\alpha(V)\bar{\eta}(U)\bar{\eta}(W) - d\alpha(W)\bar{\eta}(U)\bar{\eta}(V) \\
 &= 2\bar{g}_\alpha(\bar{\nabla}_U V, W) + \alpha(L_N\bar{g})(U, V)\bar{g}(N, W) + d\alpha(U)\bar{\eta}(V)\bar{g}(N, W) \\
 &\quad + \alpha\bar{\eta}(U)d\bar{\eta}(V, W) + \alpha\bar{\eta}(V)d\bar{\eta}(U, W) + d\alpha(V)\bar{\eta}(U)\bar{g}(N, W) - d\alpha(W)\bar{\eta}(U)\bar{\eta}(V),
 \end{aligned}$$

and (2) holds. □

3. Null hypersurfaces

3.1. Some preliminaries on null hypersurfaces

Let $x : (M, g) \rightarrow (\bar{M}, \bar{g})$ be a null hypersurface of (\bar{M}, \bar{g}) . A rigging for M is a vector field ζ defined on an open subset containing M such that for any $p \in M$, $\zeta_p \notin T_p M$. We say that a rigging ζ is a null rigging for M when the restriction of ζ on M is lightlike. Therefore, if N is a null vector field on \bar{M} anywhere transversal to M , then N is a null rigging for M . Determining conditions for the existence of a rigging for a given null hypersurface is still an open problem. However, it is clear that when the ambient manifold is a time-orientable Lorentzian manifold, any null hypersurface has a rigging, e.g., a timelike vector field globally defined.

Let ζ be a rigging for M and $\bar{\eta}$ be the 1-form \bar{g} -metrically equivalent to ζ (namely $\bar{\eta} = \bar{g}(\zeta, \cdot)$), and let $\eta = x^*\bar{\eta}$ be the restriction of $\bar{\eta}$ on M . The associated metric \tilde{g} is given by

$$\tilde{g} = g + \eta \otimes \eta. \tag{3}$$

The following is easy to prove.

Lemma 3.1 [2] *The associated metric \tilde{g} is nondegenerate.*

The associated rigged vector field is the vector field \tilde{g} -metrically equivalent to the 1-form η and denoted ξ . As \bar{g} is nondegenerate, it holds that

$$\eta(v) \neq 0, \quad \forall v \in T_x M^\perp \setminus \{0\}. \tag{4}$$

Lemma 3.2 *The rigged field ξ is the unique section of TM^\perp such that $\eta(\xi) = 1$.*

Proof Let $v \in T_x M$. By definition of ξ , $\eta(v) = \tilde{g}(\xi, v) = g(\xi, v) + \eta(\xi)\eta(v)$, which implies that $g(\xi, v) = \eta(v)(1 - \eta(\xi))$. In particular, when $v \in T_x M^\perp \setminus \{0\}$, the latter gives $\eta(v)(1 - \eta(\xi)) = 0$. Hence, $\eta(\xi) = 1$,

since from (4) $\eta(v) \neq 0$. It also follows that $g(\xi, v) = 0$ for every $v \in T_x M$. Hence, ξ is a section of TM^\perp and the uniqueness follows from the fact that TM^\perp is a rank 1 distribution. \square

From now on, $\zeta = N$ is a null rigging and ξ is the associated rigged vector field. We will consider perturbations (1) of the ambient metric along this null rigging. Setting $\mathcal{S}(TM) = \ker(\eta)$ and $tr(TM) = span(N)$, it is easy to prove that $\mathcal{S}(TM)$ is a screen distribution and the following decompositions hold:

$$T\overline{M}|_M = TM \oplus tr(TM) = \mathcal{S}(TM) \oplus_{orth} (TM^\perp \oplus tr(TM)). \tag{5}$$

It also holds that

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = 0, \quad \forall X \in \Gamma(\mathcal{S}(TM)). \tag{6}$$

Let ∇ be the connection on M induced from $\overline{\nabla}$ through the projection along the transversal bundle $tr(TM) = span(N)$. To avoid confusion, we may denote the induced connection by ∇^N . For every section X of TM , one has $\overline{g}(\overline{\nabla}_X \xi, \xi) = 0$, which shows that $\overline{\nabla}_X \xi \in \Gamma(TM)$. The Weingarten map is the endomorphism field

$$\chi : \begin{matrix} \Gamma(TM) & \rightarrow & \Gamma(TM) \\ X & \mapsto & \overline{\nabla}_X \xi \end{matrix}.$$

Gauss–Weingarten formulas of the immersion $x : (M, g) \rightarrow (\overline{M}, \overline{g})$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{7}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{9}$$

$$\nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \tag{10}$$

for all $X, Y \in \Gamma(TM)$, where $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathcal{S}(TM)$ induced by ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}(TM))$ along ξ . B and C are the local second fundamental forms of M and $\mathcal{S}(TM)$, respectively; A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathcal{S}(TM)$, respectively; and the rotation 1-form τ is given by $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. It is easy to check that $\overset{\star}{A}_\xi$ and A_N are $\mathcal{S}(TM)$ -valued.

Shape operators and second fundamental forms are related by

$$B(X, Y) = g(\overset{\star}{A}_\xi X, Y), \tag{11}$$

$$C(X, PY) = g(A_N X, Y). \tag{12}$$

Using (6), (7), and (11), it is straightforward to show that $\overset{\star}{A}_\xi$ is g -symmetric and $\overset{\star}{A}_\xi(\xi) = 0$. On the contrary, A_N is not necessarily g -symmetric. However, A_N is g -symmetric on the screen distribution as a consequence of the following lemma.

Lemma 3.3 *For any sections X, Y of the tangent bundle TM , one has*

$$g(A_N X, Y) - g(X, A_N Y) = \tau(X)\eta(Y) - \tau(Y)\eta(X) - d\eta(X, Y).$$

Proof One just computes $d\eta(X, Y)$ by using the covariant derivative and Gauss–Weingarten equations. \square

The mean curvatures of M and $S(TM)$ are respectively given by (cf. [4, 5])

$$\overset{\star}{H} = \sum_{i=2}^n \varepsilon_i B(E_i, E_i) \quad \text{and} \quad H = \sum_{i=2}^n \varepsilon_i C(E_i, E_i), \tag{13}$$

where (E_2, \dots, E_n) is an orthonormal basis of the screen distribution and $\varepsilon_i = g(E_i, E_i) = \pm 1$.

Proposition 3.1 *If the screen distribution $S(TM)$ is integrable and L is a leaf then $\vec{H} = H\xi + \overset{\star}{H}N$ is the mean curvature vector of the immersion $L \rightarrow (\overline{M}, \overline{g})$.*

Proof For every $x \in L$, one has $T_x L = \text{span}(N_x, \xi_x)$ and the Gauss formula of the immersion $L \rightarrow (\overline{M}, \overline{g})$ is given by

$$\overline{\nabla}_X Y = \overset{\star}{\nabla}_X Y + \overset{\star}{\nabla}_X^\perp Y = \overset{\star}{\nabla}_X Y + C(X, Y)\xi + B(X, Y)N, \tag{14}$$

for all $X, Y \in \Gamma(S(TM))$. Using (13), one has

$$\vec{H} = \sum_{i=2}^n \varepsilon_i \overset{\star}{\nabla}_{E_i}^\perp E_i = \sum_{i=2}^n \varepsilon_i C(E_i, E_i)\xi + \sum_{i=2}^n \varepsilon_i B(E_i, E_i)N = H\xi + \overset{\star}{H}N.$$

\square

A null hypersurface M is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function ρ on M such that at each point $x \in M$ and for all $X, Y \in T_x M$, $B_x(X, Y) = \rho(x)g_x(X, Y)$ (resp. B vanishes identically on M). It is equivalent to write $\overset{\star}{A}_\xi = \rho P$ and $\overset{\star}{A}_\xi = 0$, respectively. Notice that these are intrinsic notions on any null hypersurface in the sense that total umbilicity and total geodesibility of M do not depend on the chosen rigging. Also, the screen distribution $S(TM)$ is totally umbilical (resp. totally geodesic) if there exists a smooth function λ on M such that $C_x(X, PY) = \lambda(x)g_x(X, Y)$ for all $X, Y \in T_x M$ (resp. $C = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$). We say that the rigged null hypersurface $x : (M, g, N) \rightarrow (\overline{M}, \overline{g})$ (or the rigging N) has a conformal screen distribution when there exists a nowhere vanishing smooth function φ on M such that

$$A_N = \varphi \overset{\star}{A}_\xi.$$

When the 1-form η is closed, we say that (M, g, N) is a null hypersurface with closed rigging. The following technical lemma* will be used in the sequel.

Lemma 3.4 *For any null hypersurface with closed rigging and conformal screen distribution, the rotation 1-form vanishes on the screen distribution.*

*Fotsing TH, Ngakeu F. Null hypersurfaces and trapping horizons, 12 June 2017, arxiv: 1706.03861v1 [math.DG].

For all sections X, Y, Z, T of TM , the so-called Gauss–Codazzi equations of (M, g, N) are given by

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, PT) &= g(R(X, Y)Z, PT) \\ &\quad + B(X, Z)C(Y, PT) - B(Y, Z)C(X, PT), \end{aligned} \tag{15}$$

$$\bar{g}(\bar{R}(X, Y)Z, N) = \bar{g}(R(X, Y)Z, N), \tag{16}$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X), \end{aligned} \tag{17}$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \end{aligned} \tag{18}$$

$$\bar{g}(\bar{R}(X, Y)\xi, N) = C(Y, \overset{\star}{A}_\xi X) - C(X, \overset{\star}{A}_\xi Y) - d\tau(X, Y), \tag{19}$$

where $\nabla_X C$ is defined by $(\nabla_X C)(Y, PZ) = X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \overset{\star}{\nabla}_X PZ)$.

3.2. α -Associated metric and α -twisted metric

For $\alpha \in C^\infty(\bar{M})^*$ a nowhere vanishing smooth function, the restriction on M of the α -twisted metric (1) is given by

$$g_\alpha = g + \alpha\eta \otimes \eta, \tag{20}$$

where we have denoted again by α the restriction of α on M . We call g_α the α -associated metric of (M, g, N) . One can observe that the 1-associated metric g_1 is just the associated metric \tilde{g} .

Lemma 3.5 *The pair (M, g_α) is a semi-Riemannian manifold of index*

$$\nu_\alpha = q - \frac{1}{2}(1 + \text{sign}(\alpha)) = \begin{cases} q - 1 & \text{if } \alpha > 0 \\ q & \text{if } \alpha < 0. \end{cases}$$

Proof Let $x \in M$ and $u \in T_x M$ such that $g_\alpha(u, v) = 0$ for all $v \in T_x M$. In particular, $0 = g_\alpha(u, \xi_x) = \alpha(x)\eta_x(u) \Rightarrow \eta_x(u) = 0$ since $\alpha(x) \neq 0$. Thus, $u \in S(T_x M)$. One then has $g(u, v) = 0$ for all $v \in S(T_x M)$, and hence $u = 0$ since the restriction of g on the screen distribution is nondegenerate. Thus, (M, g_α) is a semi-Riemannian manifold. For the index, one just remarks that g is of index $q - 1$ on $S(TM)$ and $g_\alpha(\xi, \xi) = \alpha$. \square

We now know that $x_\alpha : (M, g_\alpha) \rightarrow (\bar{M}, \bar{g}_\alpha)$ is a nondegenerate hypersurface of the semi-Riemannian manifold $(\bar{M}, \bar{g}_\alpha)$. The Gauss map of the isometric immersion x_α is given by

$$\delta_\alpha = \sqrt{|\alpha|}N - \frac{\text{sign}(\alpha)}{\sqrt{|\alpha|}}\xi. \tag{21}$$

In fact, $g(X, \xi) = 0 \implies \bar{g}_\alpha(X, \delta_\alpha) = 0, \forall X \in \Gamma(TM)$, and also $\bar{g}_\alpha(\delta_\alpha, \delta_\alpha) = -\text{sign}(\alpha)$. It follows that $(\bar{M}, \bar{g}_\alpha)$ is a semi-Riemannian manifold of index q , since (M, g) is of index $\nu_\alpha = q - \frac{1}{2}(1 + \text{sign}(\alpha))$. For the rest of this subsection, we assume that the rigging N is closed. This means that its equivalent 1-form $\bar{\eta}$ is

closed. It is easy to check that this is equivalent to

$$\bar{g}(\bar{\nabla}_U N, V) = \bar{g}(U, \bar{\nabla}_V N), \quad \forall U, V \in \Gamma(T\bar{M}). \tag{22}$$

Using (2), one has

$$\bar{\nabla}_X \delta_\alpha = \bar{\nabla}_X \delta_\alpha + \frac{1}{2} [\alpha (L_N \bar{g})(X, \delta_\alpha) + d\alpha(X) \bar{\eta}(\delta_\alpha) + d\alpha(\delta_\alpha) \eta(X)] N - \frac{1}{2} \eta(X) \bar{\eta}(\delta_\alpha) d\alpha^{\# \bar{g}_\alpha}.$$

Using (7)–(22) and by direct calculations, one gets

$$\begin{aligned} \bar{\nabla}_X \delta_\alpha &= \frac{\text{sign}(\alpha)}{\sqrt{|\alpha|}} \left[-\alpha A_N X + \overset{\star}{A}_\xi X + \tau(X) \xi \right] + (X \cdot \sqrt{|\alpha|} + \sqrt{|\alpha|} \tau(X)) N + \frac{d\alpha(X)}{2\alpha} \xi, \\ (L_N \bar{g})(X, N) &= 0, \quad d\alpha^{\# \bar{g}_\alpha} = d\alpha^{\# g_\alpha} - \text{sign}(\alpha) d\alpha(\delta_\alpha) \delta_\alpha, \\ (L_N \bar{g})(X, \xi) &= 2\tau(X), \quad \bar{\eta}(\delta_\alpha) = -\frac{\text{sign}(\alpha)}{\sqrt{|\alpha|}}. \end{aligned}$$

Thus,

$$\bar{\nabla}_X \delta_\alpha = \frac{\text{sign}(\alpha)}{\sqrt{|\alpha|}} \left[-\alpha A_N X + \overset{\star}{A}_\xi X + \tau(X) \xi + \frac{d\alpha(X)}{2\alpha} \xi + \frac{\eta(X)}{2\sqrt{|\alpha|}} \left(\sqrt{|\alpha|} d\alpha^{\# g_\alpha} + d\alpha(\delta_\alpha) \xi \right) \right].$$

The shape operator of the immersion x_α is then given by

$$A_{\delta_\alpha}(X) = \frac{\text{sign}(\alpha)}{\sqrt{|\alpha|}} \left[\alpha A_N X - \overset{\star}{A}_\xi X - \tau(X) \xi - \frac{d\alpha(X)}{2\alpha} \xi - \frac{\eta(X)}{2\sqrt{|\alpha|}} \left(\sqrt{|\alpha|} d\alpha^{\# g_\alpha} + d\alpha(\delta_\alpha) \xi \right) \right].$$

If α is constant on each leaf of the screen distribution and the screen distribution is conformal with conformal factor $1/\alpha$, then the shape operator of the isometric immersion x_α is given by

$$A_{\delta_\alpha}(X) = -\frac{\text{sign}(\alpha)}{2\sqrt{|\alpha|}} \eta(X) [2\tau(\xi) + \eta(d\alpha^{\# g_\alpha}) + d\alpha(N)] \xi.$$

We then have the following result.

Theorem 3.1 *Let $x : (M, g, N) \rightarrow (\bar{M}^{n+1}, \bar{g})$ be a null hypersurface with closed rigging and conformal screen distribution with conformal factor $1/\alpha$ constant on leaves of the screen distribution. Then the isometric immersion $x_\alpha : (M, g_\alpha) \rightarrow (\bar{M}, \bar{g}_\alpha)$ (\bar{g}_α being defined by (20)) is a nondegenerate hypersurface with at most two principal curvatures: 0 with multiplicity $n - 1$ and eigenvectors the sections of $S(TM)$, and $-\frac{\text{sign}(\alpha)}{2\sqrt{|\alpha|}} [2\tau(\xi) + \eta(d\alpha^{\# g_\alpha}) + d\alpha(N)]$ with multiplicity 1 and eigenvectors the sections of $Rad(TM)$.*

3.3. Induced metric and α -associated metric

In this subsection, we will relate some geometrical objects of the α -associated metric g_α with those of the induced metric g . From here on, N is strictly a null rigging for M , meaning that we do not require N to be lightlike globally on \bar{M} , but on M . Recall that ∇_α is the Levi-Civita connection of the α -associated semi-Riemannian manifold (M, g_α) and ∇ is the connection on the rigged null hypersurface $x : (M, g, N) \rightarrow (\bar{M}, \bar{g})$ induced from $\bar{\nabla}$ through the projection along N .

Proposition 3.2 *The connections ∇_α and ∇ are related by*

$$\begin{aligned} \nabla_\alpha X Y &= \nabla_X Y - \frac{1}{2} \eta(X) \eta(Y) d\alpha^{\#g_\alpha} + \frac{\alpha}{2} [\eta(X)(i_Y d\eta)^{\#g_\alpha} + \eta(Y)(i_X d\eta)^{\#g_\alpha}] \\ &+ \frac{1}{2\alpha} [\alpha(L_N \bar{g})(X, Y) + 2B(X, Y) + d\alpha(X)\eta(Y) + d\alpha(Y)\eta(X)] \xi. \end{aligned} \tag{23}$$

Proof Reasoning as in the proof of (2), one has

$$\begin{aligned} 2g_\alpha(\nabla_\alpha X Y, Z) &= \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z) + \alpha X \cdot (\eta(Y)\eta(Z)) + d\alpha(X)\eta(Y)\eta(Z) \\ &+ \bar{g}(\bar{\nabla}_Y Z, X) + \bar{g}(Z, \bar{\nabla}_Y X) + \alpha Y \cdot (\eta(X)\eta(Z)) + d\alpha(Y)\eta(X)\eta(Z) \\ &- \bar{g}(\bar{\nabla}_Z X, Y) - \bar{g}(X, \bar{\nabla}_Z Y) - \alpha Z \cdot (\eta(X)\eta(Y)) - d\alpha(Z)\eta(X)\eta(Y) \\ &+ \bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z) + \alpha\eta(\bar{\nabla}_X Y - \bar{\nabla}_Y X)\eta(Z) - \bar{g}(\bar{\nabla}_Y Z - \bar{\nabla}_Z Y, X) \\ &- \alpha\eta([Y, Z])\eta(X) + \bar{g}(\bar{\nabla}_Z X - \bar{\nabla}_X Z, Y) + \alpha\eta([Z, X])\eta(Y) \\ &= 2g_\alpha(\nabla_X Y, Z) + 2B(X, Y)\eta(Z) + \alpha(L_N \bar{g})(X, Y)\eta(Z) + d\alpha(X)\eta(Y)\eta(Z) \\ &+ \alpha\eta(X)d\eta(Y, Z) + \alpha\eta(Y)d\eta(X, Z) + d\alpha(Y)\eta(X)\eta(Z) - d\alpha(Z)\eta(X)\eta(Y). \end{aligned}$$

From here, using the fact that

$$\alpha\eta(X) = g_\alpha(X, \xi) \quad \forall X \in \Gamma(TM), \tag{24}$$

one obtains (23). □

From here on, we assume that the rigging N is closed. Then using (22), (9), and (12), one has

$$(L_N \bar{g})(X, Y) = 2\tau(X)\eta(Y) - 2C(X, PY),$$

and equation (23) becomes

$$\begin{aligned} \nabla_\alpha X Y &= \nabla_X Y - \frac{1}{2} \eta(X) \eta(Y) d\alpha^{\#g_\alpha} \\ &+ \frac{1}{2\alpha} [2B(X, Y) - 2\alpha C(X, PY) + 2\alpha\tau(X)\eta(Y) + d\alpha(X)\eta(Y) + d\alpha(Y)\eta(X)] \xi. \end{aligned} \tag{25}$$

From now on, we use the following range of indexes:

$$i, j = 0, 1, \dots, n; \quad a, b = 1, \dots, n \quad k, l = 2, \dots, n,$$

for summations (often with Einstein summation convention). For free indexes, we shall use

$$\beta, \gamma = 1, \dots, n.$$

Let $(E_1 = \frac{1}{\sqrt{|\alpha|}}\xi, E_2, \dots, E_n)$ be a g_α -orthonormal frame field of TM such that (E_2, \dots, E_n) is a frame field of $S(TM)$. The matrix of g_α in this frame is given by

$$g_\alpha = (g_\alpha(E_a, E_b)),$$

and we set (g_α^{ab}) to be the inverse matrix. Note that $g_\alpha^{ab} = \varepsilon^a \delta^{ab}$, with $\varepsilon^a := \pm 1$.

Proposition 3.3 *One has:*

- ① for all X, Y sections of TM , $(L_\xi g_\alpha)(X, Y) = -2B(X, Y) + \eta(X)\eta(Y)d\alpha(\xi)$;
- ② in particular, $div^{g_\alpha}(\xi) = \frac{1}{2|\alpha|}d\alpha(\xi) - \overset{\star}{H}$;
- ③ if ξ is g_α -Killing conformal (or g_α -Killing) with conformal factor φ , then (M, g, N) is totally umbilical (or geodesic) with umbilical factor $\rho = -\frac{1}{2}\varphi$.

Proof Since ∇_α is the Levi-Civita connection of g_α , one has

$$(L_\xi g_\alpha)(X, Y) = g_\alpha(\nabla_X \xi, Y) + g_\alpha(X, \nabla_Y \xi). \tag{26}$$

Using (25), the latter becomes

$$(L_\xi g_\alpha)(X, Y) = g_\alpha(\nabla_X \xi, Y) + g_\alpha(X, \nabla_Y \xi) + \eta(X)\eta(Y)d\alpha(\xi) + \alpha[\eta(X)\tau(Y) + \eta(Y)\tau(X)].$$

From here, using (20) and Gauss–Weingarten formulas, the first item holds. By definition and using (26), one has

$$div^{g_\alpha}(\xi) = tr(\nabla_\alpha \xi) = \varepsilon^k g_\alpha(\nabla_{E_k} \xi, E_k) = \frac{1}{2}\varepsilon^k (L_\xi g_\alpha)(E_k, E_k).$$

From here, using the first item, we obtain the second item. For the last item, let us assume that ξ is g_α -conformal Killing with conformal factor φ . Then the first item says that for all X, Y sections of the tangent bundle TM ,

$$-2B(X, Y) + \eta(X)\eta(Y)d\alpha(\xi) = \varphi g(X, Y) + \alpha\varphi\eta(X)\eta(Y). \tag{27}$$

Setting $X = Y = \xi$, one finds $d\alpha(\xi) = \alpha\varphi$, and (27) becomes

$$-2B(X, Y) = \varphi g(X, Y),$$

which completes the proof. □

With the above proof we see that when ξ is g_α -Killing, α is necessarily constant along integral lines of ξ . We have two connections on M , namely the induced connection ∇ and the α -associated connection ∇_α . A natural question is to ask if both connections can coincide. The following result gives a necessary and sufficient condition to have an affirmative answer.

Theorem 3.2 *Let $x : (M, g, N) \rightarrow (\overline{M}, \overline{g})$ be a null hypersurface with closed rigging.*

- ① *Let α be a nowhere vanishing function constant on each leaf of the screen distribution. Then the induced connection is the Levi-Civita connection of the α -associated metric if and only if*

$$\overset{\star}{A}_\xi = \alpha A_N \quad \text{and} \quad 2\alpha\tau(\xi) + d\alpha(\xi) = 0. \tag{28}$$

- ② *Let α be a nonzero real number. Then the induced connection is the Levi-Civita connection of the α -associated metric if and only if*

$$\overset{\star}{A}_\xi = \alpha A_N \quad \text{and} \quad \tau \equiv 0. \tag{29}$$

Proof If α is constant along the leaves of the screen distribution, then

$$d\alpha(X) = \eta(X)d\alpha(\xi) \quad \text{and} \quad \alpha d\alpha^{\#g_\alpha} = d\alpha(\xi)\xi,$$

and equation (25) becomes

$$\nabla_\alpha X Y = \nabla_X Y + \frac{1}{2\alpha} [2B(X, Y) - 2\alpha C(X, PY) + 2\alpha\tau(X)\eta(Y) + \eta(X)\eta(Y)d\alpha(\xi)] \xi. \quad (30)$$

Now $\nabla_\alpha = \nabla$ if and only if

$$2B(X, Y) - 2\alpha C(X, PY) + 2\alpha\tau(X)\eta(Y) + \eta(X)\eta(Y)d\alpha(\xi) = 0. \quad (31)$$

Replacing X and Y by ξ in the latter, one obtains $d\alpha(\xi) + 2\alpha\tau(\xi) = 0$. The latter together with (31) allows us to conclude that if α is constant along the leaves of the screen distribution then (28) holds. Now if α is constant on M then the screen distribution is conformal and $\tau(\xi) = 0$, which by the Lemma 3.4 implies that τ identically vanishes. The converse is straightforward by using (30). \square

Thus, given a null rigging N , to find an α -associated perturbation of g for which the coincidence of connections happens, we have to solve equation (28) with α as unknown. By using Theorem 4.1 in [2], the proof of the following result is a straightforward computation.

Proposition 3.4 *Let $(M, g, N) \rightarrow (\overline{M}, \overline{g})$ be a rigged null hypersurface. If α is a function such that (28) holds, then the same equations hold for any change of rigging $\tilde{N} = \phi N$, with ϕ constant on each leaf of the screen distribution, and for $\tilde{\alpha} = \frac{\alpha}{\phi^2}$.*

We notice that for another nowhere vanishing function ϕ on M , the α -associated metric along N coincides with the $\frac{\alpha}{\phi^2}$ -associated metric along $\tilde{N} = \phi N$. Therefore, if $\nabla_\alpha^N = \nabla$, then we also have $\nabla_{\frac{\tilde{\alpha}}{\phi^2}}^{\tilde{N}} = \nabla$ without the condition that ϕ is constant along leaves of the screen distribution or the condition that \tilde{N} is closed.

4. Curvature of the α -associated metric

In this section, $x : (M, g, N) \rightarrow (\overline{M}, \overline{g})$ is a rigged null hypersurface with closed null rigging N of a semi-Riemannian manifold, and α is a nowhere vanishing smooth function on M constant on each leaf of the screen distribution. Let X, Y, Z be sections of TM . We recall that the Riemannian curvature R_α of the α -associated metric g_α is given by

$$R_\alpha(X, Y)Z = \nabla_\alpha X \nabla_\alpha Y Z - \nabla_\alpha Y \nabla_\alpha X Z - \nabla_{[X, Y]} Z. \quad (32)$$

It is straightforward to relate each of the three terms of the right-hand side of the above relation with tools of the lightlike metric. Using equation (25) and Gauss–Weingarten equations, one finds

$$\begin{aligned} \nabla_{\alpha X} \nabla_{\alpha Y} Z &= \nabla_X \nabla_Y Z - \left[\frac{1}{\alpha} B(Y, Z) - C(Y, PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha} \eta(Y)\eta(Z) d\alpha(\xi) \right] \star_{\xi} X \\ &+ \left\{ \frac{1}{\alpha} B(\nabla_Y X, Z) - C(X, P\nabla_Y Z) + \tau(X)\eta(\nabla_Y Z) + \frac{1}{2\alpha} \eta(X)\eta(\nabla_Y Z) d\alpha(\xi) \right. \\ &\quad - \left[\frac{1}{\alpha} B(Y, Z) - C(Y, PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha} \eta(Y)\eta(Z) d\alpha(\xi) \right] \tau(X) \\ &\quad - \frac{d\alpha(X)}{\alpha^2} \eta(Y)\eta(Z) d\alpha(\xi) + \frac{1}{2\alpha} X \cdot (\eta(Y)\eta(Z)) d\alpha(\xi) \\ &\quad \left. - \frac{d\alpha(X)}{2\alpha^2} B(Y, Z) + \frac{1}{\alpha} X \cdot B(Y, Z) - X \cdot C(Y, PZ) + X \cdot (\tau(Y)\eta(Z)) \right\} \xi. \end{aligned}$$

Similarly, we express the two other terms of (32) to obtain the following:

Proposition 4.1 *Riemannian curvatures of the connections ∇_{α} and ∇ are related by*

$$\begin{aligned} R_{\alpha}(X, Y)Z &= R(X, Y)Z - \left[\frac{1}{\alpha} B(Y, Z) - C(Y, PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha} \eta(Y)\eta(Z) d\alpha(\xi) \right] \star_{\xi} X \\ &+ \left[\frac{1}{\alpha} B(X, Z) - C(X, PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha} \eta(X)\eta(Z) d\alpha(\xi) \right] \star_{\xi} Y + d\tau(X, Y)\eta(Z) \\ &+ \left\{ \frac{1}{\alpha} (\nabla_X B)(Y, Z) - \frac{1}{\alpha} (\nabla_Y B)(X, Z) + (\nabla_Y C)(X, PZ) - (\nabla_X C)(Y, PZ) \right. \\ &\quad - \left[\frac{1}{\alpha} B(Y, Z) - 2C(Y, PZ) \right] \tau(X) + \left[\frac{1}{\alpha} B(X, Z) - 2C(X, PZ) \right] \tau(Y) \\ &\quad \left. + \frac{d\alpha(\xi)}{2\alpha^2} [\eta(Y)(2B(X, Z) - \alpha C(X, PZ)) - \eta(X)(2B(Y, Z) - \alpha C(Y, PZ))] \right\} \xi. \end{aligned}$$

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{T}$ be sections of the screen distribution. Using the above proposition, one finds

$$\begin{aligned} g_{\alpha}(R_{\alpha}(X, Y)Z, \mathcal{X}) &= g(R(X, Y)Z, \mathcal{X}) + \left[\frac{1}{\alpha} B(X, Z) - C(X, PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha} \eta(X)\eta(Z) d\alpha(\xi) \right] B(Y, \mathcal{X}) \\ &\quad - \left[\frac{1}{\alpha} B(Y, Z) - C(Y, PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha} \eta(Y)\eta(Z) d\alpha(\xi) \right] B(X, \mathcal{X}). \end{aligned}$$

Now using equation (15), this becomes

$$\begin{aligned} g_{\alpha}(R_{\alpha}(X, Y)Z, \mathcal{X}) &= \bar{g}(\bar{R}(X, Y)Z, \mathcal{X}) - B(X, Z)C(Y, \mathcal{X}) + B(Y, Z)C(X, \mathcal{X}) \\ &\quad + \left[\frac{1}{\alpha} B(X, Z) - C(X, PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha} \eta(X)\eta(Z) d\alpha(\xi) \right] B(Y, \mathcal{X}) \tag{33} \\ &\quad - \left[\frac{1}{\alpha} B(Y, Z) - C(Y, PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha} \eta(Y)\eta(Z) d\alpha(\xi) \right] B(X, \mathcal{X}). \end{aligned}$$

Using equations (16)–(17) and the above proposition, we obtain

$$\begin{aligned} \bar{g}(R_\alpha(\xi, \mathcal{X})\mathcal{Y}, N) &= \frac{1}{\alpha} (\nabla_\xi B)(\mathcal{X}, \mathcal{Y}) - \frac{1}{\alpha} (\nabla_{\mathcal{X}} B)(\xi, \mathcal{Y}) - \left[\frac{1}{\alpha} B(\mathcal{X}, \mathcal{Y}) - C(\mathcal{X}, \mathcal{Y}) \right] \tau(\xi) \\ &\quad - \frac{d\alpha(\xi)}{2\alpha^2} [2B(\mathcal{X}, \mathcal{Y}) + \alpha C(\mathcal{X}, \mathcal{Y})] - C(\xi, \mathcal{Y})\tau(\mathcal{X}). \end{aligned} \tag{34}$$

Equation (34) together with Gauss–Codazzi equation (18) gives

$$\begin{aligned} \bar{g}(R_\alpha(\xi, \mathcal{X})\mathcal{Y}, N) &= \frac{1}{\alpha} \bar{g}(\bar{R}(\xi, \mathcal{X})\mathcal{Y}, \xi) - \left[\frac{2}{\alpha} B(\mathcal{X}, \mathcal{Y}) - C(\mathcal{X}, \mathcal{Y}) \right] \tau(\xi) \\ &\quad - \frac{d\alpha(\xi)}{2\alpha^2} [2B(\mathcal{X}, \mathcal{Y}) + \alpha C(\mathcal{X}, \mathcal{Y})] - C(\xi, \mathcal{Y})\tau(\mathcal{X}). \end{aligned} \tag{35}$$

In Proposition 4.1, we have given relationships between Riemannian curvatures of the connections ∇_α and ∇ . Since ∇ is not a g -metric connection, the $(1, 3)$ -tensor R does not have all Riemannian curvature symmetries and does not allow us to define the classical Ricci tensor. However, if one defines a Ricci tensor as $Ric(X, Y) = tr(Z \mapsto R(Z, X)Y)$, this gives a nonnecessarily symmetric tensor and the definition of the scalar curvature becomes ambiguous. For this reason, we will relate the Ricci tensor of ∇_α with the one of $\bar{\nabla}$ for sections of TM . In [8], such a relationship was found for $\alpha = 1$ and by assuming that M is totally geodesic. We are going to relate this Ricci tensor for a function α constant on the leaves of the screen distribution and without the total geodesic condition. Let us start with sections of the screen distribution.

Proposition 4.2 *For all \mathcal{X}, \mathcal{Y} sections of the screen distribution, one has*

$$\begin{aligned} Ric_\alpha(\mathcal{X}, \mathcal{Y}) &= \bar{Ric}(\mathcal{X}, \mathcal{Y}) - \bar{g}(\bar{R}(\xi, \mathcal{X})\mathcal{Y}, N) - \bar{g}(\bar{R}(\xi, \mathcal{Y})\mathcal{X}, N) + \frac{1}{\alpha} \bar{g}(\bar{R}(\xi, \mathcal{X})\mathcal{Y}, \xi) - C(\xi, \mathcal{Y})\tau(\mathcal{X}) \\ &\quad + \frac{1}{\alpha} g(\overset{\star}{A}_\xi \mathcal{X}, \overset{\star}{A}_\xi \mathcal{Y}) - g(\overset{\star}{A}_\xi \mathcal{X}, A_N \mathcal{Y}) - g(A_N \mathcal{X}, \overset{\star}{A}_\xi \mathcal{Y}) + B(\mathcal{X}, \mathcal{Y}) \left(H - \frac{1}{\alpha} \overset{\star}{H} \right) \\ &\quad + C(\mathcal{X}, \mathcal{Y}) \overset{\star}{H} - \left[\frac{2}{\alpha} B(\mathcal{X}, \mathcal{Y}) - C(\mathcal{X}, \mathcal{Y}) \right] \tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2} [2B(\mathcal{X}, \mathcal{Y}) + \alpha C(\mathcal{X}, \mathcal{Y})]. \end{aligned} \tag{36}$$

Proof By definition,

$$Ric_\alpha(\mathcal{X}, \mathcal{Y}) = tr(\mathcal{Z} \mapsto R_\alpha(\mathcal{Z}, \mathcal{X})\mathcal{Y}) = \sum_{k=2}^n \varepsilon^k g(R_\alpha(E_k, \mathcal{X})\mathcal{Y}, E_k) + \bar{g}(R_\alpha(\xi, \mathcal{X})\mathcal{Y}, N).$$

Let us compute each term of the latter. Using (33), one has

$$\begin{aligned} \varepsilon^k g(R_\alpha(E_k, \mathcal{X})\mathcal{Y}, E_k) &= \varepsilon^k \bar{g}(\bar{R}(E_k, \mathcal{X})\mathcal{Y}, E_k) - B(A_N \mathcal{X}, \mathcal{Y}) + B(\mathcal{X}, \mathcal{Y})H \\ &\quad + \frac{1}{\alpha} B(\overset{\star}{A}_\xi \mathcal{X}, \mathcal{Y}) - B(\mathcal{X}, A_N \mathcal{Y}) - \left[\frac{1}{\alpha} B(\mathcal{X}, \mathcal{Y}) - C(\mathcal{X}, \mathcal{Y}) \right] \overset{\star}{H}. \end{aligned}$$

Again by definition,

$$\bar{Ric}(\mathcal{X}, \mathcal{Y}) = \varepsilon^k g(\bar{R}(E_k, \mathcal{X})\mathcal{Y}, E_k) + \bar{g}(\bar{R}(\xi, \mathcal{X})\mathcal{Y}, N) + \bar{g}(\bar{R}(\xi, \mathcal{Y})\mathcal{X}, N),$$

where we have used the quasi-orthonormal basis $(N, \xi, E_2, \dots, E_n)$. Hence,

$$\begin{aligned} \varepsilon^k g(R_\alpha(E_k, \mathcal{X})\mathcal{Y}, E_k) &= \overline{Ric}(\mathcal{X}, \mathcal{Y}) - \bar{g}(\overline{R}(\xi, \mathcal{X})\mathcal{Y}, N) - \bar{g}(\overline{R}(\xi, \mathcal{Y})\mathcal{X}, N) \\ &\quad - B(A_N\mathcal{X}, \mathcal{Y}) + B(\mathcal{X}, \mathcal{Y})H \\ &\quad + \frac{1}{\alpha} B(\overset{\star}{A}_\xi \mathcal{X}, \mathcal{Y}) - B(\mathcal{X}, A_N\mathcal{Y}) - \left[\frac{1}{\alpha} B(\mathcal{X}, \mathcal{Y}) - C(\mathcal{X}, \mathcal{Y}) \right] \overset{\star}{H}. \end{aligned}$$

Then one obtains (36) by summing the latter with (35). □

To complete the computation of the Ricci of all two sections of TM , it remains to compute $Ric_\alpha(\xi, \xi)$ and $Ric_\alpha(\xi, \mathcal{X})$.

Proposition 4.3 *For any function α constant on each leaf of the screen distribution of the rigged null hypersurface $(M, g, N) \rightarrow (\overline{M}, \bar{g})$ with closed null rigging N , the following hold:*

- ① $Ric_\alpha(\xi, \xi) = \overline{Ric}(\xi, \xi) - [\tau(\xi) + \frac{1}{2\alpha} d\alpha(\xi)] \overset{\star}{H}$.
- ② For any section \mathcal{X} of $S(TM)$,

$$Ric_\alpha(\xi, \mathcal{X}) = \overline{Ric}(\xi, \mathcal{X}) + d\tau(\xi, \mathcal{X}) + g(A_N\xi, \mathcal{X}) \overset{\star}{H}.$$

Proof By definition, $Ric_\alpha(\xi, X) = \varepsilon^k g_\alpha(R_\alpha(E_k, \xi)X, E_k)$. Equation (33) gives

$$\begin{aligned} \varepsilon^k g_\alpha(R_\alpha(E_k, \xi)X, E_k) &= \varepsilon^k \bar{g}(\overline{R}(E_k, \xi)X, E_k) - \varepsilon^k B(E_k, X)C(\xi, E_k) \\ &\quad + \varepsilon^k \left[C(\xi, PX) - \tau(\xi)\eta(X) - \frac{1}{2\alpha}\eta(X)d\alpha(\xi) \right] B(E_k, E_k). \end{aligned} \tag{37}$$

Replacing X by ξ and summing over k , one finds

$$Ric_\alpha(\xi, \xi) = \sum \varepsilon_k \bar{g}(\overline{R}(E_k, \xi)\xi, E_k) - \left[\tau(\xi) + \frac{1}{2\alpha} d\alpha(\xi) \right] \overset{\star}{H}.$$

Since $\overline{Ric}(\xi, \xi) = \sum \varepsilon_k g(\overline{R}(E_k, \xi)\xi, E_k)$, the first item holds. Now replacing X by \mathcal{X} in (37) and summing, one finds

$$Ric_\alpha(\xi, \mathcal{X}) = \overline{Ric}(\xi, \mathcal{X}) - \bar{g}(\overline{R}(\xi, \mathcal{X})\xi, N) + g(A_N\xi, \overset{\star}{A}_\xi \mathcal{X}) + g(A_N\xi, \mathcal{X}) \overset{\star}{H},$$

since $\overline{Ric}(\xi, \mathcal{X}) = \varepsilon^k g(\overline{R}(E_k, \xi)\mathcal{X}, E_k) + \bar{g}(\overline{R}(N, \xi)\mathcal{X}, \xi) = \varepsilon^k g(\overline{R}(E_k, \xi)\mathcal{X}, E_k) + \bar{g}(\overline{R}(\xi, \mathcal{X})\xi, N)$. Then using Gauss–Codazzi equation (19), the second item follows. □

The following relates sectional curvatures of ∇_α and $\overline{\nabla}$. Recall that the sectional curvature of a plane $\Pi = span(X, Y)$ is given by

$$K_\alpha(\Pi) = \frac{g_\alpha(R_\alpha(X, Y)X, Y)}{g_\alpha(X, X)g_\alpha(Y, Y) - g_\alpha(X, Y)^2}.$$

By using equation (33), the proof of the following proposition is a straightforward calculation.

Proposition 4.4 *Let \mathcal{X} and \mathcal{Y} be two orthogonal sections of the screen structure. Let us consider the planes $\Pi_0 = \text{span}(\xi, \mathcal{X})$ and $\Pi = \text{span}(\mathcal{X}, \mathcal{Y})$. Then:*

$$\begin{aligned} \textcircled{1} \quad & K_\alpha(\Pi_0) = \frac{1}{\alpha g(\mathcal{X}, \mathcal{X})} [\overline{K}_\xi(\Pi_0) + [\tau(\xi) + \frac{1}{2\alpha} d\alpha(\xi)] B(\mathcal{X}, \mathcal{X})]; \\ \textcircled{2} \quad & \end{aligned}$$

$$\begin{aligned} K_\alpha(\Pi) &= \overline{K}(\Pi) + \frac{B(\mathcal{X}, \mathcal{X})B(\mathcal{Y}, \mathcal{Y}) - B(\mathcal{X}, \mathcal{Y})^2}{\alpha g(\mathcal{X}, \mathcal{X})g(\mathcal{Y}, \mathcal{Y})} \\ &+ \frac{2B(\mathcal{X}, \mathcal{Y})C(\mathcal{X}, \mathcal{Y}) - B(\mathcal{X}, \mathcal{X})C(\mathcal{Y}, \mathcal{Y}) - B(\mathcal{Y}, \mathcal{Y})C(\mathcal{X}, \mathcal{X})}{g(\mathcal{X}, \mathcal{X})g(\mathcal{Y}, \mathcal{Y})}. \end{aligned}$$

Let us now relate scalar curvatures of (M, g_α) and $(\overline{M}, \overline{g})$.

Theorem 4.1 *Let $(M, g, N) \rightarrow (\overline{M}, \overline{g})$ be a null hypersurface with closed rigging of a semi-Riemannian manifold and g_α the semi-Riemannian metric on M defined as in (20). The scalar curvatures s_α and \overline{s} of (M, g_α) and $(\overline{M}, \overline{g})$, respectively, are related (on M) by*

$$\begin{aligned} s_\alpha &= \overline{s} - 4\overline{Ric}(\xi, N) + 2\overline{K}(\xi, N) + \frac{2}{\alpha} \overline{Ric}(\xi, \xi) - 2tr \left(\overset{\star}{A}_\xi \circ A_N \right) + \frac{1}{\alpha} tr \left(\overset{\star 2}{A}_\xi \right) \\ &+ \left(2H - \frac{1}{\alpha} \overset{\star}{H} \right) \overset{\star}{H} - \tau(A_N \xi) + \left(H - \frac{3}{\alpha} \overset{\star}{H} \right) \tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2} \left(H + 3 \overset{\star}{H} \right). \end{aligned}$$

Proof By definition,

$$s_\alpha = g_\alpha^{aa} Ric_\alpha(E_a, E_a) = \varepsilon^k Ric_\alpha(E_k, E_k) + \frac{1}{\alpha} Ric_\alpha(\xi, \xi).$$

Let us compute each term of the latter. Replacing \mathcal{X} and \mathcal{Y} by E_k in equation (36) and summing over k , one obtains

$$\begin{aligned} \varepsilon^k Ric_\alpha(E_k, E_k) &= \overline{s} - 4\overline{Ric}(\xi, N) + 2\overline{K}(\xi, N) + \frac{1}{\alpha} \overline{Ric}(\xi, \xi) - 2tr \left(\overset{\star}{A}_\xi \circ A_N \right) + \frac{1}{\alpha} tr \left(\overset{\star 2}{A}_\xi \right) \\ &+ \left(2H - \frac{1}{\alpha} \overset{\star}{H} \right) \overset{\star}{H} - \tau(A_N \xi) + \left(H - \frac{2}{\alpha} \overset{\star}{H} \right) \tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2} \left(H + 2 \overset{\star}{H} \right). \end{aligned} \tag{38}$$

The first item of Proposition 4.3 gives:

$$\frac{1}{\alpha} Ric_\alpha(\xi, \xi) = \frac{1}{\alpha} \overline{Ric}(\xi, \xi) - \frac{1}{\alpha} \left[\tau(\xi) + \frac{1}{2\alpha} d\alpha(\xi) \right] \overset{\star}{H}. \tag{39}$$

One obtains the announced result by summing (38) and (39). □

5. Application on Monge null hypersurfaces of \mathbb{R}_q^{n+1}

Let us now set $(\overline{M}, \overline{g}) = \mathbb{R}_q^{n+1}$, the real standard semi-Euclidean space with its canonical metric

$$\overline{g} = \varepsilon_i (dx^i)^2,$$

with Einstein's summation and where (x^0, \dots, x^n) is the rectangular coordinate of \mathbb{R}^{n+1} , and we have set

$$\varepsilon^i = \varepsilon_i := \begin{cases} -1 & \text{if } 0 \leq i \leq q-1 \\ +1 & \text{if } q \leq i \leq n \end{cases}.$$

Let \mathcal{D} be an open subset of \mathbb{R}_{q-1}^n and let $F : \mathcal{D} \rightarrow \mathbb{R}$ be a nowhere vanishing smooth function. Let us consider the immersion

$$x : \begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathbb{R}_q^{n+1} \\ p = (u^1, \dots, u^n) & \mapsto & x(p) = (x^0 = F(p), x^1 = u^1, \dots, x^n = u^n). \end{array} \tag{40}$$

Then $M = x(\mathcal{D})$ is called a Monge hypersurface. It is easy to check that a vector field $X = X^i \frac{\partial}{\partial x^i}$ (Einstein's summation) on \mathbb{R}_q^{n+1} is tangent to M if and only if $X^0 = X^a F'_{u^a}$, so $\mathbf{n} = \frac{\partial}{\partial x^0} + \varepsilon^a F'_{u^a} \frac{\partial}{\partial x^a}$ is normal to M . The Monge hypersurface M is a null hypersurface if and only if \mathbf{n} is a null vector field. It is equivalent to write

$$\varepsilon^a (F'_{u^a})^2 = \|\nabla F\|^2 = 1, \tag{41}$$

where ∇F is the gradient of F in the semi-Euclidean space \mathbb{R}_{q-1}^n . Then taking the partial derivative of (41) with respect to x^β leads to

$$\varepsilon^a F'_{u^a} F''_{u^a u^\beta} = 0. \tag{42}$$

5.1. Generic UCC-normalization on a Monge null hypersurface

Let us endow the Monge null hypersurface $x : M \rightarrow \mathbb{R}_q^{n+1}$ with the (physically and geometrically) relevant rigging

$$\mathcal{N}_F = \frac{1}{\sqrt{2}} \left[-\frac{\partial}{\partial x^0} + \varepsilon^a F'_{u^a} \frac{\partial}{\partial x^a} \right]. \tag{43}$$

The corresponding rigged vector field is given by

$$\xi_F = \frac{1}{\sqrt{2}} \mathbf{n} = \frac{1}{\sqrt{2}} \left[\frac{\partial}{\partial x^0} + \varepsilon^a F'_{u^a} \frac{\partial}{\partial x^a} \right]. \tag{44}$$

We show below that this is a closed normalization with vanishing rotation 1-form τ and conformal screen distribution with unit conformal factor $\varphi = 1$. Let us consider the natural (global) parametrization of M given by

$$\begin{cases} x^0 = F(u^1, \dots, u^n) \\ x^\alpha = u^\alpha \\ \alpha = 1, \dots, n \end{cases} \quad (u^1, \dots, u^n) \in \mathcal{D}. \tag{45}$$

Then $\Gamma(TM)$ is spanned by $\{\frac{\partial}{\partial u^\beta}\}_\beta$ with

$$\frac{\partial}{\partial u^\beta} = F'_{u^\beta} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^\beta}. \tag{46}$$

Now taking the covariant derivative of \mathbf{n} by the flat connection $\bar{\nabla}$ and using (42), we obtain

$$\begin{aligned} \bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathbf{n} &= \varepsilon^a F''_{u^\beta u^a} \frac{\partial}{\partial x^a} \\ &= \varepsilon^a F''_{u^\beta u^a} \left(F'_{u^a} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^a} \right), \\ \bar{\nabla}_{\frac{\partial}{\partial u^\alpha}} \mathbf{n} &= \varepsilon^a F''_{u^\beta u^a} \frac{\partial}{\partial u^a}. \end{aligned} \tag{47}$$

Using (42) again we have

$$\bar{g}(\bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathbf{n}, \mathcal{N}_F) = \varepsilon^a F'_{x^\beta} F''_{x^\beta x^a} = 0.$$

Hence, $\bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathbf{n}$ is a section of the screen distribution.

Proposition 5.1 *Let $x : (M, g, \mathcal{N}_F) \rightarrow \mathbb{R}_q^{n+1}$ be a Monge null hypersurface graph of a function F endowed with the rigging \mathcal{N}_F as in (43). Then the following hold:*

1. *The rigging \mathcal{N}_F is closed and the corresponding rotation 1-form $\tau^{\mathcal{N}_F}$ vanishes identically.*
2. *The screen distribution is conformal with $\varphi = 1$ as conformal factor.*
3. *The screen distribution is integrable with leaves the level sets of the function F .*
4. *The induced connection ∇ coincides with the Levi-Civita connection of the associated metric g_1 , i.e.*

$$\nabla_1 = \nabla.$$

5. *In the natural basis $\{\frac{\partial}{\partial u^a}\}_a$, the divergence (with respect to the induced connection) of a vector field $X = X^a \frac{\partial}{\partial u^a}$ takes the form*

$$\text{div} X = \frac{\partial X^a}{\partial u^a}$$

(as in the usual Euclidean case).

Proof Since $\bar{\nabla}$ is a flat connection and the difference between both of the vectors \mathcal{N}_F , ξ_F and $\frac{1}{\sqrt{2}}\mathbf{n}$ is a constant vector, then

$$\bar{\nabla} \cdot \mathcal{N}_F = \bar{\nabla} \cdot \xi_F = \frac{1}{\sqrt{2}} \bar{\nabla} \cdot \mathbf{n}.$$

Then by using (47) and (10), $\tau^{\mathcal{N}_F}$ identically vanishes and

$$A_{\mathcal{N}_F} \left(\frac{\partial}{\partial u^\alpha} \right) = \star A_{\xi_F} \left(\frac{\partial}{\partial u^\beta} \right) = -\frac{1}{\sqrt{2}} \varepsilon^a F''_{u^\beta u^a} \frac{\partial}{\partial u^a}. \tag{48}$$

Hence, $\star A_{\xi_F} = A_{\mathcal{N}_F}$, which shows that the screen distribution is conformal with conformal factor $\varphi = 1$. The 1-form η is given by

$$\eta = \sqrt{2} F'_{u^a} du^a.$$

Using the Gauss lemma it follows that

$$d\eta = \sqrt{2}F''_{u^a u^b} du^b \wedge du^a = \sqrt{2} \sum_{a \neq b} (F''_{u^a u^b} - F''_{u^b u^a}) du^b \otimes du^a = 0,$$

which shows that the rigging \mathcal{N}_F is closed. Then the screen distribution is integrable. Let us now show that the leaves of the screen distribution are really the level sets of F . Let $c \in \text{Im}(F)$ be a regular value of F and $M_c = F^{-1}(c)$ the c -level set of F in \mathbb{R}^n_{q-1} . Then $\psi_c : M_c \rightarrow \mathbb{R}^n_{q-1}$ is a semi-Riemannian hypersurface of the semi-Euclidean space \mathbb{R}^n_{q-1} and the Gauss map is the gradient ∇F of F . We take ψ_c to be the inclusion map and M_c is a subset of \mathcal{D} . We then have the following diagram:

$$\begin{array}{ccc} M_c \xrightarrow{\psi_c} \mathcal{D} & \xrightarrow{\psi} & M \xrightarrow{i} \mathbb{R}^{n+1}_q, \\ p(u^1, \dots, u^n) & \mapsto & x(x^0 = F(u^1, \dots, u^n), x^1 = u^1, \dots, x^n = u^n). \end{array} \tag{49}$$

We denote by $\overset{\circ}{\nabla}$ and ∇_c the Levi-Civita connections of \mathbb{R}^n_{q-1} and M_c , respectively. Taking the Jacobian matrix of ψ , it is easy to check that for any $X \in \Gamma(TM_c)$, $\psi_*(\psi_{c*}X) = \psi_*(X) = (\langle X, \nabla F \rangle, X) = (0, X)$ and

$$\begin{aligned} \langle \psi_*(X), \xi_F \rangle &= (1/\sqrt{2})\langle (0, X), (1, \nabla F) \rangle = (1/\sqrt{2})(-0 + \langle X, \nabla F \rangle) = 0, \\ \langle \psi_*(X), \mathcal{N}_F \rangle &= (1/\sqrt{2})\langle (0, X), (-1, \nabla F) \rangle = (1/\sqrt{2})(0 + \langle X, \nabla F \rangle) = 0. \end{aligned}$$

Thus, the level sets $\psi(M_c)$ are the leaves of the screen distribution $\mathcal{S}(\mathcal{N}_F)$ of M (endowed with the normalization (43)). Conversely, let $L \rightarrow (M, g) \rightarrow \mathbb{R}^{n+1}_q$ be a (connected) leaf of the screen distribution. We need to show that L is a level set of F . For every $x = (F(p), p) \in L$, for every $X = X^i \frac{\partial}{\partial x^i} \in T_x \mathbb{R}^{n+1}$,

$$X \in T_x L \iff \langle X, \xi_F \rangle = 0 = \langle X, \mathcal{N}_F \rangle \iff X^0 = 0.$$

Hence, for every $\beta = 1, \dots, n$, $\frac{\partial}{\partial x^\beta} \in T_x L$. In addition, since $\langle \frac{\partial}{\partial x^\beta}, \xi_F \rangle = \frac{1}{\sqrt{2}}F'_{u^\beta}$, it follows that $F'_{u^\beta}(p) = 0, \quad \forall x = (F(p), p) \in L, \forall \beta = 1, \dots, n$, so there exists $c \in \mathbb{R}$ such that

$$F(p) = c, \quad \forall x = (F(p), p) \in L.$$

Thus, the leaf L is defined on M by $F = c$.

Since $\tau^{\mathcal{N}_F}$ identically vanishes and $\overset{\star}{A}_{\xi_F} = A_{\mathcal{N}_F}$, ∇ is the Levi-Civita connection of the (semi-Riemannian) associated metric g_1 (see Theorem 4.1 in [2]). Let $X = X^a \frac{\partial}{\partial u^a}$ be a section of TM :

$$X = X^a \frac{\partial}{\partial u^a} = X^0 \frac{\partial}{\partial x^1} + X^a \frac{\partial}{\partial x^a},$$

with $X^0 = F'_{u^a} X^a$. We have

$$\bar{\nabla}_{\partial_{u^b}} X = \partial_{u^b}(X^0) \partial x^0 + \partial_{u^b}(X^a) \partial x^a.$$

By using (7) and (43) the left-hand side of the above equation gives

$$\begin{aligned} \bar{\nabla}_{\partial_{u^b}} X &= \nabla_{\partial_{u^b}} X + B(\partial_{u^b}, X) \mathcal{N}_F \\ &= f^a \partial_{u^a} + B(\partial_{u^b}, X) \mathcal{N}_F \\ &= \left(F'_{u^a} f^a - \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^0} \\ &\quad + \sum_{a=1}^{q-1} \left(f^a - F'_{u^a} \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^a} + \sum_{a=q}^n \left(f^a + F'_{u^a} \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^a}. \end{aligned}$$

After identification, one gets

$$f^a = \begin{cases} \partial_{u^b}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) & \text{if } 1 \leq a \leq q-1 \\ \partial_{u^b}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) & \text{if } q \leq a \leq n \end{cases}.$$

Hence,

$$\nabla_{\partial_{u^b}} X = \sum_{a=1}^{q-1} \left(\partial_{u^b}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) \right) \partial_{u^a} + \sum_{a=q}^n \left(\partial_{u^b}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) \right) \partial_{u^a}.$$

The above relation together with equation (41) leads to

$$\begin{aligned} \operatorname{div} X &= \operatorname{tr}(\nabla X) \\ &= \sum_{a=1}^{q-1} \left(\partial_{u^a}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^a}, X) \right) + \sum_{a=q}^n \left(\partial_{u^a}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^a}, X) \right) \\ &= \partial_{u^a}(X^a) + \frac{1}{\sqrt{2}} \sum_{a=1}^{q-1} F'_{u^a} B(F'_{u^a} \partial_{x^0} + \partial_{x^a}, X) - \frac{1}{\sqrt{2}} \sum_{a=q}^n F'_{u^a} B(F'_{u^a} \partial_{x^0} + \partial_{x^a}, X) \\ &= \partial_{u^a}(X^a) - B(\xi_F, X) \\ &= \partial_{u^a}(X^a). \end{aligned}$$

□

Hence, on any Monge null hypersurface, our rigging \mathcal{N}_F has many good properties: the screen distribution is integrable, the 1-form τ identically vanishes, and

$$A_{\mathcal{N}_F} = A_{\xi_F}^* . \tag{50}$$

On a Monge null hypersurface, the rigging (43) is called generic unitary conformally closed (UCC)-rigging, since it is closed and has a conformal screen with conformal factor $\varphi = 1$. Recall that a hypersurface of a semi-Riemannian manifold is said to be totally geodesic when its shape operator identically vanishes. The above proposition together with Theorem 3.1 gives the following result.

Theorem 5.1 *For any Monge null hypersurface $(M, g, \mathcal{N}_F) \rightarrow \mathbb{R}_q^{n+1}$ endowed with its generic UCC-rigging (43), the isometric immersion $x_1 : (M, g_1) \rightarrow (\mathbb{R}^{n+1}, \bar{g}_1)$ into the twisted semi-Riemannian manifold $(\mathbb{R}^{n+1}, \bar{g}_1)$ with the metric (20) is a totally geodesic semi-Riemannian hypersurface.*

5.2. A special rigging on a Monge null hypersurface of \mathbb{R}_q^{n+1}

Let us now consider for $x : M \rightarrow \mathbb{R}_q^{n+1}$, as in (40), the rigging

$$\mathcal{N}_F = \frac{1}{2x^0} \left[\frac{\partial}{\partial x^0} - \varepsilon^a F'_{u^a} \frac{\partial}{\partial x^a} \right] \tag{51}$$

with corresponding rigged vector field

$$\xi_F = -x^0 \mathbf{n} = x^0 \left[-\frac{\partial}{\partial x^0} - \varepsilon^a F'_{u^a} \frac{\partial}{\partial x^a} \right]. \tag{52}$$

These two vector fields are defined on $\mathbb{R}^* \times \mathcal{D}$, which is an open subset containing our Monge null hypersurface M . However, they are lightlike only along M . Since \mathcal{N}_F is pointwise conformal to the generic UCC-rigging, the rigging \mathcal{N}_F also has integrable screen distribution and corresponding leaves are the level sets of the function F . Furthermore, for this rigging,

$$\bar{\eta} = -\frac{1}{2x^0} \left[dx^0 + F'_{u^a} dx^a \right] \tag{53}$$

and

$$\eta = -\frac{1}{x^0} F'_{u^a} du^a \tag{54}$$

since $dx^0 = F'_{u^a} du^a$ along M . Let us set for this subsection $\alpha = 2(x^0)^2$, which is constant along the leaves of the screen distribution. After a direct calculation we obtain

$$g_\alpha = [\varepsilon_a + (F'_{u^a})^2] (du^a)^2 + 2 \sum_{a < b} F'_{u^a} F'_{u^b} du^a du^b, \tag{55}$$

where $2dx^i dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i$. Notice that g_α is a semi-Riemannian metric of index $q - 1$ on M , but since \mathcal{N}_F is lightlike only along M , the metric \bar{g}_α is not necessary nondegenerate. The problem is to find integers n and q for which this metric is nondegenerate, in order to apply results of Section 3 to the Monge null hypersurface M endowed with this rigging. For example, by a calculation of determinant, one shows that for $n = 3$ and $q = 2$, this metric \bar{g}_α is nondegenerate for any F .

Using (46) one has

$$\bar{\nabla}_{\frac{\partial}{\partial u^a}} \xi_F = -x^0 \bar{\nabla}_{\frac{\partial}{\partial u^a}} \mathbf{n} - \frac{\partial x^0}{\partial u^a} \mathbf{n} = -x^0 \bar{\nabla}_{\frac{\partial}{\partial u^a}} \mathbf{n} + \frac{F'_{u^a}}{x^0} \xi_F.$$

The latter together with (10) and (54) gives

$$A_\xi^* = x^0 \bar{\nabla} \cdot \mathbf{n} \quad \text{and} \quad \tau = \eta. \tag{56}$$

We also have

$$\bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathcal{N}_F = -\frac{1}{2x^0} \bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathbf{n} - (2x^0) \frac{\partial 1/(2x^0)}{\partial u^\beta} \mathcal{N}_F = -\frac{1}{2x^0} \bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \mathbf{n} - \frac{F'_{u^\beta}}{x^0} \mathcal{N}_F,$$

which allows us to find

$$A_N = \frac{1}{2x^0} \bar{\nabla} \cdot \mathbf{n}. \tag{57}$$

Then the screen distribution is conformal with

$$\star A_\xi = 2(x^0)^2 A_N \quad \text{and} \quad \tau = \eta. \tag{58}$$

From here, it is easy to check that (28) holds. By Theorem 3.2, the induced connection is the Levi-Civita connection of the α -associated metric g_α . Thus, $\nabla = \nabla_\alpha$, where $\alpha = 2(x^0)^2$.

Remark 5.1 *It is noteworthy that for all changes of rigging $\tilde{N} = \phi\mathcal{N}_F$, the Levi-Civita connection of the $\frac{\alpha}{\phi^2}$ -associated metric coincides with the induced connection $\nabla^{\tilde{N}}$.*

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References

- [1] Atindogbe C. Blaschke type normalization on light-like hypersurfaces. *Journal of Geometry and Physics* 2010; 6 (4): 362-382.
- [2] Atindogbe C, Ezin JP, Tossa J. Pseudo-inversion of degenerate metrics. *International Journal of Mathematics and Mathematical Sciences* 2003; 55: 3479-3501.
- [3] Atindogbe C, Fotsing TH. Newton transformations on null hypersurfaces. *Communication in Mathematics* 2015; 23 (1): 57-83.
- [4] Dong J, Liu X. Totally umbilical lightlike hypersurfaces in Robertson-Walker spacetimes. *ISRN Geometry* 2014; 2014: 974695.
- [5] Duggal KL, Bejancu A. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Application*. Dordrecht, the Netherlands: Springer-Netherlands, 1996.
- [6] Duggal KL, Jin DH. *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*. Singapore: World Scientific Publishing Company, 2007.
- [7] Duggal KL, Sahin B. *Differential Geometry of Lightlike Submanifolds*. New York, NY, USA: Springer Science & Business Media, 2010.
- [8] Gutierrez M, Olea B. Induced Riemannian structures on null hypersurfaces. *Mathematische Nachrichten* 2015; 289: 1219-1236.
- [9] Haesen S. Vacuum rank 1 space-time perturbations. *General Relativity and Gravitation* 2004; 36 (8): 1791-1804.
- [10] Kupeli DN. *Singular Semi-Riemannian Geometry*. Dordrecht, the Netherlands: Kluwer Academic, 1996.
- [11] Olea B. Canonical variation of a Lorentzian metric. *Journal of Mathematical Analysis and Applications* 2014; 419: 156-171.
- [12] Olea B. Null hypersurfaces on Lorentzian manifolds and rigging techniques. In: *International Meeting on Lorentzian Geometry*; 2017. pp. 237-251.