Turkish Journal of Physics

Volume 36 | Number 1 Article 10

1-1-2012

The differential transform method for solving heat-like and wave-like equations with variable coefficients

Khatereh TABATABAEI

ERCAN ÇELİK

Raziyeh TABATABAEI

Recommended Citation

TABATABAEI, Khatereh; ÇELİK, ERCAN; and TABATABAEI, Raziyeh (2012) "The differential transform method for solving heat-like and wave-like equations with variable coefficients," *Turkish Journal of Physics*: Vol. 36: No. 1, Article 10. https://doi.org/10.3906/fiz-1102-6

Available at: https://journals.tubitak.gov.tr/physics/vol36/iss1/10

Turk J Phys
36 (2012) , 87 – 98.
© TÜBİTAK
doi:10.3906/fiz-1102-6

The differential transform method for solving heat-like and wave-like equations with variable coefficients

Khatereh TABATABAEI[1], **Ercan ÇELİK**[2] **and Raziyeh TABATABAEI**[1]
[1] *Young Researchers Club, Azad University, Department of Mathematics, Bonab-IRAN*
[2] *Atatürk University, Faculty of Science, Department of Mathematics, 25240 Erzurum-TURKEY*
e-mail: ercelik@atauni.edu.tr

Received: 10.02.2011

Abstract

In this article, Differential transform method is presented for solving heat-like and wave-like equations with variable coefficients. We applied these methods to six examples. This powerful method gives an exact solution. These examples are prepared to show the efficiency and simplicity of the method.

Key Words: Heat-like equation, wave-like equation, differential transform method

1. Introduction

The heat-like and wave-like equations can be found in a wide variety of engineering and scientific applications. In recent years, numerous works have focused on the development of more advanced and efficient methods for heat-like and wave-like equations such as the Adomian decomposition method [1], variational iteration method [2] and the Adomian method [3, 4].

Differential transform method (DTM) is based on Taylor series expansion [5, 6]. In 1986, the differential transform method (DTM) was first introduced by Zhou [7] to solve linear and nonlinear initial value problems associated with electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylo's series method, which requires symbolic competition of the necessary derivatives of the data functions. This method has been successfully applied to solve many types of partial differential equations [8, 9, 10]. All of the previous applications of the differential transform method deal with solutions without discontinuity. However, many partial differential equations have different types of discontinuity. As the DTM is more effective than the other methods, we further apply it to solvethe heat-like and wave-like equations.

The main goal of this work is to obtain exact solutions to heat-like and wave-like equations with variable coefficients. This paper outlines the application of DTM to the heat-like and wave-like equations.Three problems for heat-like equation and three problem for wave-like equation are solved to make clear the application of the transform.

Heat-like equation

We consider a heat-like equation with variable coefficients described by a three-dimensional initial boundary value problem (IBVP) of the form

$$\begin{aligned} u_t = \;& f(x,y,z)\,u_{xx} + g(x,y,z)\,u_{yy} + h(x,y,z)\,u_{zz}\,, \\ & 0<x<a\,,\ 0<y<b\,,\ 0<z<c\,, t>0\,, \end{aligned} \tag{1.1}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,z,t) &= f_1(y,z,t)\,, \qquad u_x(a,y,z,t) = f_2(y,z,t)\,, \\ u_y(x,0,z,t) &= g_1(x,z,t)\,, \qquad u_y(x,b,z,t) = g_2(x,z,t)\,, \\ u_z(x,y,0,t) &= h_1(x,y,t)\,, \qquad u_z(x,y,c,t) = h_2(x,y,t)\,, \end{aligned} \tag{1.2}$$

and the initial condition

$$u(x,y,z,0) = \varphi(x,y,z)\ . \tag{1.3}$$

Wave-like equation

We consider a wave-like equation with variable coefficients obeying a three-dimensional initial boundary value problem (IBVP) of the form

$$\begin{aligned} u_{tt} = \;& f(x,y,z)\,u_{xx} + g(x,y,z)\,u_{yy} + h(x,y,z)\,u_{zz}\,, \\ & 0<x<a\,,\ \ 0<y<b\,,\ \ 0<z<c\,,\ \ t>0\,, \end{aligned} \tag{1.4}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,z,t) &= f_1(y,z,t)\,, \qquad u_x(a,y,z,t) = f_2(y,z,t)\,, \\ u_y(x,0,z,t) &= g_1(x,z,t)\,, \qquad u_y(x,b,z,t) = g_2(x,z,t)\,, \\ u_z(x,y,0,t) &= h_1(x,y,t)\,, \qquad u_z(x,y,c,t) = h_2(x,y,t)\,, \end{aligned} \tag{1.5}$$

and the initial condition

$$u(x,y,z,0) = \psi(x,y,z)\ , \quad u_t(x,y,z,0) = \theta(x,y,z)\ . \tag{1.6}$$

2. N-Dimensional differential transfor

Differential transform of function $w(x_1,x_2,...,x_n)$ is defined as

$$W(k_1,k_2,...,k_n) = \frac{1}{k_1!k_2!...k_n!}\left[\frac{\partial^{k_1+k_2+...+k_n}\ w(x_1,x_2,...,x_n)}{\partial x_1^{k_1}\,\partial x_2^{k_2}...\,\partial x_n^{k_n}}\right]_{x_1=0,x_2=0,...,x_n=0}. \tag{2.1}$$

In equation (2.1), $w(x_1,x_2,...,x_n)$ is the original function and $W(k_1,k_2,...,k_n)$ is the transformed function. The Differential inverse transform of $W(k_1,k_2,...,k_n)$ is defined as

$$w(x_1,x_2,...,x_n) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty} W(k_1,k_2,...,k_n)\,x_1^{k_1}\,x_2^{k_2}\,...\,x_n^{k_n}. \tag{2.2}$$

In actual applications, the function $w(x_1, x_2, ..., x_n)$ is expressed by a finite series and equation (2.2) can be written as

$$w(x_1, x_2, ..., x_n) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} ... \sum_{k_n=0}^{m_n} W(k_1, k_2, ..., k_n)\, x_1^{k_1}\, x_2^{k_2} ...\, x_n^{k_n}. \tag{2.3}$$

The fundamental mathematical operations performed by n-Dimensional Differential Transform are listed in Table 1.

Table 1. The fundamental operations of n-dimensional DTM.

Original function	Transformed function
$w(x_1, ..., x_n) = \alpha\, u(x_1, ..., x_n) \pm \beta\, v(x_1, ..., x_n)$	$W(k_1, ..., k_n) = \alpha\, U(k_1, ..., k_n) \pm \beta\, V(k_1, ..., k_n)$
$w(x_1, x_2, ..., x_n) = \frac{\partial^{r_1+r_2+...+r_n} u(x_1,x_2,...,x_n)}{\partial x_1^{r_1} ... \partial x_n^{r_n}}$	$W(k_1, ..., k_n) = (k_1+1)...(k_1+r_1)(k_2+1)...$ $(k_2+r_2)...(k_n+1)...(k_n+r_n)$ $U(k_1+r_1, k_2+r_2..., k_n+r_n)$
$w(x_1, x_2, ..., x_n) = x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$	$W(k_1, ..., k_n) = \delta(k_1-e_1, k_2-e_2, ..., k_n-e_n)$

Theorem 1. If $w(x_1, x_2, ..., x_n) = u(x_1, x_2, ..., x_n)\, v(x_1, x_2, ..., x_n)$, then
$$W(k_1, k_2, ..., k_n) = \sum_{r_1=0}^{k_1} \sum_{r_2=0}^{k_2} ... \sum_{r_n=0}^{k_n} U(r_1, ..., r_{n-1}, k_n - r_n)\, V(k_1-r_1, k_2-r_2, ..., k_{n-1}-r_{n-1}, r_n)\,.$$

Proof. By substituting the $w(x_1, x_2, ..., x_n) = u(x_1, x_2, ..., x_n)\, v(x_1, x_2, ..., x_n)$ in equation (2.1), we get

$$k_1 = k_2 = ... = k_{n-1} = 0,$$

$$\begin{aligned} k_n = 0: \quad W(0,0,...,0) &= \frac{1}{0!\,0!...0!} \frac{\partial^0}{\partial x_1^0 \partial x_2^0 ... \partial x_n^0} u(x_1, x_2, ..., x_n)\, v(x_1, x_2, ..., x_n) \\ &= U(0,0,....,0)V(0,0,...,0); \end{aligned}$$

$$\begin{aligned} k_n = 1: \quad W(0,0,...,1) &= \frac{1}{0!\,0!...1!} \frac{\partial}{\partial x_1^0 \partial x_2^0 ... \partial x_n} u(x_1, x_2, ..., x_n)\, v(x_1, x_2, ..., x_n) \\ &= \frac{\partial u(x_1,x_2,...,x_n)}{\partial x_n} v(x_1, x_2, ..., x_n) + u(x_1, x_2, ..., x_n)\, \frac{\partial v(x_1,x_2,...,x_n)}{\partial x_n}\Bigg|_{x_1=0,x_2=0,...,x_n=0} \\ &= U(0,0,...,0,1)V(0,0,...,0) + U(0,0,...,0,0)V(0,0,...,0,1), \end{aligned}$$

$$\begin{aligned} k_n = 2: \quad W(0,0,\ldots,2) &= \frac{1}{0!\,0!\cdots 2!} \frac{\partial^2}{\partial x_1^0 \partial x_2^0 \cdots \partial x_n^2} u(x_1, x_2, \ldots, x_n)\, v(x_1, x_2, \ldots x_n)\Big|_{x_1=0,x_2=0,\ldots,x_n=0} \\ &= \frac{1}{2!} \frac{\partial}{\partial x} \Bigg[\frac{\partial u(x_1,x_2,...,x_n)}{\partial x_n} v(x_1\ , x_2, \ldots, x_n) \\ &\quad + u(x_1, x_2, \ldots, x_n)\ \frac{\partial v(x_1,x_2,...,x_n)}{\partial x_n} \Bigg]\ \Bigg|_{x_1=0,x_2=0,\ldots,x_n=0} \\ &= U(0,0,\ldots,0,2)V(0,0,\ldots,0) + U(0,0,\ldots,0,1)V(0,0,\ldots,0,1) \\ &\quad + U(0,0,\ldots,0,0)V(0,0,\ldots,0,2) \\ &\quad \vdots \\ W(0,0,\ldots,k_n) &= \sum U(0,0,\ldots,k_n - r_n)\, V(0,0,\ldots,0,r_n). \end{aligned}$$

In the general form we have

$$W(k_1,k_2,...,k_n)=\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\cdots\sum_{r_n=0}^{k_n}U(r_1,\ldots,r_{n-1},k_n-r_n)\,V(k_1-r_1,k_2-r_2,\ldots,k_{n-1}-r_{n-1},r_n)\,.$$

3. Numerical example

3.1. Heat-like models

In this section, three Heat-like models from each type will be tested by using the DTM.

Example 1. We first consider the one-dimensional heat-like model

$$u_t=\frac{1}{2}x^2u_{xx},\quad 0<t<1\ ,\ t>0\ , \tag{3.1}$$

Subject to the boundary conditions

$$u(0,t)=0,\quad u(1,t)=e^t, \tag{3.2}$$

and the initial condition

$$u(x,0)=x^2\ . \tag{3.3}$$

Taking the two dimensional differential transform of (3.1), we can obtain

$$U(k,h+1)=\frac{1}{2(h+1)}\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-2,h-s)\,(k-r+2)\,(k-r+1)\,U(k-r+2,s), \tag{3.4}$$

and by applying the differential transform to boundary conditions (3.2), we have

$$U(0,h)=0,\quad U(1,h)=\frac{1}{h!}. \tag{3.5}$$

From initial condition (3.3), we can write

$$U(k,0)=\delta(k-2)=\begin{cases}1 & k=2\ ,\\ 0 & k\neq 2\ .\end{cases} \tag{3.6}$$

For each k,h, substituting equations (3.5) and (3.6) into equation (3.4) and by recursive method, the values $U(k,\ h)$ can be evaluated as:

$$U(k,h+1)=0,\quad k=0,1,3,4,5,....,\quad h=0,1,2,3,...\quad ,U(2,h)=\frac{1}{h!} \tag{3.7}$$

By using the inverse transformation rule for two dimensional in equation (2.2), the following solution can be obtained:

$$\begin{aligned}u(x,t)\ &=\ \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h=U(2,0)\,x^2t^0+U(2,1)\,x^2t^1+U(2,2)\,x^2t^2+U(2,3)\,x^2t^3\\ &\quad +U(2,4)\,x^2t^4+...+U(2,n)\,x^nt^n.\end{aligned} \tag{3.8}$$

Therefore, the closed form solution can be obtained as

$$\begin{aligned} u(x,t) &= x^2 + x^2t^1 + \tfrac{1}{2!}x^2t^2 + \tfrac{1}{3!}x^2t^3 + \tfrac{1}{4!}x^2t^4 + \tfrac{1}{5!}x^2t^5 + ... \\ &= x^2(1 + t^1 + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3 + \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + ...) = x^2\, e^t\,. \end{aligned} \tag{3.9}$$

Example 2. We consider the two-dimensional heat-like model

$$u_t = \frac{1}{2}(y^2\, u_{x\,x} + x^2\, u_{y\,y}), \quad 0 < x\ ,\ y < 1\ ,\ t > 0 \tag{3.10}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,t) = 0, &\qquad u_x(1,y,t) = 2\,\sinh t\ , \\ u_y(x,0,t) = 0, &\qquad u_y(x,1,t) = 2\,\cosh t\ , \end{aligned} \tag{3.11}$$

and the initial condition

$$u(x,y,0) = y^2. \tag{3.12}$$

Taking the three dimensional differential transform of (3.10), we can obtian:

$$\begin{aligned} U(k,h,m+1) \;=\; & \tfrac{1}{2(m+1)}\Bigg[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}\delta(r,s-2,m-l)\,(k-r+1)\,(k-r+2)\,U(k-r+2,h-s,l) \\ & + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}\delta(r-2,s,m-l)\,(h-s+1)\,(h-s+2)\,U(k-r,h-s+2,l)\Bigg]\ , \end{aligned} \tag{3.13}$$

From the boundary conditions (3.11), we can write

$$U(1,h,m) = 0, \qquad U(2,h,m) = \begin{cases} 0 & \text{for } m \text{ is even}\,, \\ \frac{2}{m!} & \text{for } m \text{ is odd}; \end{cases} \tag{3.14}$$

$$U(k,1,m) = 0, \qquad U(k,2,m) = \begin{cases} \frac{2}{m!} & \text{even}\, m, \\ 0 & \text{odd}\, m. \end{cases} \tag{3.15}$$

From the initial condition (3.12), we can write

$$U(k,h,0) = \delta(k,h-2,m) = \begin{cases} 1 & k = m = 0 \text{ and } h = 2\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

For each k, h substituting equations (3.14), (3.15) and (3.16) into equation (3.13), and by the recursive method, the value of $U(k,h)$ can be evaluated as follows:

$$\begin{aligned}
&U(k,h,m+1)=0, \quad \text{for } k=1,3,4,5,6,7,\ldots,\ h=0,1,2,3,\ldots,\ \textit{and}\ m=0,1,2,3,\ldots,\\
&U(0,h,m+1)=0, \quad \text{for } \ h=0,1,3,4,\ldots \quad \text{and} \quad m=0,1,2,3,\ldots,\\
&U(2,h,m+1)=0, \quad \text{for } \ h=1,2,3,4,\ldots \quad \text{and} \quad m=0,1,2,3,\ldots,\\
&U(0,2,m)=\begin{cases}\frac{2}{m!} & \text{even } m\\ 0 & \text{odd } m,\end{cases}\\
&U(2,0,m)=\begin{cases}0 & \text{even } m\\ \frac{2}{m!} & \text{odd } m.\end{cases}
\end{aligned} \tag{3.17}$$

By using the inverse transformation rule for three dimensional in equation (2.2), the following solution can be obtained:

$$\begin{aligned}
u(x,y,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty} x^k y^h t^m U(k,h,m) = 2y^2+y^2t^2+\tfrac{2}{4!}y^2t^4+\tfrac{2}{6!}y^2t^6+\cdots\\
&+2x^2t^1+\tfrac{2}{3!}x^2t^3+\tfrac{2}{5!}x^2t^5+\tfrac{2}{7!}x^2t^7+\cdots .
\end{aligned} \tag{3.18}$$

When we rearrange the solution , we get the following closed form solution:

$$\begin{aligned}
u(x,y,t) &= 2y^2(1+\tfrac{t^2}{2!}+\tfrac{t^4}{4!}+\tfrac{t^6}{6!}+\cdots)+2x^2(t^1+\tfrac{t^3}{3!}+\tfrac{t^5}{5!}+\tfrac{t^7}{7!}+\cdots)\\
&= 2y^2\cosh t+2x^2\sinh t\ .
\end{aligned} \tag{3.19}$$

Example 3. We consider the three-dimensional heat-like model

$$u_t = x^4y^4z^4+\frac{1}{36}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz})\ ,\quad 0<x,y,z<1,\ t>0\,, \tag{3.20}$$

subject to the boundary conditions

$$\begin{aligned}
u(0,y,z,t)=0, &\qquad u(1,y,z,t)=y^4z^4(e^t-1),\\
u(x,0,z,t)=0, &\qquad u(x,1,z,t)=x^4z^4(e^t-1),\\
u(x,y,0,t)=0, &\qquad u(x,y,1,t)=x^4y^4(e^t-1),
\end{aligned} \tag{3.21}$$

and the initial condition

$$u(x,y,z,0)=0\,. \tag{3.22}$$

Taking the four dimensional differential transform of (3.20), we can obtain:

$$\begin{aligned}
&U(k,h,n,m+1)=\tfrac{1}{m+1}\delta(k-4,h-4,n-4,m)\\
&+\tfrac{1}{36(m+1)}\Bigg[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r-2,s,\lambda,m-l)(k-r+1)(k-r+2)U(k-r+2,h-s,n-\lambda,l)\\
&+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s-2,\lambda,m-l)(h-s+1)(h-s+2)U(k-r,h-s+2,n-\lambda,l)\\
&+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s,\lambda-2,m-l)(n-\lambda+1)(n-\lambda+2)U(k-r,h-s,n-\lambda+2,l)\Bigg]\,.
\end{aligned} \tag{3.23}$$

By applying the differential transform to boundary conditions (3.21), we have

$$\begin{aligned}
&U(0,h,n,m)=0\,,\\
&U(1,h,n,m)=\left(\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s-4,\lambda-4,m-l).\tfrac{1}{(k-r+h-s+n-\lambda+l)!}\right)-\delta(r,s-4,\lambda-4,m-l)\,,\\
&U(k,0,n,m)=0\,,\\
&U(k,1,n,m)=\left(\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r-4,s,\lambda-4,m-l).\tfrac{1}{(k-r+h-s+n-\lambda+l)!}\right)-\delta(r-4,s,\lambda-4,m-l)\,,\\
&U(k,h,0,m)=0\,,\\
&U(k,h,1,m)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r-4,s-4,\lambda,m-l).\tfrac{1}{(k-r+h-s+n-\lambda+l)!}-\delta(r-4,s-4,\lambda,m-l)\,.
\end{aligned} \tag{3.24}$$

From the initial condition (3.22), we can write

$$U(k,h,n,0)=0\,. \tag{3.25}$$

For each k,h substituting equations (3.24) and (3.25) into equation (3.23) and by recursive method, the values $U(k,\ h)$ can be evaluated as follows:

$$U(k,h,n,m+1)=0\,,\quad for\ \ k=0,1,2,3,5,6,\dots\ ,\ \ h=n=m=0,1,2,... \tag{3.26}$$

$$U(4,4,4,m)=\begin{cases}0 & m=0\\ \frac{1}{m!} & m\geq 1.\end{cases} \tag{3.27}$$

By using the inverse transformation rule for four dimensional in equation (2.2), we obtain series for $u(x,y,z,t)$. When we rearrange the solution, we get the following closed form solution:

$$\begin{aligned}
u(x,y,z,t) &= x^4y^4z^4\left(t+\tfrac{t^2}{2!}+\tfrac{t^3}{3!}+\tfrac{t^4}{4!}+...\right)\\
&= x^4y^4z^4(e^t-1)\,.
\end{aligned} \tag{3.28}$$

3.2. Wave-like models

In what follows we illustrate our analysis by examining the following three wave-like equations.

Example 4. Consider the one-dimensional wave-like model

$$u_{tt}=\frac{1}{2}x^2u_{xx}\,,\qquad 0<x<1\,,\ t>0\,, \tag{3.29}$$

subject to the boundary conditions

$$u(0,t)=0,\qquad u(1,t)=1+\sinh t, \tag{3.30}$$

and initial conditions

$$u(x,0) = x, \qquad u_t(x,0) = x^2 \,. \tag{3.31}$$

Taking the one dimensional differential transform, we can obtain

$$U(k,h+2) = \frac{1}{2(h+1)(h+2)} \left(\sum_{r=0}^{k} \sum_{s=0}^{h} \delta(r-2,h-s)\,(k-r+2)\,(k-r+1)\,U(k-r+2,s) \right), \tag{3.32}$$

then by applying the differential transform to boundary conditions (3.30), we have

$$U(0,h) = 0, \quad U(1,h) = \delta(k)\,\delta(h) \, + \begin{cases} 0 & \text{for h is even}\,, \\ \frac{1}{h!} & \text{for } h \text{ is odd}\,. \end{cases} \tag{3.33}$$

Applying the differential transform to initial conditions (3.31), we have

$$U(k,0) = \delta(k-1) = \begin{cases} 1 & k = 1\,, \\ 0 & k \neq 1\,. \end{cases} \qquad U(k,1) = \delta(k-2) = \begin{cases} 1 & k = 2\,, \\ 0 & k \neq 2\,. \end{cases} \tag{3.34}$$

For each k, h, substituting equation (3.34) into equation (3.32), and by recursive method, $U(k,h)$ can be evaluated as follows:

$$\begin{aligned} U(1,h) &= \begin{cases} 1 & \text{for } h = 0 \\ 0 & \text{for } h = 1,2,3,... \end{cases} \\ U(2,h) &= \begin{cases} 0 & \text{for even } h \\ \frac{1}{h!} & \text{for odd } h, \end{cases} \\ U(k,h) &= 0, \;\; \text{for } k = 3,4,5,..., \;\; h = 0,1,2,3,\ldots\,. \end{aligned} \tag{3.35}$$

Using the inverse transformation rule, we obtain a series for $u(x,t)$. Rearranging the solution, we get the following closed form solution:

$$\begin{aligned} u(x,t) &= x + x^2 (t + \tfrac{t^3}{3!} + \tfrac{t^5}{5!} + \tfrac{t^7}{7!} + \cdots) \\ &= x + x^2 (\sinh t)\,. \end{aligned} \tag{3.36}$$

Example 5. We next consider the two-dimensional wave-like model

$$u_{tt} = \frac{1}{12}(x^2\, u_{xx} + y^2\, u_{yy}), \quad 0 < x\,,\; y < 1\,,\; t > 0\,, \tag{3.37}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,t) &= 0, \qquad u_x(1,y,t) = 4\cosh ht\,, \\ u_y(x,0,t) &= 0, \qquad u_y(x,1,t) = 4\sinh ht\,, \end{aligned} \tag{3.38}$$

and the initial conditions

$$u(x,y,0) = x^4\,, \qquad u_t(x,y,0) = y^4\,. \tag{3.39}$$

Taking the three dimensional differential transform of (3.37), we can obtain

$$U(k,h,m+2) = \frac{1}{12(m+1)(m+2)} \times \left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m} \delta(r-2,s,m-l)\,(k-r+1)\,(k-r+2)U(k-r+2,h-s,l) + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m} \delta(r,s-2,m-l)\,(h-s+1)\,(h-s+2)\,U(k-r,h-s+2,l) \right], \tag{3.40}$$

From the boundary conditions (3.38), we can write

$$U(1,h,m) = 0$$
$$U(2,h,m) = \begin{cases} \frac{4}{m!} & \text{for even } m, \\ 0 & \text{for odd } m, \end{cases} \tag{3.41}$$

$$U(k,1,m) = 0$$
$$U(k,2,m) = \begin{cases} 0 & \text{for even } m \\ \frac{4}{m!} & \text{for odd } m. \end{cases} \tag{3.42}$$

From the initial condition (3.39), we can write

$$U(k,h,0) = \delta(k-4,h,m) = \begin{cases} 1 & h=m=0 \text{ and } k=4, \\ 0 & \text{otherwise.} \end{cases}$$
$$U(k,h,1) = \delta(k,h-4,m) = \begin{cases} 1 & k=m=0 \text{ and } h=4, \\ 0 & \text{otherwise.} \end{cases} \tag{3.43}$$

For each k, h, substituting equation (3.43) into equation (3.40), and by the method of recursion, the value $U(k,h)$ can be evaluated as:

$$U(k,h,m+1) = 0, \text{for } k = 1,2,3,\ldots,\ h,m = 0,1,2,3,\ldots,$$
$$U(0,4,m) = \begin{cases} 0 & \text{for even } m \\ \frac{1}{m!} & \text{for odd } m, \end{cases} \tag{3.44}$$
$$U(4,0,m) = \begin{cases} \frac{1}{m!} & \text{for even } m \geq 2 \\ 0 & \text{for } m=0 \text{ and odd } m. \end{cases}$$

By using the inverse transformation rule, the following solution can be obtained:

$$u(x,y,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty} x^k y^h t^m U(k,h,m) = y^4 t + \frac{1}{3!}y^4t^3 + \frac{1}{5!}y^4t^5 + \frac{1}{7!}y^4t^7 + \cdots + x^4 + \frac{1}{2!}x^4t^2 + \frac{1}{4!}x^4t^4 + \frac{1}{6!}x^4t^6 + \cdots \tag{3.45}$$

When we rearrange the solution, we get the following closed form solution:

$$u(x,y,t) = x^4 \cosh t + y^4 \sinh t\ . \tag{3.45}$$

Example 6. We consider the three-dimensional wave-like model

$$u_{tt} = \left(x^2 + y^2 + z^2\right) + \frac{1}{2}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz})\,, \quad 0 < x, y, z < 1,\ t > 0\,, \tag{3.46}$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,z,t) &= y^2(e^t-1) + z^2(e^{-t}-1), & u(1,y,z,t) &= \left(1+y^2\right)(e^t-1) + z^2(e^{-t}-1),\\ u(x,0,z,t) &= x^2(e^t-1) + z^2(e^{-t}-1), & u(x,1,z,t) &= \left(1+x^2\right)(e^t-1) + z^2(e^{-t}-1),\\ u(x,y,0,t) &= \left(x^2+y^2\right)(e^t-1), & u(x,y,1,t) &= \left(x^2+y^2\right)(e^t-1) + (e^{-t}-1), \end{aligned} \tag{3.47}$$

and the initial conditions

$$u(x,y,z,0) = 0\,, \qquad u_t(x,y,z,0) = x^2 + y^2 - z^2. \tag{3.48}$$

Taking the four dimensional differential transform of (3.46), we can obtain:

$$U(k,h,n,m+2) = \tfrac{1}{(m+1)(m+2)}\left[\delta(k-2,h,n,m) + \delta(k,h-2,n,m) + \delta(k,h,n-2,m)\right]$$

$$+\tfrac{1}{2(m+1)(m+2)}\left[\begin{array}{l} \sum\limits_{r=0}^{k}\sum\limits_{s=0}^{h}\sum\limits_{\lambda=0}^{n}\sum\limits_{l=0}^{m}\delta(r-2,s,\lambda,m-l)(k-r+1)(k-r+2)U(k-r+2,h-s,n-\lambda,l)\\ +\sum\limits_{r=0}^{k}\sum\limits_{s=0}^{h}\sum\limits_{\lambda=0}^{n}\sum\limits_{l=0}^{m}\delta(r,s-2,\lambda,m-l)(h-s+1)(h-s+2)U(k-r,h-s+2,n-\lambda,l)\\ +\sum\limits_{r=0}^{k}\sum\limits_{s=0}^{h}\sum\limits_{\lambda=0}^{n}\sum\limits_{l=0}^{m}\delta(r,s,\lambda-2,m-l)(n-\lambda+1)(n-\lambda+2)U(k-r,h-s,n-\lambda+2,l) \end{array}\right] \tag{3.49}$$

By applying the differential transform to boundary conditions (3.47), we get

$$\begin{aligned} U(0,h,n,m) \;=\;& \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s-2,\lambda,m-l)\cdot\tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h-2,n,m)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s,\lambda-2,m-l)\cdot\tfrac{(-1)^{k-r+h-s+n-\lambda+l}}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h,n-2,m), \end{aligned}$$

$$\begin{aligned} U(1,h,n,m) \;=\;& \tfrac{1}{m!} - \delta(k,h,n,m) + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s-2,\lambda,m-l)\cdot\tfrac{1}{(k-r+h-s+n-\lambda+l)!}\\ &-\delta(k,h-2,n,m) + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s,\lambda-2,m-l)\cdot\tfrac{(-1)^{k-r+h-s+n-\lambda+l}}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h,n-2,m), \end{aligned}$$

$$\begin{aligned} U(k,0,n,m) \;=\;& \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r-2,s,\lambda,m-l)\cdot\tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k-2,h,n,m)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s,\lambda-2,m-l)\cdot\tfrac{(-1)^{k-r+h-s+n-\lambda+l}}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h,n-2,m), \end{aligned}$$

$$\begin{aligned} U(k,0,n,m) \;=\;& \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r-2,s,\lambda,m-l)\cdot\tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k-2,h,n,m)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m}\delta(r,s,\lambda-2,m-l)\cdot\tfrac{(-1)^{k-r+h-s+n-\lambda+l}}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h,n-2,m), \end{aligned}$$

$$
\begin{aligned}
U(k,1,n,m) &= \tfrac{1}{m!} - \delta(k,h,n,m) + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r-2,s,\lambda,m-l)\cdot \tfrac{1}{(k-r+h-s+n-\lambda+l)!} \\
&\quad + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r,s,\lambda-2,m-l)\cdot \tfrac{(-1)^{k-r+h-s+n-\lambda+l}}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h,n-2,m),
\end{aligned}
$$

$$
\begin{aligned}
U(k,h,0,m) &= \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r-2,s,\lambda,m-l)\cdot \tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k-2,h,n,m) \\
&\quad + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r,s-2,\lambda,m-l)\cdot \tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h-2,n,m),
\end{aligned}
$$

$$
\begin{aligned}
U(k,h,1,m) &= \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r-2,s,\lambda,m-l)\cdot \tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k-2,h,n,m) \\
&\quad + \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{\lambda=0}^{n}\sum_{l=0}^{m} \delta(r,s-2,\lambda,m-l)\cdot \tfrac{1}{(k-r+h-s+n-\lambda+l)!} - \delta(k,h-2,n,m) \\
&\quad + \tfrac{(-1)^m}{m!} - \delta(k,h,n,m).
\end{aligned}
\tag{3.50}
$$

From the initial condition (3.48), we can write

$$
\begin{aligned}
&U(k,h,n,0) = 0, \\
&U(k,h,n,1) = \delta(k-2,h,n,m) + \delta(k,h-2,n,m) - \delta(k,h,n-2,m) \\
&= \begin{cases} 1 & k=2, h=n=m=0 \\ 0 & \text{otherwise} \end{cases} \\
&+ \begin{cases} 1 & h=2,\ \ k=n=m=0 \\ 0 & \text{otherwise} \end{cases} \\
&+ \begin{cases} 1 & n=2,\ \ k=h=m=0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{3.51}
$$

For each k, h, substituting equations (3.50) and (3.51) into equation (3.49), and via the recursive method, the values $U(k,h)$ can be evaluated as follows:

$$
U(k,h,n,m+1) = 0\,, \quad for\ \ k = 1,3,4,5,6,\dots\ ,\ \ h = n = m = 0,1,2,\dots
$$

$$
\begin{aligned}
U(0,0,2,m) &= \begin{cases} 0 & m=0 \\ \frac{(-1)^m}{m!} & m \geq 1 \end{cases} \\
U(0,2,0,m) &= \begin{cases} 0 & m=0 \\ \frac{1}{m!} & m \geq 1 \end{cases} \\
U(2,0,0,m) &= \begin{cases} 0 & m=0 \\ \frac{1}{m!} & m \geq 1. \end{cases}
\end{aligned}
\tag{3.52}
$$

Using the inverse transformation rule for four dimensional in equation (2.2), we obtain a series for $u(x,y,z,t)$.

On rearranging the solution, we get the following closed form solution:

$$\begin{aligned} u(x,y,z,t) &= z^2(-t+\tfrac{t^2}{2!}-\tfrac{t^3}{3!}+\tfrac{t^4}{4!}-\tfrac{t^5}{5!}\cdots)+y^2(t+\tfrac{t^2}{2!}+\tfrac{t^3}{3!}+\tfrac{t^4}{4!}+\tfrac{t^5}{5!}\cdots)+x^2(t+\tfrac{t^2}{2!}+\tfrac{t^3}{3!}+\tfrac{t^4}{4!}+\tfrac{t^5}{5!}\cdots) \\ &= z^2(e^{-t}-1)+(y^2+x^2)(e^t-1)\ . \end{aligned} \tag{3.53}$$

That is,

$$u(x,y,z,t)=z^2\sum_{j=1}\frac{(-t)^j}{j!}+y^2\sum_{j=1}\frac{t^j}{j!}+x^2\sum_{j=1}\frac{t^j}{j!}=z^2(e^{-t}-1)+(y^2+x^2)(e^t-1)\ . \tag{3.54}$$

4. Conclusion

In this study, the differential transform method is successfully expanded for the solution of heat-like and wave-like equations. In the first three examples various kinds of heat-like equations. The final two problems were considered for various kinds of wave-like equations. Since the Differential Transform Method (DTM) gives rapidly converging series solutions, the differential transform method is more effective than other methods. The accuracy of the obtained solution can be improved by taking more terms in the solution. Exact closed form solution is obtained for all examples presented in this paper.

References

[1] A. M. Wazwaz, A. Gorguis, *Appl. Math. Comput.*, **149**, (2004), 15.

[2] Da Hua Shou, Ji Huan He, *Physics Letters A*, **372**, (2008), 233.

[3] Sh. Momani, *Appl. Math. Comput.*, **165**, (2005), 459.

[4] N. Bildik, A. Konuralp, *Int. Journal of Comput.Math.*, **83**, (2006), 973.

[5] M. J. Jang, C. L. Chen, Y. C. Liu, *Appl. Math. Comput.*, **121**, (2001), 261.

[6] V. S. Ertürk, S. H. Momeni, *Int. Journal. Math. Manuscript*, **1**, (2007), 65.

[7] J. K. Zhou, *Differential Transform and its Applications for Electrical Circuits,* (Huazhong University Press, Wuhan. 1986).

[8] M. J. Jang, C. L. Chen, Y. C. Liu, *Appl. Math. Comput.*, **121**, (2001), 261.

[9] F. Ayaz, *Appl. Math. Comput.*, **143**, (2003), 361.

[10] F. Ayaz, *Appl. Math. Comput.*, **147**, (2004), 547.