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Companion sequences associated to the r-Fibonacci sequence: algebraic and combinatorial properties

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Abstract: It is well known that the companion sequence of the Fibonacci sequence is Lucas's sequence. For the generalized Fibonacci sequences, the companion sequence is not unique. Several authors proposed different definitions, and they are in a certain sense all good. Our purpose is to introduce a family of companion sequences for some generalized Fibonacci sequence: the r-Fibonacci sequence. We evaluate the generating functions and give some applications, and we exhibit convolution relations that generalize some known identities such as Cassini's. Afterwards, we calculate the sums of their terms using matrix methods. Next, we propose a q-analogue and extend the definition to negative ns. Also, we define the incomplete associated sequences using a Euler–Seidel-like approach.

Key words: r-Fibonacci sequence, companion sequences, recurrence relation, convolution, hyper-r-Lucas polynomial, incomplete r-Lucas polynomial, q-analogues

1. Introduction

This work was intended as an attempt to introduce the family $\{(V_n^{(r,s)})_{n\geq 0}, 1 \leq s \leq r\}$ of r companion sequences associated to the generalized r-Fibonacci polynomial sequence. According to the parameter s, each sequence of this family satisfies the same recurrence relation of order $(r+1)$, with the initial term $V_0 = s+1$ $(s = 1, \dots, r)$. The classical Fibonacci $(U_n) = (U_n(x,y))$ and Lucas polynomials $(V_n(x,y)) = (V_n)$ for $n \geq 2$ and x, y variables are given respectively by

$$\begin{cases} U_0 = 0, U_1 = 1, U_n = xU_{n-1} + yU_{n-2}, \\ V_0 = 2, V_1 = x, V_n = xV_{n-1} + yV_{n-2}. \end{cases}$$

In what follows, the sequences we deal with are sequences of bivariate polynomials of r-Fibonacci and r-Lucas. For convenience we will use r-Fibonacci polynomials and r-Lucas polynomials. There are some particular cases of these sequences and we provide a few of these: Fibonacci $(F_n)_n$, Pell $(P_n)_n$, and Jacobsthal $(J_n)_n$ and their companion sequences Lucas $(L_n)_n$, Pell-Lucas $(Q_n)_n$, and Jacobsthal-Lucas $(j_n)_n$, respectively, $(F_n, L_n) =$

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$(U_n(1,1), V_n(1,1))$, $(P_n, Q_n) = (U_n(2,1), V_n(2,1))$, and $(J_n, j_n) = (U_n(1,2), V_n(1,2))$. For combinatorial and arithmetic properties see, for instance, [5, 6].

Some well-known generalizations of Fibonacci numbers can be found in the following:
Dickinson [13], $S_m = S_{m-c+a} + S_{m-c}$ (a, c are positive integers);
Miles [21], $f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k}$ ($k \geq 2$ integer);
Raab [23], $U_n = aU_{n-1} + bU_{n-r-1}$ (a, b real numbers, r integer ≥ 1).

Definition 1 *For any integer $r \geq 1$, the bivariate r-Fibonacci polynomial $(U_n^{(r)}(x,y))_n$, denoted briefly as $(U_n^{(r)})_n$, is defined by the following recurrence relation*

$$\begin{cases} U_0^{(r)} = 0, \ U_k^{(r)} = x^{k-1} \ (1 \leq k \leq r), \\ U_{n+1}^{(r)} = xU_n^{(r)} + yU_{n-r}^{(r)} \ (n \geq r). \end{cases}$$

For $n \geq 0$, we have (see [23])

$$U_{n+1}^{(r)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k} x^{n-(r+1)k} y^k. \tag{1.1}$$

For $x = y = 1$, the sequence $(U_n^{(r)}(x,y))_n$ is reduced to the Fibonacci p-numbers. Properties of these numbers have been studied by several authors; for more details, see [18] and the references therein.

In Section 2, we establish an explicit formulation of $V_n^{(r,s)}$ and we give its companion matrix. Afterwards, we produce the generating function. Section 3 is devoted to some applications: convolution relations and sums of their general terms using the matrix method. Section 4 suggests the q-analogue of each $V_n^{(r,s)}$, $s = 1, \ldots, r$. Section 5 is devoted to a combinatorial interpretation for r-Fibonacci numbers and their companion sequences. In Section 6, we extend $V_n^{(r,s)}$ to negative ns. Finally, in Section 7, we introduce the incomplete r-Lucas polynomials and the hyper r-Lucas polynomials.

2. The companion sequences associated to the r-Fibonacci polynomial sequence

In this section, we define the companion sequences family related to the r-Fibonacci sequence, and then we give an explicit formulation expressing its general term.

Definition 2 *For any integers n, r, and s $(1 \leq s \leq r)$, the companion sequence family of $(U_n^{(r)})_n$ is defined by the following recurrence:*

$$\begin{cases} V_0^{(r,s)} = s+1, \ V_k^{(r,s)} = x^k \ (1 \leq k \leq r), \\ V_{n+1}^{(r,s)} = xV_n^{(r,s)} + yV_{n-r}^{(r,s)} \ (n \geq r). \end{cases} \tag{2.1}$$

The sequence $(V_n^{(r,s)})$ is called an r*-Lucas polynomial of type* s.

Remark 1 *Note that when $s = 0$, we get the shifted r-Fibonacci polynomial.*

According to [7], for $s=1$, we name $(V_n^{(r,1)})$ the generalized r-Lucas polynomial of the second kind, and for $s=r$, we name $(V_n^{(r,r)})$ the generalized r-Lucas polynomial of the first kind.

The following theorem gives us an explicit formulation for $V_n^{(r,s)}$ in terms of s and $U_n^{(r)}$.

Theorem 1 *Let r and s be nonnegative integers such that $1 \leq s \leq r$, and x, y are elements of an unitary ring $\mathcal{A}$. We suppose that y is reversible in $\mathcal{A}$. We have for $n \geq r$ the following:*

$$V_n^{(r,s)} = U_{n+1}^{(r)} + syU_{n-r}^{(r)}, \tag{2.2}$$

We also get the explicit form for $n \geq 1$:

$$V_n^{(r,s)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \frac{n-(r-s)k}{n-rk} \binom{n-rk}{k} x^{n-(r+1)k} y^k. \tag{2.3}$$

In [4], Belbachir and Bencherif gave a formula expressing the general term of a linear recurrent sequence.

Lemma 1 *[4] Let $(u_n)_{n>-m}$ be a sequence of elements of $\mathcal{A}$, defined by*

$$\begin{cases} u_{-j} = \alpha_j \quad (0 \leq j \leq m-1), \\ u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} \ (n \geq 1). \end{cases} \tag{2.4}$$

Let $(\lambda_j)_{0 \leq j \leq m-1}$ and $(y_n)_{n>-m}$ be the sequences of elements of $\mathcal{A}$ defined by $\lambda_j = -\sum_{k=0}^{m-1-j} a_k \alpha_{k+j}$ with $a_0 = -1$ and $y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$. Then, for $n > -m$, we have

$$u_n = \lambda_0 y_n + \lambda_1 y_{n+1} + \cdots + \lambda_{m-1} y_{n+m-1}. \tag{2.5}$$

Proof [of Theorem 1] We consider the sequence $(V_n^{(r,s)})_{n\geq 0}$ given by (2.1), which corresponds in (2.4) to $a_1 = x, a_{r+1} = y$ and $a_2 = a_3 = \cdots = a_r = 0$. Observing that, for $0 \leq j \leq r$, $u_{-j} = \alpha_j = y^{-1}(U_{r-j+1} - xU_{r-j})$, we get $\alpha_0 = s+1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{r-1} = 0$, and $\alpha_r = -sxy^{-1}$. Consequently, $(\lambda_j)_{0\leq j\leq r}$ is defined by $\lambda_0 = s+1$ and $\lambda_j = -\sum_{k=j}^{r} a_{k-j}\alpha_k = -a_{r-j}\alpha_r$ for $1 \leq j \leq r$, with $a_0 = -1$, and then $\lambda_1 = \lambda_2 = \cdots = \lambda_{r-2} = 0$, $\lambda_{r-1} = sx^2y^{-1}$, and $\lambda_r = -sxy^{-1}$. Finally, we get for $n \geq 0$, $y_n = \sum_{k_1+(r+1)k_{r+1}=n} \binom{k_1+k_{r+1}}{k_1,k_{r+1}} x^{k_1} y^{k_{r+1}} = \sum_{k\geq 0} \binom{n-rk}{k} x^{n-(r+1)k} y^k = U_{n+1}^{(r)}$. Applying formula (2.5), we obtain the expression of $V_n^{(r,s)}$ in terms of $s, x, y, \lambda_0, \ldots, \lambda_r$, and $U_n^{(r)}$. We have for any integer $n \geq r$ the following:

$$\begin{aligned} V_n^{(r,s)} &= \lambda_0 U_{n+1}^{(r)} + \lambda_1 U_{n+2}^{(r)} + \cdots + \lambda_r U_{n+r+1}^{(r)} = \lambda_0 U_{n+1}^{(r)} + \lambda_{r-1} U_{n+r}^{(r)} + \lambda_r U_{n+r+1}^{(r)} \\ &= (s+1)U_{n+1}^{(r)} + sxy^{-1}(xU_{n+r}^{(r)} - U_{n+r+1}^{(r)}) = (s+1)U_{n+1}^{(r)} - sxU_n^{(r)} = syU_{n-r}^{(r)} + U_{n+1}^{(r)}. \end{aligned}$$

Then

$$V_n^{(r,s)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \left(1 + s\frac{k}{n-rk}\right) \binom{n-rk}{k} x^{n-(r+1)k} y^k = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \frac{n-(r-s)k}{n-rk} \binom{n-rk}{k} x^{n-(r+1)k} y^k. \qquad \square$$

In the following theorem we give the generating function of the r-Lucas polynomial of type s.

Theorem 2 *We suppose that $\mathcal{A}=\mathbb{R}$ or $\mathbb{C}$ for $z\in\mathbb{C}$, the generating function of the sequence $(V_n^{(r,s)})_{n\geq 0}$, is given by*

$$V(z)=\sum_{n\geq 0}V_n^{(r,s)}z^n=\frac{(1+s)-sxz}{1-xz-yz^{r+1}}. \tag{2.6}$$

Proof We have

$$V(z) = \sum_{n\geq 0}{V_n}^{(r,s)}z^n=\sum_{n\geq 0}({U_{n+1}}^{(r)}+sy{U_{n-r}}^{(r)})z^n=U(z)+s\sum_{n\geq 0}(U_{n+1}-xU_n)z^n$$

$=(1+s)U(z)-sxzU(z)$, where $U(z)=\frac{1}{1-xz-yz^{r+1}}$, then $V(z)=\frac{(1+s)-sxz}{1-xz-yz^{r+1}}$. □

Let $\alpha_1,\alpha_2,\ldots,\alpha_{r+1}$ be the roots of the characteristic polynomial $P(t)=t^{r+1}-xt^r-y$ associated with $(U_n^{(r)})_{n\geq 0}$ and $(V_n^{(r,s)})_{n\geq 0}$ such that $y\neq(-1/r)\left(rx/r+1\right)^{r+1}$. Then for integers $1\leq s\leq r$, we have the Binet formulae:

$$U_{n+1}^{(r)}=\sum_{k=1}^{r+1}\frac{\alpha_k^{n+1}}{(r+1)\alpha_k-rx}\quad\text{and}\quad V_n^{(r,s)}=\sum_{k=1}^{r+1}\alpha_k^n\frac{(s+1)\alpha_k-sx}{(r+1)\alpha_k-rx}.$$

Remark 2 *The companion matrix of order $(r+1)$ associated to $(V_n^{(r,s)})_n$ and its n-powers are $A_r(x,y):=$ $\begin{pmatrix} 0 & 0 & \cdots & 0 & y\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1 & x\end{pmatrix}$ and $A_r^n(x,y)=\begin{pmatrix} yU_{n-r}^{(r)} & \cdots & yU_{n-1}^{(r)} & yU_n^{(r)}\\ yU_{n-r-1}^{(r)} & \cdots & yU_{n-2}^{(r)} & yU_{n-1}^{(r)}\\ \vdots & \vdots & \ddots & \vdots\\ yU_{n-2r+1}^{(r)} & \cdots & yU_{n-r}^{(r)} & yU_{n-r+1}^{(r)}\\ U_{n-r+1}^{(r)} & \cdots & U_n^{(r)} & U_{n+1}^{(r)}\end{pmatrix}$. Also, there is a determinantal representation of the terms of r-Fibonacci sequences of the form $U_{(r+1)n+r}^{(r)}$. Kiliç and Arikan presented an approach in [19] to evaluate Hessenberg determinants. They evaluated the Hessenberg determinant whose entries consist of the terms of the sequence $\binom{n+m-1}{m}$. Setting $m=r$, the value of this determinant is equal to $\sum_{m=0}^{n}\binom{(r+1)n+r(1-k)}{k}=U_{(r+1)n+r}^{(r)}$.*

3. Applications

3.1. Some convoluted relations

We give some convoluted relations according to r-Fibonacci and r-Lucas polynomials.

Theorem 3 *Let $(U_n^{(r)})_{n\geq 0}$ and $(V_n^{(r,s)})_{n\geq 0}$ be respectively the r-Fibonacci polynomial and the generalized r-Lucas polynomials of type s. Then, for integers $n,m\geq r$, we have*

$$y\sum_{j=1}^{r}U_{n-j}^{(r)}U_{m+j}^{(r)}=U_{n+m+r}^{(r)}-U_n^{(r)}U_{m+r+1}^{(r)}\quad\text{and}\quad y\sum_{j=1}^{r}U_{m-r+j}^{(r)}V_{n-j}^{(r,s)}=V_{n+m}^{(r,s)}-U_{m+1}^{(r)}V_n^{(r,s)}.$$

Proof We have $A_r^{n+m}(x,y)=A_r^n(x,y)\times A_r^m(x,y)$. Consequently, an element of $A_r^{n+m}(x,y)$ is the product of a row of $A_r^n(x,y)$ and a column of $A_r^m(x,y)$. □

For example, for $(x, y) = (1, 1)$ and $r = 1$, Theorem 3 gives Cassini's identities $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$ and $L_{n+m} = F_mL_{n-1} + F_{m+1}L_n$.

Corollary 1 *For all integers $n \geq 0$, we have*

$$U_{2n}^{(r)} = 2y \sum_{j=0}^{\lfloor r/2 \rfloor} U_{n-j}^{(r)} U_{n-r+j}^{(r)} + x^{2(r/2-\lfloor r/2 \rfloor)} U_n^{(r)} U_{n+1-2(r/2-\lfloor r/2 \rfloor)}^{(r)}, \tag{3.1}$$

and

$$U_{2n+1}^{(r)} = 2y \sum_{j=0}^{\lfloor r/2 \rfloor} U_{n+1-j}^{(r)} U_{n-r+j}^{(r)} + (U_{n+1}^{(r)})^2 + 2y(r/2 - \lfloor r/2 \rfloor)(U_{n-(r-1)/2}^{(r)})^2. \tag{3.2}$$

For $(x, y) = (1, 1)$ and $r = 1$, we obtain the known identities for Fibonacci sequences given in [20], $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ and $F_{2n+1} = F_{n+1}^2 + F_n^2$.

3.2. Sums of finite terms of r-Fibonacci polynomials and the related companion sequences

In this part, we give an explicit formula for the sums of the terms of the r-Fibonacci polynomial and its companion sequences using a matrix approach, which was used by Kiliç in [17]. Let the sums $S_n^{(r)}$ of r-Fibonacci polynomials and $S_n^{(r,s)}$ of r-Lucas polynomials be defined by

$$S_n^{(r)} := \sum_{j=1}^{n} U_j^{(r)} \text{ and } S_n^{(r,s)} := \sum_{j=1}^{n} V_j^{(r,s)}.$$

We extend the matrix representation of $(U_n^{(r)})$ and we define the generating matrix of the sum of r-Fibonacci polynomials.

Let $T_r(x, y)$ and $R_n(x, y)$ be square matrices of order $(r + 2)$ defined by

$$T_r(x, y) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x & 0 & \cdots & 0 & y \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

and

$$R_n(x, y) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ S_n & U_{n+1}^{(r)} & yU_{n-r+1}^{(r)} & \cdots & yU_{n-1}^{(r)} & yU_n^{(r)} \\ S_{n-1} & U_n^{(r)} & yU_{n-r}^{(r)} & \cdots & yU_{n-2}^{(r)} & yU_{n-1}^{(r)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{n-r+1} & U_{n-r+2}^{(r)} & \ddots & \cdots & yU_{n-r}^{(r)} & yU_{n-r+1}^{(r)} \\ S_{n-r} & yU_{n-r+1}^{(r)} & yU_{n-2r+1}^{(r)} & \cdots & yU_{n-r-1}^{(r)} & yU_{n-r}^{(r)} \end{pmatrix}.$$

We define an $(r+2)\times(r+2)$ matrix $G_r(x,y)$ as follows:

$$G_n(x,y) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{1-x-y} & \alpha_1^r & \alpha_2^r & \cdots & \alpha_r^r & \alpha_{r+1}^r \\ \frac{1}{1-x-y} & \alpha_1^{r-1} & \alpha_2^{r-1} & \cdots & \alpha_r^{r-1} & \alpha_{r+1}^{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{1-x-y} & \alpha_1 & \ddots & \vdots & \alpha_r & \alpha_{r+1} \\ \frac{1}{1-x-y} & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \tag{3.3}$$

where $1,\alpha_1,\alpha_2,\dots,\alpha_{r+1}$ are the eigenvalues of $T_r(x,y)$, which are all distinct when we suppose that the discriminant of the corresponding characteristic polynomial of the matrix $A_r(x,y)$ is different from zero.

Theorem 4 *Let $S_n^{(r)}$ be the sum of the terms of r-Fibonacci polynomials from 1 to n, and $P(t)=t^{r+1}-xt^r-y$ the characteristic polynomial such that $P(1)\neq 0$. Then*

$$S_n^{(r)} = \frac{1}{1-x-y}(1-U_{n+1}^{(r)} - y\sum_{j=1}^{r} U_{n-r+j}^{(r)}). \tag{3.4}$$

Proof We have

$$T_r(x,y)\times G_r(x,y) = G_r(x,y)\times M, \tag{3.5}$$

where $M = Diag(1,\alpha_1,\alpha_2,\dots,\alpha_{r+1})$, and by the definition of the Vandermonde matrix $det(G_r(x,y)) = \prod_{j,k}(\alpha_j-\alpha_k)\neq 0$ for $j\neq k$ (so $G_r(x,y)$ is invertible), so we can write identity (3.5) as follows: $G_r^{-1}(x,y)\times T_r(x,y)\times G_r(x,y) = M$. Then $G_r^{-1}(x,y)\times T_r^n(x,y)\times G_r(x,y) = M^n$, since $T_r^n(x,y) = R_n(x,y)$, and equating the corresponding entries, the identity is realized. □

Now, we deduce the expression for the sum of the terms of $(V_n^{(r,s)})$.

Theorem 5 *Let $S_n^{(r,s)}$ be the sum of the terms of $(V_n^{(r,s)})$ from 1 to n, and $P(t)=t^{r+1}-xt^r-y$ the corresponding characteristic polynomial such that $P(1)\neq 0$. Then*

$$S_n^{(r,s)} = \frac{1}{1-x-y}(1+sy-V_{n+1}^{(r,s)} - y\sum\nolimits_{j=1}^{r} V_{n-r+j}^{(r,s)}) - 1. \tag{3.6}$$

Proof First, we calculate $\sum_{j=r}^{n} V_j^{(r,s)}$, and using relation (3.4) and characterization (2.2) of the sequence $(V_n^{(r,s)})$, we have $\sum_{j=r}^{n} V_j^{(r,s)} = \sum_{j=r}^{n}(U_{j+1}^{(r)} + syU_{j-r}^{(r)}) = \sum_{j=r+1}^{n+1} U_j^{(r)} + sy\sum_{j=1}^{n-r} U_j^{(r)} = S_{n+1}^{(r)} - \sum_{j=1}^{r} U_j^{(r)} + syS_{n-r}^{(r)}$, since $\sum_{j=1}^{r} U_j^{(r)} = \sum_{j=1}^{r} x^{j-1} = \sum_{j=0}^{r-1} x^j = 1+\sum_{j=1}^{r-1} V_j^{(r,s)}$, and then the complete sum is evaluated. □

4. The q-analogue of the sequence $(V_n^{(r,s)})$

In this section, we propose a q-analogue of the r-Lucas polynomials of type s, inspired by the explicit formula of the sequence $(V_n^{(r,s)})_{n\geq 0}$ given by relation (2.2) in Theorem 1. First, we give some notations. Let $q\in\mathbb{R}$, $[n]_q := 1+q+\cdots+q^{n-1}$ and $[n]_q! := [1]_q[2]_q\cdots[n]_q$. We have $[n]_q = [k]_q + q^k[n-k]_q = q^{n-k}[k]_q + [n-k]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

In an unpublished work, Belbachir et al. gave a generalized q-analogue of r-Fibonacci polynomial $\mathbf{U}_{n+1}^{(r)}(z,m)$, which is a unified approach of Carlitz and Cigler [12]. They defined

$$\mathbf{U}_{n+1}^{(r)}(z,m) := \sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_q z^k, \tag{4.1}$$

with $\mathbf{U}_0^{(r)}(z,m)=0$. These polynomials satisfy the following recurrences:

$$\mathbf{U}_{n+1}^{(r)}(z,m) = \mathbf{U}_n^{(r)}(qz,m) + qz\mathbf{U}_{n-r}^{(r)}(zq^{m+1},m), \tag{4.2}$$

and

$$\mathbf{U}_{n+1}^{(r)}(z,m) = \mathbf{U}_n^{(r)}(z,m) + q^{n-r}z\mathbf{U}_{n-r}^{(r)}(zq^{m-r},m). \tag{4.3}$$

Definition 3 *For nonnegative integers r,s such that $1 \le s \le r$, the q-analogue of the r-Lucas polynomials of type s of the first kind and second kind, respectively, are defined, for $n \ge 0$, by $\mathbf{V}_0^{(r,s)}(z,m) = \mathbb{V}_0^{(r,s)}(z,m) = s+1$ and*

$$\mathbf{V}_n^{(r,s)}(z,m) := \sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_q (1+s\frac{[k]_q}{[n-rk]_q})z^k, \tag{4.4}$$

and

$$\mathbb{V}_n^{(r,s)}(z,m) := \sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_q (1+sq^{(n-(r+1)k)}\frac{[k]_q}{[n-rk]_q})z^k. \tag{4.5}$$

Some specializations follow.

For $s=r=1$, we obtain the q-Lucas polynomials of the first kind and the q-Lucas polynomials of the second kind given in [7].

For $s=1$, we obtain the q-analogue of the r-Lucas polynomials of the first kind and the q-analogue of the r-Lucas polynomials of the second kind defined in an unpublished work.

Now we establish some links with the initial r-Fibonacci polynomial.

Theorem 6 *For nonnegative integers r,s, the polynomials $\mathbf{V}_n^{(r,s)}(z,m)$ and $\mathbb{V}_n^{(r,s)}(z,m)$ satisfy the following recurrences:*

1. *Expression of $V_n^{(r,s)}$ s in terms of $\mathbf{U}_{n+1}^{(r)}$ and $\mathbf{U}_{n-r}^{(r)}$ without weight:*

$$\mathbf{V}_n^{(r,s)}(z,m) = \mathbf{U}_{n+1}^{(r)}(z/q,m) + sz\mathbf{U}_{n-r}^{(r)}(zq^m,m), \tag{4.6}$$

and

$$\mathbb{V}_n^{(r,s)}(z,m) = \mathbf{U}_{n+1}^{(r)}(z,m) + sq^{n-r}z\mathbf{U}_{n-r}^{(r)}(zq^{m-r},m). \tag{4.7}$$

2. *Expression of* $V_n^{(r,s)}$*s in terms of* $\mathbf{U}_{n+1}^{(r)}$ *and* $\mathbf{U}_n^{(r)}$ *weighted by* $(s+1)$*:*

$$\mathbf{V}_n^{(r,s)}(z,m) = (1+s)\mathbf{U}_{n+1}^{(r)}(z/q,m) - s\mathbf{U}_n^{(r)}(z,m), \tag{4.8}$$

and

$$\mathbb{V}_n^{(r,s)}(z,m) = (1+s)\mathbf{U}_{n+1}^{(r)}(z,m) - s\mathbf{U}_n^{(r)}(z,m). \tag{4.9}$$

3. *Expression of* $V_n^{(r,s)}$*s in terms of* $\mathbf{U}_n^{(r)}$ *and* $\mathbf{U}_{n-r}^{(r)}$*:*

$$\mathbf{V}_n^{(r,s)}(z,m) = \mathbf{U}_n(z,m) + (1+s)z\mathbf{U}_{n-r}(zq^m,m), \tag{4.10}$$

and

$$\mathbb{V}_n^{(r,s)}(z,m) = \mathbf{U}_n^{(r)}(z,m) + (1+s)q^{n-r}z\mathbf{U}_{n-r}^{(r)}(zq^{m-r},m). \tag{4.11}$$

Proof

We give the proof of the two first relations. The approach is similar for the others.

$$\begin{aligned}\mathbf{V}_n^{(r,s)}(z,m) &= \textstyle\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k}{2}(m+1)}\begin{bmatrix}n-rk\\k\end{bmatrix}_q z^k + s\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k}{2}(m+1)}\begin{bmatrix}n-rk\\k\end{bmatrix}_q \frac{[k]_q}{[n-rk]_q} z^k\\ &= \mathbf{U}_{n+1}^{(r)}(z/q,m) + sz\mathbf{U}_{n-r}^{(r)}(zq^m,m),\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}_n^{(r,s)}(z,m) &= \textstyle\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}n-rk\\k\end{bmatrix} z^k + s\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}n-rk\\k\end{bmatrix} q^{(n-(r+1)k)}\frac{[k]_q}{[n-rk]_q} z^k\\ &= \mathbf{U}_{n+1}^{(r)}(z,m) + s\textstyle\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}+n-(r+1)k}\begin{bmatrix}n-rk-1\\k-1\end{bmatrix}_q z^k\\ &= \mathbf{U}_{n+1}^{(r)}(z,m) + szq^{n-r}\textstyle\sum_{k=0}^{\lfloor n/(r+1)\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}n-r(k+1)-1\\k\end{bmatrix}_q (q^{m-r}z)^k\\ &= \mathbf{U}_{n+1}^{(r)}(z,m) + szq^{n-r}\mathbf{U}_{n-r}^{(r)}(zq^{m-r},m).\end{aligned}$$

□

Corollary 2 *For nonnegative integers* r,s *such that* $1 \leq s \leq r$*, the polynomials* $\mathbf{V}_n^{(r,s)}$ *and* $\mathbb{V}_n^{(r,s)}$ *satisfy the following recurrences:*

$$\mathbf{V}_{n+1}^{(r,s)}(z,m) = \mathbf{V}_n^{(r,s)}(qz,m) + qz\mathbf{V}_{n-r}^{(r,s)}(zq^{m+1},m), \tag{4.12}$$

and

$$\mathbb{V}_{n+1}^{(r,s)}(z,m) = \mathbb{V}_n^{(r,s)}(z,m) + q^{n-r}z\mathbb{V}_{n-r}^{(r,s)}(zq^{m-r},m). \tag{4.13}$$

5. Combinatorial interpretation of sequences $(U_n^{(r)})$ and $(V_n^{(r,s)})$

In this section, we propose a combinatorial interpretation for the r-Fibonacci numbers and their companion sequences by using linear tiling via "*square* and $(r+1)$-*omino*"; see, for instance, Benjamin and Quinn [8] for $r=1$.

For nonnegative integers, the generalized Fibonacci number U_{n+1} counts the number of ways to tile an n-board with colored squares and colored dominos where there are x different colors for squares and y different colors for dominos.

Our purpose is to deal with circular tiling of length n (also called an n-bracelet) by squares or $(r+1)$-ominos (see Figure 1). An n-bracelet is in-phase if the zero position of the n-bracelet is bordered either by squares or the position 0 or $(r+1)$ of the $(r+1)$-omino. It is out of phase if the $(r+1)$-omino covers position zero at level t for $1 \leq t \leq r$ of the corresponding $(r+1)$-omino. In our case, the only possible positions able to be out of phase are $1, \ldots, s$ (without losing the generality, we can accept any chosen s positions from 1 to r). For example, as illustrated in Figures 2, 3, and 4, when $n = 5$ and $r = 3$, the first 3 bracelets are in-phase and the others correspond to an unphased bracelet where the 4-omino is covering cells 1 and 5 at the first s authorized positions, for $1 \leq s \leq 3$.

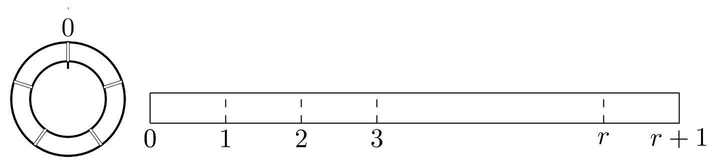

Figure 1. n-bracelet and $(r+1)$-omino.

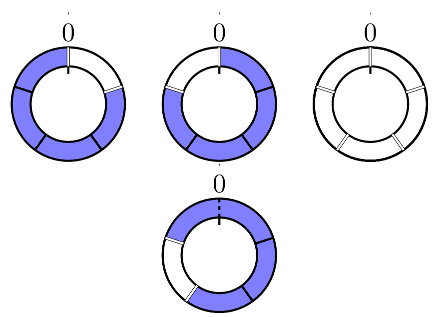

Figure 2. Circular tiling of length 5 for $s = 1$.

We have by relation (1.1) and Theorem 1 the following:

$$U_{n+1}^{(r)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} U(n,k) \quad \text{with} \quad U(n,k) := \binom{n-rk}{k} x^{n-(r+1)k} y^k, \tag{5.1}$$

and

$$V_n^{(r,s)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} V(n,k) \quad \text{with} \quad V(n,k) := \frac{n-(r-s)k}{n-rk} \binom{n-rk}{k} x^{n-(r+1)k} y^k. \tag{5.2}$$

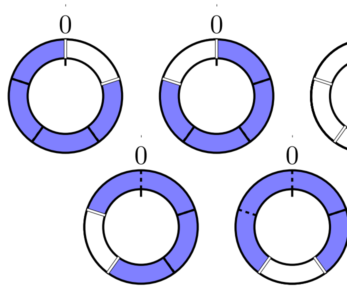

Figure 3. Circular tiling of length 5 for $s = 2$.

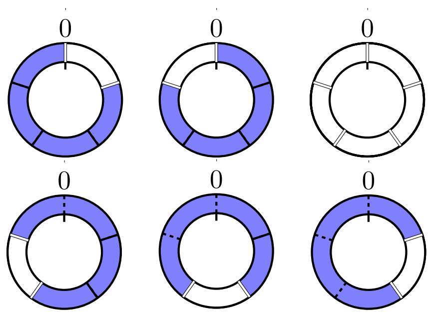

Figure 4. Circular tiling of length 5 for $s = 3$.

Proposition 1 *Let x, y and n be nonnegative integers. The r-Fibonacci numbers $U_{n+1}^{(r)}$ are interpreted as the number of ways to tile an n-board with colored squares and colored $(r+1)$-ominos, where there are x different colors for squares and y different colors for $(r+1)$-ominos.*

Proof For $n \geq r$ and $(k = 0, \ldots, \lfloor n/(r+1) \rfloor)$, if there are k $(r+1)$-ominos to tile, then there must be $n-(r+1)k$ squares. Hence, there are $\binom{n-rk}{k}$ ways to choose k $(r+1)$-ominos from the tiles of weight y^k and the rest of the tiles of squares are of weight $x^{n-(r+1)k}$. Thus, the number of ways to cover the n-board is $\binom{n-rk}{k} x^{n-(r+1)k} y^k$. □

Now we give a combinatorial interpretation of $V_n^{(r,s)}$ $(1 \leq s \leq r)$ being a nonnegative integer.

Theorem 7 *Let n, r, and s be nonnegative integers, and let $V_n^{(r,s)}$ count the number of ways to tile an n-bracelet with colored squares of parameter color x and $(r+1)$-ominos of parameter color y, with the first s authorized positions in the zero fixed point of the bracelet.*

Proof We have $V_n^{(r,s)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} V(n,k)$, where $V(n,k)$ counts the number of ways to tile an n-bracelet with squares of parameter color x and exactly k $(r+1)$-ominos of parameter color y, with the first s authorized positions in the zero fixed point of the bracelet. Note that the number $V(n,k)$ is given by $V(n,k) = \frac{n-(r-s)k}{n-rk} \binom{n-rk}{k} x^{n-(r+1)k} y^k$. First, if there is no $(r+1)$-omino covering cells 1 and n, the bracelet is in-phase, and then the number of ways to tile the n-bracelet is the same as the number of ways to tile an n-board, which is given by $\binom{n-rk}{k} x^{n-(r+1)k} y^k$ possibilities. Second, if there is an $(r+1)$-omino occupying cells

1 and n, without losing the generality, we accept only the first s alternate positions of the $(r+1)$-ominos to cover cells 1 and n. Thus, there are s possibilities, $1 \leq s \leq r$. Then we have to tile $n-(r+1)$ cells with $(k-1)$ $(r+1)$-ominos, which gives exactly $\binom{n-rk-1}{k-1}x^{n-(r+1)k}y^k$ possibilities. Finally, we have

$$\binom{n-rk}{k}x^{n-(r+1)k}y^k + s\binom{n-rk-1}{k-1}x^{n-(r+1)k}y^k = V(n,k),$$

which produces the result by summing these numbers. □

6. Extension of $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ to negative indices

Note that the Binet formula permits us to extend the definitions of $U_n^{(r)}$ and $V_n^{(r,s)}$ to negative ns.

Proposition 2 *Let x, y be reversible elements of a unitary ring $\mathcal{A}$. For integers $n \geq 0$ and $r \geq 1$, the terms of the sequence $(U_{-n}^{(r)})_n$ satisfy the following recurrence relation:*

$$U_{-n}^{(r)} = y^{-1}U_{-n+r+1}^{(r)} - xy^{-1}U_{-n+r}^{(r)} \quad (n \geq r+1). \tag{6.1}$$

Proof We replace $n \to -n+r$ in the Binet formula. □

Lemma 2 *For any integers m, r, we have*

$$\sum_{j=1}^{r}(-1)^{j+1}\binom{k}{m+j} = (-1)^{(r+1)}\binom{k-1}{m+r} + \binom{k-1}{m}. \tag{6.2}$$

Proof The proof is easy by induction and Pascal's rule. □

Theorem 8 *For $n \in \mathbb{N}$, $(U_{-n}^{(r)})_{n\geq 1}$ satisfy the two following equivalent identities:*

$$U_{-n}^{(r)} = \sum_{k}\binom{k-1}{n-rk}(-x)^{-n-1+(r+1)k}y^{-k}, \text{ with } U_0 = 0, \tag{6.3}$$

$$U_{-n}^{(r)} = \sum_{k}\binom{(n-k-r)/r}{k}(-x)^{(n-r-(r+1)k)/r}y^{(-n+k)/r}, \tag{6.4}$$

and we may restrict the first sum to integers $k \geq 1$ ranging between $\lfloor (n+1)/(r+1) \rfloor$ and $\lfloor (n-1)/r \rfloor$; the second summation is limited to those integers k lying between 0 and $\lfloor (n-r)/(r+1) \rfloor$, which satisfy r divides $(n-k)$.

Proof Using Lemma 1, we consider the sequence W_n defined by $W_n = U_{-n}^{(r)}$. Thus $W_n = y^{-1}W_{n-r-1}^{(r)} - xy^{-1}W_{n-r}^{(r)}$, with $a_1 = a_2 = \cdots = a_{r-1} = 0$, $a_r = -xy^{-1}$, and $a_{r+1} = y^{-1}$. Notice also that for $1 \leq j \leq r$ we have $W_{-j} = U_j^{(r)} = x^{j-1}$. Consequently, the sequence $(\lambda_j)_{0\leq j\leq r}$ is defined by $\lambda_j = -\sum_{k=0}^{r-j} a_k U_{k+j}^{(r)}$ for

$0 \le j \le r$, with $a_0 = -1$, so $\lambda_0 = x^r y^{-1}$ and $\lambda_j = x^{j-1}$ for $1 \le j \le r$. Finally, the sequence $(y_n)_n$ is given by the expression

$$y_n = \sum_{rk_r+(r+1)k_{r+1}=n} \binom{k_r+k_{r+1}}{k_r, k_{r+1}} a_r^{k_r} a_{r+1}^{k_{r+1}} = \sum_k \binom{k}{n-rk} (-x)^{(r+1)k-n} y^{-k}.$$

Now, applying Lemma 1, we get the expression of $(W_n)_n$ in terms of (λ_n) and (y_n) for $n \ge 0$:

$$\begin{aligned}
W_n &= \lambda_0 y_n + \lambda_1 y_{n+1} + \cdots + \lambda_r y_{n+r}, \\
&= x^r y^{-1} y_n + \textstyle\sum_{j=1}^{r} x^{j-1} y_{n+j}, \\
&= x^r y^{-1} \sum_k \binom{k}{n-rk} (-x)^{(r+1)k-n} y^{-k} + x^{j-1} \textstyle\sum_{j=1}^{r} \sum_k \binom{k}{n+j-rk} (-x)^{(r+1)k-n-j} y^{-k} \\
&= \sum_k (-x)^{rk+k-n-1} y^{-k} (\textstyle\sum_{j=1}^{r} (-1)^{j+1} \binom{k}{n+j-rk} + (-1)^{(r)} \binom{k-1}{n+r-rk})) \\
&= \sum_k \binom{k-1}{n-rk} (-x)^{-n-1+(r+1)k} y^{-k} (\text{using Lemma } 2) \\
&= \sum_{k;\ r|(n-k)} \binom{(n-k-r)/r}{k} (-x)^{(n-r-(r+1)k)/r} y^{(-n+k)/r}.
\end{aligned}$$

□

The positiveness of rs corresponds to the fact that elements $\binom{-n+rk}{k}$ are lying over traversals of finite support as nonvanishing values. Now we give an expression similar to relation (2.2) for the sequences $(V_{-n}^{(r,s)})_{n \ge 1}$ in terms of s and $U_{-n}^{(r)}$ using the corresponding Binet formula.

Theorem 9 *Letting r and s be nonnegative integers such that $1 \le s \le r$, we have for $n \ge 1$*

$$V_{-n}^{(r,s)} = U_{-n+1}^{(r)} + syU_{-n-r}^{(r)}, \tag{6.5}$$

and also, we get the explicit form for $n \ge 1$:

$$V_{-n}^{(r,s)} = \sum_{k}^{\lfloor (n-1)/r \rfloor} \frac{n-(r-s)k}{n-rk} \binom{k-1}{n-1-rk} (-x)^{-n+(r+1)k} y^{-k} + s(-x)^{n/r} y^{-n/r} [r \mid n]. \tag{6.6}$$

Equivalently,

$$V_{-n}^{(r,s)} = \sum_{k,\ r|(n-k)} \frac{sn+(r-s)k}{rk} \binom{(n-k-r)/r}{k-1} (-x)^{(n-(r+1)k)/r} y^{(-n+k)/r} + s(-x)^{n/r} y^{-n/r} [r \mid n], \tag{6.7}$$

with $V_0^{(r,s)} = s+1$, and $[r \mid n] = 1$ for r dividing n and 0 otherwise; for the notation, see [16].

We may restrict the first sum to integers k ranging between $\lfloor n/(r+1) \rfloor$ and $\lfloor (n-1)/r \rfloor$; the second summation is limited to those integers k lying between 1 and $\lfloor n/(r+1) \rfloor$, which satisfy r divides $(n-k)$.

Proof We give the proof of the first identity given by relation (6.6). The proof of the second one, (6.7), can be obtained easily using the same approach. Then, applying relations (6.5) and (6.3), we obtain

$$\begin{aligned} V_{-n}^{(r,s)} &= U_{-n+1}^{(r)} + syU_{-n-r}^{(r)} \\ &= \sum_{k}^{\lfloor (n-1)/r \rfloor} \tbinom{k-1}{n-1-rk}(-x)^{-n+(r+1)k}y^{-k} + s\sum_{k}^{\lfloor (n-1)/r \rfloor} \tbinom{k-1}{n+r-rk}(-x)^{-n-r-1+(r+1)k}y^{-k+1} \\ &= \sum_{k}^{\lfloor (n-1)/r \rfloor} \tbinom{k-1}{n-1-rk}(-x)^{-n+(r+1)k}y^{-k} + s\sum_{k}^{\lfloor n/r \rfloor} \tbinom{k}{n-rk}(-x)^{-n+(r+1)k}y^{-k} \\ &= \sum_{k}^{\lfloor (n-1)/r \rfloor} (1+s\tfrac{k}{n-rk})\tbinom{k-1}{n-1-rk}(-x)^{-n+(r+1)k}y^{-k} + s\tbinom{\lfloor n/r \rfloor}{n-r\lfloor n/r \rfloor}(-x)^{-n+(r+1)\lfloor n/r \rfloor}y^{-\lfloor n/r \rfloor}, \end{aligned}$$

which gives the result. □

We deduce that characterization (2.2) of sequences $(V_n^{(r,s)})$ given in Theorem 1 is satisfied for $n \in \mathbb{Z}$.

Theorems 8 and 9 given previously allow us to produce some applications. For instance, we have the following:

Application Let us take the 2-Fibonacci numbers $(U_n^{(2)})_n$ for $n \geq 1$. We have

$$U_{-n}^{(2)} = \sum_k \binom{k-1}{n-2k}(-1)^{-n-1+3k}.$$

Its companion sequences at negative indices for $s = 1, 2$ are given by the following identities:

$$V_{-n}^{(2,1)} = \sum_k^{\lfloor (n-1)/2 \rfloor} \frac{n-k}{n-2k}\binom{k-1}{n-1-2k}(-1)^{-n+3k} + \frac{1}{2}(1+(-1)^n)(-1)^{\lfloor n/2 \rfloor},$$

and

$$V_{-n}^{(2,2)} = \sum_k^{\lfloor (n-1)/2 \rfloor} \frac{n}{n-2k}\binom{k-1}{n-1-2k}(-1)^{-n+3k} + (1+(-1)^n)(-1)^{\lfloor n/2 \rfloor}.$$

7. Incomplete r-Lucas and hyper r-Lucas polynomials of type s

In this section, we define the incomplete r-Lucas and hyper-r-Lucas polynomials of type s. For simplicity of notation, we introduce:

U_n, V_n for the r-Fibonacci and r-Lucas polynomials,

$U_n(k), V_n(k)$ for the incomplete r-Fibonacci and r-Lucas polynomials,

$U_n^{[m]}, V_n^{[m]}$ for the hyper-r-Fibonacci and hyper-r-Lucas polynomials.

7.1. Incomplete r-Lucas polynomials of type s

In [25], the incomplete r-Fibonacci polynomials are given by

$$U_{n+1}(k) = \sum_{j=0}^{k} \binom{n-rj}{j} x^{n-(r+1)j}y^j \ , 0 \leq k \leq \lfloor n/(r+1) \rfloor . \tag{7.1}$$

In this subsection, we give a unifying expression of the incomplete r-Lucas polynomials of type s, which generalizes identity (1.2) given by Tasci et al. in [25].

Definition 4 *For r, s nonnegative integers such that $1 \leq s \leq r$, the incomplete r-Lucas polynomials of type s is defined as*

$$V_n(k) = \sum_{j=0}^{k} \frac{n-(r-s)j}{n-rj}\binom{n-rj}{j} x^{n-(r+1)j} y^j, \; 0 \leq k \leq \lfloor n/(r+1) \rfloor. \tag{7.2}$$

For $k = \lfloor n/(r+1) \rfloor$ $V_n(k) = V_n$, we obtain the r-Lucas polynomials of type s.

For $x = y = 1$, $s = r = 1$, we get the incomplete Lucas numbers [15].

For $x = 2$, $y = 1$, and $s = r = 1$, we get the incomplete Jacobsthal–Lucas numbers [14]

.

In the following, we give a recurrence relation for the incomplete r-Lucas polynomials of type s.

Theorem 10 *The incomplete r-Lucas polynomials of type s satisfy the following recurrence:*

$$V_n(k+1) = xV_{n-1}(k+1) + yV_{n-r-1}(k). \tag{7.3}$$

The following theorem gives a relationship between the incomplete r-Lucas polynomials of type s and the incomplete r-Fibonacci polynomials.

Theorem 11 *Let r, s be nonnegative integers, such that $1 \leq s \leq r$. The incomplete r-Lucas polynomials of type s satisfy the following recurrence relation:*

$$V_n(k) = U_{n+1}(k) + syU_{n-r}(k-1). \tag{7.4}$$

Proof Using relation (7.2), we have

$V_n(k) = \sum_{j=0}^{k} \binom{n-rj}{j} x^{n-(r+1)j} y^j + s \sum_{j=0}^{k} \binom{n-rj-1}{j-1} x^{n-(r+1)j} y^j = U_{n+1}(k) + syU_{n-r}(k-1)$. □

The following theorem provides a nonhomogeneous relation for the incomplete r-Lucas polynomials of type s.

Theorem 12 *Let r, s be nonnegative integers, such that $1 \leq s \leq r$. The incomplete r-Lucas polynomials of type s satisfy the following nonhomogeneous relation:*

$$V_{n+1}(k) \;=\; xV_n(k) + yV_{n-r}(k) - \frac{n-r-(r-s)k}{n-r(k+1)}\binom{n-r(k+1)}{k} x^{n-k-(r+1)k} y^{k+1}. \tag{7.5}$$

Proof The proof is done using relations (7.2) and (7.3). □

7.1.1. Generating function of the incomplete r-Lucas polynomials of type s

The lemma given below allows us to introduce the generating function of the incomplete r-Lucas polynomials of type s.

Lemma 3 *([22]) Let* $(s_n)_{n\geq 0}$ *be a complex sequence satisfying the following nonhomogeneous recurrence relation* $s_n = xs_{n-1} + ys_{n-r-1} + \alpha_n, \quad n > r$, *where* (α_n) *is a given complex sequence. Then the generating function* $S_r^k(x, y; t)$ *of the sequence* (s_n) *is*

$$S_r^k(x, y; t) = \frac{(s_0 - \alpha_0 + \sum_{i=1}^{r}(s_i - xs_{i-1} - \alpha_i)t^i + G(t))}{(1 - xt - yt^{r+1})}, \tag{7.6}$$

where $G(t)$ *is the generating function of* (α_n).

Theorem 13 *The generating function of the incomplete* r*-Lucas polynomials of type* s *is*

$$\sum_{n\geqslant 0} V_n(k)t^n = \frac{t^{k(r+1)}}{(1 - xt - yt^{r+1})}\left[V_{k(r+1)} + \sum_{i=1}^{r}(V_{k(r+1)+i} - xV_{k(r+1)+i-1})t^i - \frac{y^{k+1}t^{r+1}[s(1-xt)+1]}{(1-xt)^{k+1}}\right].$$

Proof From (7.2), we have $V_n(k) = 0$ for $0 \leqslant n < k(r+1)$, and then for $n \geqslant k(r+1)$ we have $s_0 = V_{k(r+1)}(k) = V_{k(r+1)}$, $\quad s_1 = V_{k(r+1)+1}(k) = V_{k(r+1)+1}$, and $s_r = V_{k(r+1)+r}(k) = V_{k(r+1)+r}$.

Also let $\alpha_0 = \alpha_1 = \cdots = \alpha_r = 0$ and $\alpha_n = \frac{n-r-1-(r-s)k}{n-1-r(k+1)}\binom{n-1-r(k+1)}{k}x^{n-(r+1)(k+1)}y^{k+1}$. The generating function of the sequence (α_n) is given by $G(t) = \frac{y^{k+1}t^{r+1}[s(1-xt)+1]}{(1-xt)^{k+1}}$ (see [24], page 355). Thus, from Lemma 3, we find the generating function of sequences $(V_n(k))$ for $1 \leqslant s \leqslant r$. □

7.2. Hyper-r-Fibonacci polynomials

Let (a_n) and $(a^{(n)})$ be two real sequences. Bahsi et al. [2] defined the symmetric infinite matrix associated to these sequences by the following recursive formula:

$$\begin{cases} a_n^{(0)} = a_n,\ a_0^{(n)} = a^{(n)}, \quad (n \geq 0), \\ a_n^{(k)} = xa_{n-1}^{(k)} + ya_n^{(k-1)}, \quad (n \geq 1, k \geq 1), \end{cases} \tag{7.7}$$

where $a_n^{(k)}$ represents the kth row and the nth column entry,

$$\begin{pmatrix} . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ . & . & . & & ya_n^{(k-1)} & . & . \\ . & . & . & & \downarrow & . & . \\ & & xa_{n-1}^{(k)} & \rightarrow & a_n^{(k)} & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

The entry $a_n^{(k)}$ has the following expression:

$$a_n^k = x^n \sum_{i=1}^{k} y^{k-i}\binom{n+k-i-1}{n-1}a_0^i + y^k \sum_{s=1}^{n} x^{n-s}\binom{n+k-s-1}{k-1}a_s^0. \tag{7.8}$$

As an application, we define the bivariate hyper r-Fibonacci polynomials as follows:

Definition 5 *For $m \geqslant 0$, the bivariate hyper r-Fibonacci polynomials are defined by the following recurrence relation:*

$$\begin{cases} U_n^{[0]} = U_n^{(r)},\ U_0^{[m]} = y^m, \\ U_{n+1}^{[m]} = xU_n^{[m]} + yU_{n+1}^{[m-1]}. \end{cases} \tag{7.9}$$

Relation (7.9) can be written as follows:

$$U_n^{[m]} = \sum_{j=0}^{n} yx^{n-j} U_j^{[m-1]}. \tag{7.10}$$

Some particular cases of hyper-r-Fibonacci polynomials are:

For $r=1,\ x=1$, and $y=1,\ U_n^{[m]}(1,1) = F_n^{[m]}$, we get hyper-Fibonacci numbers [2].

For $r=1,\ x=2$, and $y=1,\ U_n^{[m]}(2,1) = p_n^{[m]}$, we get hyper-Pell numbers [1].

For $r=1,\ x=1$, and $y=2,\ U_n^{[m]}(1,2) = j_n^{[m]}$, we get hyper-Jacobsthal numbers.

As a result of (7.8), the bivariate hyper-r-Fibonacci polynomial can be expressed as a sum of binomial coefficients and r-Fibonacci polynomial:

$$U_{n+1}^{[m]} = \sum_{k=1}^{n+1} \binom{n+m-k}{m-1} x^{n+1-k} y^m U_k. \tag{7.11}$$

7.2.1. Combinatorial interpretation

This section deals with a combinatorial interpretation of relation (7.9) and the explicit formula of the bivariate hyper-r-Fibonacci. polynomials

Theorem 14 *The hyper-r-Fibonacci polynomial $U_n^{[m]}$ is interpreted as the number of ways to tile an $[n+(r+1)m]$-board with at least m $(r+1)$-ominos, such that we distribute a weight x for each square and weight y for each $(r+1)$-omino.*

Proof Let $T_{n,m}$ count the number of ways to tile an $[n+(r+1)m]$-board using squares of weight x and at least m $(r+1)$-ominos of weight y.

For $m=0$, $T_{n,0}$ corresponds to the number of ways to tile an n-board without condition on number of ominos, which gives $U(n,k)$. For $n=0$, the number of ways to tile $(r+1)m$ with at least m $(r+1)$-ominos of weight y is given by y^m. Now let $T_{n,m}$ be a tiling of an $[n+(r+1)m]$ -board with at least m $(r+1)$-ominos. If the first tile is a square of weight x, then the weight of the $n+(r+1)m-1$ tiling with at least m $(r+1)$-ominos is $T_{(n-1),m}$, whereas if the first tile is a $(r+1)$-omino of weight y, then the weight of $n+(r+1)m-(r+1) = n+(r+1)(m-1)$ is $T_{n,m-1}$. □

The following establishes the explicit formula of the hyper-r-Fibonacci polynomial $U_n^{[m]}$.

Theorem 15 *For any $n \geqslant 0$, $m \geqslant 0$, and $k = 0, \dots, \lfloor n/(r+1) \rfloor$, we have*

$$U_{n+1}^{[m]} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n+m-rk}{m+k} x^{n-(r+1)k} y^{k+m}, \tag{7.12}$$

where the number $\binom{n+m-rk}{m+k}x^{n-(r+1)k}y^{k+m}$ *represents the weight of a linear* $n+(r+1)m$*-tiling with at least* $(k+m)$ $(r+1)$*-ominos.*

Proof An $n+(r+1)m$ tiling with at least $(k+m)$ $(r+1)$-ominos must use $n-(r+1)k$ squares, and then there are $\binom{n+m-rk}{m+k}$ ways to choose $(k+m)$ $(r+1)$-ominos of weight y^{k+m} and $x^{n-(r+1)k}$ for the rest of the tiles (squares). The proof of relation (7.12) is done using Theorem 14. □

Corollary 3 *The first terms of the hyper-*r*-Fibonacci numbers are given by*

$$U_k^{[m]} = \binom{m+k-1}{k-1}x^{k-1}y^m, \qquad (1 \leq k \leq r),$$

Theorem 16 *The hyper-*r*-Fibonacci polynomial* $U_n^{[m]}$ *satisfies the following nonhomogeneous recurrence relation:*

$$U_{n+1}^{[m]} = xU_n^{[m]} + yU_{n-r}^{[m]} + \binom{n+m-1}{m-1}x^n y^m. \tag{7.13}$$

Proof Using relations (7.9) and (7.12), we have

$$\begin{aligned}
U_{n+1}^{[m]} &= xU_n^{[m]} + yU_{n+1}^{[m-1]} \\
&= xU_n^{[m]} + y\sum_{k=0}^{\lfloor n/(r+1)\rfloor} \tbinom{n+m-1-rk}{m-1+k}x^{n-(r+1)k}y^{k+m-1} \\
&= xU_n^{[m]} + y\sum_{k=1}^{\lfloor n/(r+1)\rfloor} \tbinom{n+m-1-rk}{m-1+k}x^{n-(r+1)k}y^{k+m-1} + \tbinom{n+m-1}{m-1}x^n y^m \\
&= xU_n^{[m]} + y\sum_{k=0}^{\lfloor (n-r-1)/(r+1)\rfloor} \tbinom{n+m-1-r(k+1)}{m+k}x^{n-(r+1)(k+1)}y^{k+m} + \tbinom{n+m-1}{m-1}x^n y^m \\
&= xU_n^{[m]} + xU_{n-r}^{[m]} + \tbinom{n+m-1}{m-1}x^n y^m.
\end{aligned}$$

□

The following result implies that every r-Fibonacci polynomial can be written as a sum of hyper-r-Fibonacci polynomials and incomplete r-Fibonacci polynomials.

Theorem 17 *For* $m \geqslant 1$ *and* $n \geqslant 0$*, we have*

$$U_{n+(r+1)m} = U_n^{[m]} + U_{n+(r+1)m}(m-1). \tag{7.14}$$

Proof It follows by equations (7.1) and (7.12). □

7.3. Hyper-r-Lucas polynomials

Now we start by giving the definition of the hyper-r-Lucas polynomials of type s. Then we give an explicit formula in terms of s and the hyper-r-Fibonacci polynomials.

Definition 6 *For $m \geqslant 0$, the bivariate hyper-r-Lucas polynomials are defined by the following recurrence relation:*

$$\begin{cases} V_n^{[0]} = V_n^{(r,s)}, \; V_0^{[m]} = (s+1)y^m, \\ V_n^{[m]} = xV_{n-1}^{[m]} + yV_n^{[m-1]}. \end{cases} \tag{7.15}$$

Some particular cases of hyper-r-Lucas polynomials of type s are:

For $r = s = 1$, $x = 1$, and $y = 1$, $V_n^{[m]}(1,1) = L_n^{[m]}$, we get hyper-Lucas numbers [2].

For $r = s = 1$, $x = 2$, and $y = 1$, $V_n^{[m]}(2,1) = P_n^{[m]}$, we get hyper-Pell–Lucas numbers [1].

For $r = s = 1$, $x = 1$, and $y = 2$, $V_n^{[m]}(1,2) = J_n^{[m]}$, we get hyper-Jacobsthal–Lucas numbers.

Theorem 18 *Let r and s be nonnegative integers such that $1 \leq s \leq r$. For any $n \geq 0$ and $m \geq 1$, we have*

$$V_n^{[m]} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \frac{n+(s+1)m-(r-s)k}{n+m-rk} \binom{n+m-rk}{m+k} x^{n-(r+1)k} y^{k+m}. \tag{7.16}$$

Proof We prove the Theorem by double induction on n and m. The idea of this proof was already used by Belbachir and Belkhir in [3].

Let $V_{n+m} = V_n^{[m]}$. Relation (7.16) is clearly satisfied for $n+m=0$ and $n+m=1$. Then we suppose that it is satisfied for all $p < n+m+1$, and we prove it for $p = n+m+1$. Using (7.15), we have

$$\begin{aligned} V_{n+1}^{[m]} &= \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \tfrac{n+(s+1)m-(r-s)k}{n+m-rk} \tbinom{n+m-rk}{m-1+k} x^{n+1-(r+1)k} y^{k+m} \\ &+ \sum_{k=0}^{\lfloor (n+1)/(r+1) \rfloor} \tfrac{n+1+(s+1)(m-1)-(r-s)k}{n+m-rk} \tbinom{n+m-rk}{m-1+k} x^{n+1-(r+1)k} y^{k+m-1} \\ &= \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \tbinom{n+m-rk}{m+k} x^{n+1-(r+1)k} y^{k+m} + s \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \tbinom{n+m-rk-1}{m+k-1} x^{n+1-(r+1)k} y^{k+m} \\ &+ \sum_{k=0}^{\lfloor (n+1)/(r+1) \rfloor} \tbinom{n+m-rk}{m+k-1} x^{n+1-(r+1)k} y^{k+m} + s \sum_{k=0}^{\lfloor (n+1)/(r+1) \rfloor} \tbinom{n+m-rk-1}{m+k-2} x^{n+1-(r+1)k} y^{k+m} \\ &= \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \left[\tbinom{n+m-rk+1}{m+k} + s\tbinom{n+m-rk}{m+k-1} \right] x^{n+1-(r+1)k} y^{k+m} \\ &= \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \tfrac{n+1+(s+1)m-(r-s)k}{n+m-rk+1} \tbinom{n+m-rk+1}{m+k} x^{n-(r+1)k} y^{k+m}. \end{aligned}$$

□

As a consequence of (7.15) and (7.16), a nonhomogeneous relation of the hyper-r-Lucas polynomials of type s is given as follows:

$$V_n^{[m]} = xV_{n-1}^{[m]} + yV_{n-r-1}^{[m]} + \frac{n+(s+1)(m-1)}{n+m-1} \binom{n+m-1}{m-1} x^n y^m. \tag{7.17}$$

In the following theorem, we establish an expression of the hyper-r-Lucas polynomials in terms of s and the hyper-r-Fibonacci polynomials.

Theorem 19 *For any integers n, m, r, and s $(1 \leq s \leq r)$, we have*

$$V_n^{[m]} = U_{n+1}^{[m]} + syU_{n+1}^{[m-1]}, \tag{7.18}$$

which also gives the following for all $n \geqslant r$ and $m \geqslant 1$:

$$V_n^{[m]} = U_{n+1}^{[m]} + syU_{n-r}^{[m]} + s\binom{n+m-1}{m-1}x^n y^m. \tag{7.19}$$

Proof We obtain the proof using relations (7.12) and (7.16). □

Corollary 4 *For given nonnegative integers m, n with $x = y = 1$ and $r = s = 1$, we have*

$$V_n^{[m]} + U_n^{[m]} = 2U_{n+1}^{[m]} \text{ and } V_n^{[m]} - U_n^{[m]} = 2U_{n+1}^{[m-1]}.$$

Now we present the connection between the incomplete r-Lucas polynomials, hyper-r-Lucas polynomials, and r-Lucas polynomials.

Theorem 20 *For $m \geqslant 1$ and $n \geqslant 0$, we have*

$$V_{n+(r+1)m} = V_n^{[m]} + V_{n+(r+1)m}(m-1). \tag{7.20}$$

Proof The proof is obtained using relations (7.14) and (7.18). □

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