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Multiplication modules with prime spectrum

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Abstract: The subject of this paper is the Zariski topology on a multiplication module M over a commutative ring R . We find a characterization for the radical submodule $rad_M(0)$ and also show that there are proper ideals I_1, \dots, I_n of R such that $rad_M(0) = rad_M((I_1 \dots I_n)M)$. Finally, we prove that the spectrum $Spec(M)$ is irreducible if and only if M is the finite sum of its submodules, whose \mathcal{T} -radicals are prime in M .

Key words: Multiplication module, prime submodule, spectrum of module

1. Introduction

Throughout this study, R and M denote a commutative ring with identity and a unitary R -module, respectively. We also use $Spec(M)$ for the spectrum of prime submodules. In [4], the author investigated some properties of Zariski topology of multiplication modules. Motivated by this study, we generalize some important results in [4] and also give a characterization for the intersection of all prime submodules of M . Then M is said to be a multiplication R -module if for each submodule N of M , there exists an ideal I of R such that $N = IM$. For example, invertible ideals and projective ideals of R are multiplication R -modules. Since every cyclic module is a multiplication module and every finitely generated Artinian multiplication module is cyclic, there is a close relationship between multiplication modules and cyclic modules and so there are many studies related to these important concepts in module theory ([1, 4, 6, 8]).

A proper submodule P of an R -module M is said to be prime if for $a \in R$ and $m \in M$, $am \in P$ implies that $m \in P$ or $aM \subseteq P$. The radical of a submodule N in M denoted by $rad_M(N)$ is defined as the intersection of all prime submodules of M containing N .

In [11], $V(N)$ was defined as the set $\{P \in Spec(M) : N \subseteq P\}$ for any submodule N of an R -module M . Note that $V(M) = \emptyset$, $V(0) = Spec(M)$ and $\bigcap_{i \in \Lambda} V(N_i)$ is equivalent to $V\left(\sum_{i \in \Lambda} N_i\right)$ for any family of submodules N_i of M .

Let $\Gamma(M) = \{V(N) : N \text{ is a submodule of } M\}$. If $\Gamma(M)$ is closed under finite union, $\Gamma(M)$ satisfies the axioms of closed subsets of a topological space. Then it is said that M is a module with a Zariski topology.

A topological space X is said to be Noetherian if the closed subsets of X satisfy the descending chain condition. X is said to be irreducible if $X \neq \emptyset$ and for every decomposition $X = X_1 \cup X_2$ with closed subsets $X_1, X_2 \subseteq X$, we have $X = X_1$ or $X = X_2$. $D \subseteq X$ is said to be dense in X if for every nonempty open

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set $U \subseteq X$, $U \cap D \neq \emptyset$ holds. X is said to be quasi-compact if every open cover of X has a finite subcover ([7, 10]).

The aim of this paper is to study Zariski topology of multiplication modules over commutative ring with identity.

Section 2 is devoted to the study of a subspace associated with a submodule. We begin by giving a base for complement Zariski topology of a submodule N in a module M . We show that $rad_M(N) = rad_M(Rm_1 + \dots + Rm_n)$, where $m_i \in M$ if \mathcal{X}_N is quasicompact (Theorem 2.4). We also prove that $\mathcal{N}_N(0)$ is a prime submodule of M if and only if \mathcal{X}_N is irreducible (Theorem 2.8). Moreover, we give equivalent conditions for $Spec(M)$ (Theorem 2.10).

In Section 3, we are interested in the relationships between the complement Zariski topologies and submodules of a module M to find some algebraic and topological tools for submodules and find some characterizations for the modules. We show that there are proper ideals I_1, \dots, I_n of R such that $rad_M(0) = rad_M((I_1 \dots I_n)M)$, where M is a finitely generated multiplication R -module satisfying the \mathcal{T} -condition for every submodule (Theorem 3.9). Consequently, we prove that $\mathcal{X} = \bigcup_{i=1}^n \mathcal{X}_{I_i M}$, where $\mathcal{X}_{I_i M}$ is irreducible if and only if $M = \left(\sum_{i=1}^n I_i\right)M$ and $\mathcal{N}_{I_i M}(0)$ is a prime submodule of M (Theorem 3.10).

2. The subspace associated with a submodule

Let M be a multiplication R -module and let $N = IM$ be a submodule of M , where I is an ideal of R . Let $\mathcal{X}_N = Spec(M) \setminus V(IM)$ and $\tilde{V}(JM) = V(JM) \setminus V(IM)$, where J is an ideal of R . Then

$$\Gamma_N = \left\{ \tilde{V}(JM) : J \text{ is an ideal of } R \right\}$$

satisfies the axioms for closed sets of a topological space on \mathcal{X}_N . We name this topology as the complement Zariski topology of N in M .

Example 2.1 Let $R = \mathbb{Z}$, $M = 6\mathbb{Z}$ and $N = 30\mathbb{Z}$. Then M is a multiplication \mathbb{Z} -module. It is clear that $Spec(M) = \{6a\mathbb{Z} : a \in \mathbb{P}\}$, $V(30\mathbb{Z}) = \{30\mathbb{Z}\}$, $V(36\mathbb{Z}) = \{12\mathbb{Z}, 18\mathbb{Z}\}$ and $V(90\mathbb{Z}) = \{18\mathbb{Z}, 30\mathbb{Z}\}$, where \mathbb{P} is the set of prime numbers. Thus we have

$$\mathcal{X}_N = Spec(M) \setminus V(30\mathbb{Z}) = \{6a\mathbb{Z} : a \in \mathbb{P} \setminus \{5\}\},$$

$$\tilde{V}(36\mathbb{Z}) = V(36\mathbb{Z}) \setminus V(30\mathbb{Z}) = \{12\mathbb{Z}, 18\mathbb{Z}\} \setminus \{30\mathbb{Z}\} = \{12\mathbb{Z}, 18\mathbb{Z}\},$$

$$\tilde{V}(90\mathbb{Z}) = V(90\mathbb{Z}) \setminus V(30\mathbb{Z}) = \{18\mathbb{Z}, 30\mathbb{Z}\} \setminus \{30\mathbb{Z}\} = \{18\mathbb{Z}\}.$$

We fix the submodule N as $N = IM$, where I is an ideal of R , and the module M as a multiplication module in this section.

Lemma 2.2 Let $N = IM$ be a submodule of a multiplication R -module M , where I is an ideal of R . For any ideal J of R , the set $(\mathcal{X}_N)^{JM} = \mathcal{X}_N \setminus \tilde{V}(JM)$ forms a base for the complement Zariski topology of N in M on \mathcal{X}_N .

Proof If \mathcal{X}_{IM} is empty, then $(\mathcal{X}_{IM})^{JM} = \emptyset$, which is the trivial case. Therefore, we can assume that $\mathcal{X}_{IM} \neq \emptyset$.

Let $U \subset \mathcal{X}_N$ be an open set. Let $\sum_{i \in \Lambda} Rm_i = N$, where $m_i \in M$. Then $U = \mathcal{X}_N \setminus \tilde{V}(JM)$, where JM is a submodule of M . Thus,

$$\begin{aligned} U &= \mathcal{X}_{IM} \setminus \tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) = \mathcal{X}_{IM} \setminus \bigcap_{i \in \Lambda} \tilde{V}(Rm_i) \\ &= \bigcup_{i \in \Lambda} (\mathcal{X}_{IM} \setminus \tilde{V}(Rm_i)) = \bigcup_{i \in \Lambda} (\mathcal{X}_{IM})^{Rm_i}. \end{aligned}$$

Thus, it is proved that $(\mathcal{X}_{IM})^{JM}$ is a base for the complement Zariski topology of N in M . □

We are interested in the properties of the complement Zariski topology of a submodule N of a module M . The following proposition reveals some connections between \mathcal{X}_{IM} and a submodule IM . One can easily prove Proposition 2.3.

Proposition 2.3 *Let $N = IM$ be a submodule of a multiplication R -module M , where I is an ideal of R . The following statements hold:*

- i) $(\mathcal{X}_N)^{JM} = \mathcal{X}_N \setminus \tilde{V}(JM) = \text{Spec}(M) \setminus V(IJM)$ for J is an ideal of R .*
- ii) $(\mathcal{X}_N)^{J_1M} \cap (\mathcal{X}_N)^{J_2M} = (\mathcal{X}_N)^{(J_1J_2)M}$ for every ideal J_1, J_2 of R .*
- iii) $(\mathcal{X}_N)^{JM} = \emptyset$ if and only if $\text{rad}_M(IJM) \subseteq \text{rad}_M(0)$ for every ideal J of R .*
- iv) $(\mathcal{X}_N)^{J_1M} = (\mathcal{X}_N)^{J_2M}$ if and only if $\text{rad}_M(IJ_1M) = \text{rad}_M(IJ_2M)$ for every ideal J_1, J_2 of R .*
- v) If $(\mathcal{X}_N)^{JM} = \mathcal{X}_N$, then we have $\text{rad}_M(IJM) = \text{rad}_M(IM)$ for every ideal J of R .*

Let M be an R -module and let N be a proper submodule of M . Then we will say that N satisfies the condition $(*)$ if there is a finite subset Δ of Λ such that $\text{rad}_M(\{\{m_i \in M : i \in \Lambda\}\}) = \text{rad}_M(\{\{m_j : j \in \Delta\}\})$, whenever $\text{rad}_M(N) \subseteq \text{rad}_M(\{\{m_i \in M : i \in \Lambda\}\})$. It is clear that if $M/\text{rad}_M(N)$ is a Noetherian module, N satisfies the condition $(*)$.

In the following theorem, we give an algebraic property belonging to a submodule N and a topological property belonging to \mathcal{X}_N .

Theorem 2.4 *Let $N = IM$ be a proper submodule of a multiplication R -module M , where I is an ideal of R . Let $(\mathcal{X}_N)^{JM} = \mathcal{X}_N \setminus \tilde{V}(JM) = \text{Spec}(M) \setminus V(IJM)$ for any ideal J of R . Then the following statements are true.*

- i) $(\mathcal{X}_N)^{JM}$ is quasicompact for every ideal J of R .*
- ii) If \mathcal{X}_N is quasicompact, then $\text{rad}_M(N) = \text{rad}_M(Rm_1 + \dots + Rm_n)$, where $m_i \in M$.*
- iii) If N satisfies the condition $(*)$, then \mathcal{X}_N is quasicompact.*

Proof *i)* Obvious.

ii) Let \mathcal{X}_N be quasicompact.

Let $N = \langle m_i : i \in \Lambda \rangle$. Then $V(\langle \{m_i : i \in \Lambda\} \rangle) = V(N)$ and so $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) = \emptyset$. Thus,

$$\begin{aligned} \mathcal{X}_N &= \mathcal{X}_N \setminus \emptyset = \mathcal{X}_N \setminus \tilde{V}\left\{\sum_{i \in \Lambda} Rm_i\right\} = \mathcal{X}_N \setminus \left(\bigcap_{i \in \Lambda} \tilde{V}(Rm_i)\right) \\ &= \bigcup_{i \in \Lambda} \left(\mathcal{X}_N \setminus \tilde{V}(Rm_i)\right) = \bigcup_{i \in \Lambda} (\mathcal{X}_N)^{Rm_i}. \end{aligned}$$

Since \mathcal{X}_N is quasicompact, there is a finite set $\Delta = \{1, 2, \dots, n\} \subseteq \Lambda$ such that $\mathcal{X}_N = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i} = \mathcal{X}_N \setminus \tilde{V}(\langle m_1, m_2, \dots, m_n \rangle)$. Then $V(\langle m_1, m_2, \dots, m_n \rangle) \subseteq V(N)$ and so $rad_M(N) \subseteq rad_M(\langle m_1, m_2, \dots, m_n \rangle)$. On the other hand, we have $rad_M(\langle m_1, m_2, \dots, m_n \rangle) \subseteq rad_M(N)$, which means $rad_M(N) = rad_M(Rm_1 + \dots + Rm_n)$.

iii) Let N satisfy the condition (*).

Let $\{A_i : i \in \Lambda\}$ be an open cover of \mathcal{X}_N . Since A_i can be expressed as a union of the sets of $(\mathcal{X}_N)^{Rm_i}$, we may assume that $A_i = (\mathcal{X}_N)^{Rm_i}$ for every $i \in \Lambda$. Then

$$\begin{aligned} \mathcal{X}_N &= \bigcup_{i \in \Lambda} (\mathcal{X}_N)^{Rm_i} = \bigcup_{i \in \Lambda} \left(\mathcal{X}_N \setminus \tilde{V}(Rm_i)\right) \\ &= \mathcal{X}_N \setminus \bigcap_{i \in \Lambda} \tilde{V}(Rm_i) \\ &= \mathcal{X}_N \setminus \tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right). \end{aligned}$$

Thus, $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) = \emptyset$ and so $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) \subseteq V(N)$.

In this case, $rad_M(N) \subseteq rad_M\left(\sum_{i \in \Lambda} Rm_i\right)$. By the condition (*), there is a finite subset $\Delta \subseteq \Lambda$ such that $rad_M\left(\sum_{i \in \Lambda} Rm_i\right) = rad_M\left(\sum_{i \in \Delta} Rm_i\right)$. Then $V\left(\sum_{i \in \Delta} Rm_i\right) \subseteq V(N)$ and so $\tilde{V}\left(\sum_{i \in \Delta} Rm_i\right) = \emptyset$. Then

$$\begin{aligned} \mathcal{X}_N &= \mathcal{X}_N \setminus \tilde{V}\left(\sum_{i \in \Delta} Rm_i\right) = \mathcal{X}_N \setminus \bigcap_{i \in \Delta} \tilde{V}(Rm_i) \\ &= \bigcup_{i \in \Delta} \left(\mathcal{X}_N \setminus \tilde{V}(Rm_i)\right) = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i}. \end{aligned}$$

Since \mathcal{X}_N is covered by a finite number $(\mathcal{X}_N)^{Rm_i}$, \mathcal{X}_N is quasicompact. □

Theorem 2.4 also generalizes Theorem 3.7 in [4].

We now introduce the new submodule class which is a generalization of radical submodule of a module.

Definition 2.5 Let N be a submodule of an R -module M . The set $\mathcal{N}_N(T)$ is defined as the intersection of all prime submodules containing submodule T which does not contain N .

It is clear that $\mathcal{N}_N(T)$ is equivalent to the radical of a submodule T when $M = N$. Then $\mathcal{N}_N(T)$ is a generalization of radical submodule.

Example 2.6 Let $M = \mathbb{Z}$ be an \mathbb{Z} -module. Let $N = 12\mathbb{Z}$ and $T = 20\mathbb{Z}$ be submodules of M . Then $\mathcal{N}_{12\mathbb{Z}}(20\mathbb{Z}) = 5\mathbb{Z}$ but $rad_{\mathbb{Z}}(20\mathbb{Z}) = 10\mathbb{Z}$. Thus $\mathcal{N}_{12\mathbb{Z}}(20\mathbb{Z})$ is different from $rad_{\mathbb{Z}}(20\mathbb{Z})$.

The following lemma deals with algebraic properties of submodule $\mathcal{N}_N(T)$.

Lemma 2.7 Let $N = IM$ be a proper submodule of a multiplication R -module M , where I is an ideal of R . The following statements are true:

- i) $\mathcal{N}_N(T)$ is a submodule of M .
- ii) $\mathcal{N}_{N/K}(T/K) = \mathcal{N}_N(T)/K$, where $K \subseteq T$ is a submodule of M .
- iii) $\mathcal{N}_N(0) = \mathcal{N}_{rad_M(N)}(0)$.

Proof The proof is straightforward. □

The following theorem gives a connection between topological property of the complement Zariski topology \mathcal{X}_N and algebraic property of submodule $\mathcal{N}_N(0)$.

Theorem 2.8 Let $N = IM$ be a proper submodule of a multiplication R -module M and $rad_M(IM) \neq rad_M(0)$. Then $\mathcal{N}_N(0)$ is a prime submodule of M if and only if \mathcal{X}_N is irreducible.

Proof Let $\mathcal{N}_N(0)$ be a prime submodule of M and K be a nonempty open subset of \mathcal{X}_{IM} . Then $K = \mathcal{X}_N \setminus \tilde{V}(JM) = Spec(M) \setminus (V(IM) \cup V(JM))$, where JM is a submodule of M . Take $P \in K$. Then we have $P \notin V(IM) \cup V(JM)$, which means that $IM \not\subseteq P$ and $JM \not\subseteq P$. Thus, $\mathcal{N}_N(0) \subseteq P$, so $JM \not\subseteq \mathcal{N}_N(0) \subseteq P$. This implies that $\mathcal{N}_N(0) \notin V(JM)$ and by the definition of $\mathcal{N}_N(0)$, we get $\mathcal{N}_N(0) \notin V(IM)$. Thus, $\mathcal{N}_N(0) \in K$. Therefore, any nonempty open subset of \mathcal{X}_N contains $\mathcal{N}_N(0)$. This means that \mathcal{X}_N is irreducible.

Let \mathcal{X}_N be irreducible. Suppose that $\mathcal{N}_N(0)$ is not a prime submodule of M . Then there exists elements $a \in R$ and $m \in M$ such that $am \in \mathcal{N}_N(0)$, $m \notin \mathcal{N}_N(0)$ and $aM \subseteq \mathcal{N}_N(0)$.

Since $rad_M(N) = rad_M(IM) \neq rad_M(0)$ and $m \in M \setminus \mathcal{N}_N(0)$, it follows that $\tilde{V}(Rm) \neq \emptyset$ and $\tilde{V}(Rm) \neq \mathcal{X}_N$, which implies $(\mathcal{X}_N)^{Rm} \neq \emptyset$. This can also be used to prove that $(\mathcal{X}_N)^{aM}$ is a nonempty open subset. Therefore, we get

$$\begin{aligned} (\mathcal{X}_N)^{aM} \cap (\mathcal{X}_N)^{Rm} &= (\mathcal{X}_N)^{Rm \cap aM} \subseteq \mathcal{X}_N \setminus \tilde{V}(am) \\ &\subseteq \mathcal{X}_N \setminus \tilde{V}(\mathcal{N}_N(0)) \\ &= Spec(M) \setminus (V(\mathcal{N}_N(0)) \cup V(N)) = \emptyset. \end{aligned}$$

This contradicts the hypothesis. Thus, $\mathcal{N}_N(0)$ is a prime submodule of M . □

We now need a condition on the submodules \mathcal{N}_N , which helps us out with going further in finding more connections between topological space and module.

A module M is said to satisfy \mathcal{T} -condition for a submodule N , if for any chain $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \subseteq \dots$, where U_i is an ideal of R , there is an integer m such that $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i .

Theorem 2.9 *Let $N = IM$ be a proper submodule of a multiplication R -module M , where I is an ideal of R . Then the following statements are equivalent:*

- i) M satisfies the \mathcal{T} -condition.*
- ii) \mathcal{X}_N is a Noetherian topological space.*

Proof (i) \Rightarrow (ii) Assume that M satisfies the \mathcal{T} -condition. Take the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \dots$, where U_iM is a submodule of M . Then we have the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \subseteq \dots$ and there exists an integer m such that $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i since M satisfies the \mathcal{T} -condition. Therefore, we have $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i . Thus, \mathcal{X}_N is Noetherian.

(ii) \Rightarrow (i) Let \mathcal{X}_N be a Noetherian topological space. Take the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \dots$, where U_iM is a submodule of M . Then this yields the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \dots$. Since \mathcal{X}_N is Noetherian, there exists an integer m such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i . This implies $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i . Therefore M satisfies the \mathcal{T} -condition. \square

We close this section with the following theorem, which reveals the connections between algebraic and topological properties.

Theorem 2.10 *Let $N = IM$ be a submodule of a multiplication R -module M , where I is an ideal of R . Then the following are equivalent:*

- i) \mathcal{X} is a Noetherian topological space.*
- ii) \mathcal{X}_N is a Noetherian topological space for every submodule N of M .*
- iii) M satisfies the \mathcal{T} -condition.*
- iv) M satisfies ascending chain condition on the radical submodules of M .*

Proof (i) \Rightarrow (ii), (ii) \Leftrightarrow (iii) and (iv) \Leftrightarrow (i) are clear.

(ii) \Rightarrow (i) Take the sequence $V(U_1M) \supseteq V(U_2M) \supseteq V(U_3M) \supseteq \dots$, where U_i is an ideal of R . Let $I = \cap U_i$ be an ideal of R . Consider the complement Zariski topology \mathcal{X}_{IM} . Then we have the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \dots$. Since \mathcal{X}_N is Noetherian, there exists an integer m such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i . Thus, we have $V(U_mM) = V(U_{m+i}M)$ for all positive integers i . Thus, \mathcal{X} is Noetherian. \square

3. The connections between subspaces and submodules

This section deals with the relationships between the complement Zariski topologies and submodules of a module to find some algebraic and topological tools for submodules and find some characterizations for modules.

Theorem 3.1 *Let M be a multiplication R -module and let I, J , and K be proper ideals of R . Then we have the following.*

- i) Any open set of \mathcal{X} is of the form \mathcal{X}_{IM} .*
- ii) $\mathcal{X}_{IM} = \mathcal{X}_{JM}$ if and only if $rad_M(IM) = rad_M(JM)$.*
- iii) $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} = \mathcal{X}_{KM}$ if and only if $rad_M(IJM) = rad_M(KM)$.*

iv) $\mathcal{X}_{IM} \subseteq \mathcal{X}_{JM}$ if and only if $rad_M(IM) \subseteq rad_M(JM)$.

Proof It is straightforward. □

Corollary 3.2 Let M be a multiplication R -module and let I, J be proper ideals of R . Then $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} = \emptyset$ if and only if $rad_M(IJM) = rad_M(0)$.

Theorem 3.3 Let M be a multiplication R -module and let I be a proper ideal of R . Then \mathcal{X}_{IM} is dense in \mathcal{X} if and only if $rad_M(IJM) \neq rad_M(0)$ for every proper ideal J such that JM is not contained in $rad_M(0)$.

Proof Let \mathcal{X}_{IM} be dense in \mathcal{X} and let J be any proper ideal of R , where JM is not in $rad_M(0)$. Then $\mathcal{X}_{JM} = Spec(M) \setminus V(JM)$ is a nonempty open set in the Zariski topology and by the hypothesis, the intersection of \mathcal{X}_{IM} and \mathcal{X}_{JM} is nonempty. Thus, $rad_M(IJM) \neq rad_M(0)$ by Corollary 3.2.

Let $rad_M(IJM) \neq rad_M(0)$ for every proper ideal J of R , where JM is not in $rad_M(0)$. By Corollary 3.2, since $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} \neq \emptyset$, it follows that \mathcal{X}_{IM} is dense in \mathcal{X} . □

The following theorem gives a characterization for the module $M/rad_M(0)$ by using topological properties.

Theorem 3.4 Let M be a faithful multiplication R -module. The following statements are equivalent:

- i) $rad_M(0)$ is a prime submodule of M .
- ii) $Spec(M)$ is irreducible.
- iii) Every submodule of $M/rad_M(0)$ is essential.
- iv) Every open subset of $Spec(M)$ is dense.

Proof (i) \Leftrightarrow (ii) By [5], it can be easily proved.

(iii) \Rightarrow (iv) Let \mathcal{X}_{IM} and \mathcal{X}_{JM} be open subsets for any ideals I, J of R . Then $(JM + rad_M(0))/rad_M(0)$ and $(IM + rad_M(0))/rad_M(0)$ are submodules of $M/rad_M(0)$. Since $\bigcap_{i \in \Lambda} (I_i M) = \left(\bigcap_{i \in \Lambda} I_i \right) M$, we observe that

$$rad_M \left(\bigcap_{i \in \Lambda} (I_i M) \right) = rad_M \left(\left(\bigcap_{i \in \Lambda} I_i \right) M \right) = \left(rad_R \left(\bigcap_{i \in \Lambda} I_i \right) \right) M, \text{ then}$$

$$\begin{aligned} rad_M(0) &\neq rad_M [(JM + rad_M(0)) \cap (IM + rad_M(0))] \\ &= rad_M [(J + rad_R(0))M \cap (I + rad_R(0))M] \\ &= rad_M [(J + rad_R(0)) \cap (I + rad_R(0))M] \\ &= rad_R [IJ + rad_R(0)] M \\ &= rad_M [IJM + rad_M(0)] \end{aligned}$$

and so $rad_M(IJM) \neq rad_M(0)$, which means that \mathcal{X}_{IM} is dense.

(iv) \Rightarrow (ii) \Rightarrow (iii) One can prove it by the above method. □

Theorem 3.5 Let M be a finitely generated multiplication R -module and let I_i be a proper ideal of R for all $i \in \Lambda$. Then $\bigcup_{i \in \Lambda} \mathcal{X}_{I_i M} = \mathcal{X}_{DM}$ for any ideal D of R if and only if $rad_M(DM) = rad_M \left(\left(\sum_{i \in \Lambda} I_i \right) M \right)$.

Proof Assume that $\bigcup_{i \in \Lambda} \mathcal{X}_{I_i M} = \mathcal{X}_{DM}$ for any ideal D of R .

Let P be a prime submodule of M such that $DM \subseteq P$. Then $P \notin \mathcal{X}_{DM}$ and so $P \notin \bigcup_{i \in \Lambda} \mathcal{X}_{I_i M}$, implying $P \notin \mathcal{X}_{I_i M}$ for all i . Then $I_i M \subseteq P$ for all i and so $\left(\sum_{i \in \Lambda} I_i\right)M \subseteq P$, which implies $rad_M\left(\left(\sum_{i \in \Lambda} I_i\right)M\right) \subseteq rad_M(DM)$.

Let $\left(\sum_{i \in \Lambda} I_i\right)M \subseteq P$, where P is a prime submodule of M . Then $I_i M \subseteq P$ for all i , implying $P \notin \mathcal{X}_{I_i M}$ for all i . Thus, $P \notin \bigcup_{i \in \Lambda} \mathcal{X}_{I_i M} = \mathcal{X}_{DM}$ and so $DM \subseteq P$. This means that $rad_M(DM) \subseteq rad_M\left(\left(\sum_{i \in \Lambda} I_i\right)M\right)$.

Assume that $rad_M(DM) = rad_M\left(\left(\sum_{i \in \Lambda} I_i\right)M\right)$.

Let $P \in \mathcal{X}_{DM}$. Then $DM \not\subseteq P$ and $rad_M\left(\left(\sum_{i \in \Lambda} I_i\right)M\right) \not\subseteq P$ and so $\left(\sum_{i \in \Lambda} I_i\right)M \not\subseteq P$, implying $I_i M \not\subseteq P$ for all i . Then $P \in \mathcal{X}_{I_i M}$ and so $P \in \bigcup_{i \in \Lambda} \mathcal{X}_{I_i M}$.

The converse can be proved with the above method, which completes the proof. □

The following corollary is a special case of Theorem 3.5.

Theorem 3.6 *Let M be a finitely generated multiplication R -module, I_i proper ideals of R for all $i \in \Lambda$ and D a finitely generated ideal of R . Then the following statements are equivalent:*

i) $\bigcup_{i \in \Lambda} \mathcal{X}_{I_i M} = \mathcal{X}_{DM}$.

ii) There is a finite subset Δ of Λ such that $\bigcup_{i \in \Delta} \mathcal{X}_{I_i M} = \mathcal{X}_{DM}$.

iii) There is a finite subset Δ of Λ such that $rad_M\left(\left(\sum_{i \in \Delta} I_i\right)M\right) = rad_M(DM)$.

Proof (i) \Rightarrow (iii) Let $rad_M(DM) = rad_M\left(\left(\sum_{i \in \Lambda} I_i\right)M\right)$ and let D be an ideal finitely generated by the set $\{d_1, \dots, d_t\}$. For each $d_i M$, there is a positive number n_i such that $d_i^{n_i} M \subseteq \left(\sum_{i \in \Lambda} I_i\right)M$ and so there is a finite subset Δ_i of Λ such that $d_i^{n_i} M \subseteq \left(\sum_{i \in \Delta_i} I_i\right)M$. If $n = \max\{n_1, \dots, n_t\}$ and $\Delta = \bigcup_{i=1}^t \Delta_i$ then

$$rad_M\left(\left(\sum_{i \in \Delta} I_i\right)M\right) = rad_M(DM).$$

(iii) \Rightarrow (ii) By Theorem 3.5.

(ii) \Rightarrow (i) It is clear. □

The following corollary is a special case of Theorem 3.6.

Corollary 3.7 *Let M be a finitely generated multiplication R -module and let I_i be a proper ideal of R for all $i \in \Lambda$. Then the following statements are equivalent:*

i) $\bigcup_{i \in \Lambda} \mathcal{X}_{I_i M} = \text{Spec}(M)$.

ii) There is a finite subset Δ of Λ such that $\bigcup_{i \in \Delta} \mathcal{X}_{I_i M} = \text{Spec}(M)$.

iii) There is a finite subset Δ of Λ such that $\left(\sum_{i \in \Delta} I_i\right) M = M$.

Corollary 3.8 *Let M be a finitely generated multiplication R -module and let D be a finitely generated ideal of R . Then \mathcal{X}_{DM} is quasicompact.*

Using topological properties, we are now ready to prove the following characterization for $\text{rad}_M(0)$.

Theorem 3.9 *Let M be a finitely generated multiplication R -module satisfying the \mathcal{T} -condition for every submodule. Then there are proper ideals I_1, \dots, I_n of R such that $\text{rad}_M(0) = \text{rad}_M((I_1 \dots I_n) M)$.*

Proof Let $\mathcal{X} = \text{Spec}(M)$ be Noetherian topological space. By [9], \mathcal{X} has only a finite number of distinct irreducible components U_i such that $\bigcup_{i=1}^n U_i = \mathcal{X}$. It is well known that any irreducible component in a topological space is closed and so for each i , there is an ideal I_i such that $U_i = V(I_i M)$. Then

$$\emptyset = \mathcal{X} \setminus \bigcup_{i=1}^n V(I_i M) = \bigcap_{i=1}^n (\mathcal{X} \setminus V(I_i M)) = \bigcap_{i=1}^n \mathcal{X}_{I_i M}.$$

Thus, by Theorem 3.1, $\text{rad}_M(0) = \bigcap_{i=1}^n \text{rad}_M(I_i M) = \text{rad}_M((I_1 \dots I_n) M)$. □

By using Theorems 2.8 and 3.5, we close the paper with the following result.

Theorem 3.10 *Let M be a finitely generated multiplication R -module and let I_i be an ideal of R . Then $\mathcal{X} = \bigcup_{i=1}^n \mathcal{X}_{I_i M}$, where $\mathcal{X}_{I_i M}$ is irreducible, if and only if $M = \left(\sum_{i=1}^n I_i\right) M$ and $\mathcal{N}_{I_i M}(0)$ is a prime submodule of M .*

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