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An integral-boundary value problem for a partial differential equation of second order

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Abstract: An integral-boundary value problem for a hyperbolic partial differential equation in two independent variables is considered. By introducing additional functional parameters, we investigate the solvability of the problem and develop an algorithm for finding its approximate solutions. The problem is reduced to an equivalent one, consisting of the Goursat problem for a hyperbolic equation with parameters and boundary value problems with an integral condition for ODEs with respect to the parameters entered. We propose an algorithm to find an approximate solution to the original problem, which is based on the algorithm for finding a solution to the equivalent problem. The convergence of the algorithms is proved. A coefficient criterion for the unique solvability of the integral-boundary value problem is established.

Key words: Hyperbolic equation of second order, integral-boundary value problem, parameter, algorithm, approximate solution

1. Introduction

On the domain $\Omega = [0,T]\times[0,\omega]$, we consider the integral-boundary value problem for the hyperbolic equation of second order

$$\frac{\partial^2 u}{\partial t\partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{1}$$

$$P(t)u(t,0) + \int_0^a K(t,\xi)u(t,\xi)d\xi = \psi(t), \qquad t\in[0,T], \tag{2}$$

$$S(x)u(0,x) + \int_0^b M(\tau,x)u(\tau,x)d\tau = \varphi(x), \qquad x\in[0,\omega], \tag{3}$$

where $u(t,x)$ is an unknown function; the functions $A(t,x)$, $B(t,x)$, $C(t,x)$, and $f(t,x)$ are continuous on Ω; the functions $K(t,x)$ and $P(t)$, $\psi(t)$, are continuously differentiable with respect to t on Ω and $[0,T]$, respectively; the functions $M(t,x)$ and $S(x)$, $\varphi(x)$, are continuously differentiable with respect to x on Ω and $[0,\omega]$, respectively; and $0 < a \le \omega$, $0 < b \le T$. The compatibility condition is given below.

Let $C(\Omega,R)$ be a space of continuous functions $u:\Omega\to R$ with norm $||u||_0 = \max_{(t,x)\in\Omega}|u(t,x)|$.

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A function $u(t,x) \in C(\Omega, R)$, whose partial derivatives $\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial u(t,x)}{\partial t}$, and $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ belong to $C(\Omega, R)$, is called a classical solution to problem (1)–(3), if it satisfies equation (1) and integral conditions (2) and (3).

Integral-boundary value problems for hyperbolic equations often arise in the mathematical modeling of processes of heat distribution, plasma physics, moisture transfer in capillary-porous media, clean technology of silicon ores, etc. [8–15,17–26,28–31]. Some types of integral-boundary value problems for hyperbolic equations were studied in [8–31]. Solvability conditions for these problems are established in different terms. Problem (1)–(3) for $P(t) = 0$, $S(x) = 0$, $K(t,x) = M(t,x)$, and $K(t,x) = M(t,x) = 1$ was investigated in [10,13,22–26]. Under the assumption of continuous differentiability of the equation coefficients, the conditions for the unique solvability of that problem have been obtained. In [31], the contractive mapping principle is used to study problem (1)–(3) for $P(t) = 0$, $S(x) = 0$ and $K(t,x) = K(x)$, $M(t,x) = M(t)$. In [7], the unique solvability conditions for the problem, where $P(t) = 0$, $S(x) = 0$, and $K(t,x) = M(t,x)$, were established in terms of initial data.

A boundary value problem for hyperbolic equations subject to general integral conditions is one of the rarely studied problems of mathematical physics. This formulation of the problem is considered for the first time.

The aim of this work is to develop an algorithm for finding a solution to problem (1)–(3) and establish conditions for the existence and uniqueness of its classical solution.

In Section 2, a scheme of the method from [4–6] is presented, which will be used to investigate the problem. We introduce new unknown functions as linear combinations of the solutions' values on the characteristics. The problem (1)–(3) is reduced to an equivalent one consisting of the Goursat problem for hyperbolic equations with functional parameters and boundary value problems with integral conditions for ODEs with respect to the parameters entered. An algorithm for finding an approximate solution to the problem under investigation is proposed. The algorithm consists of two parts. First, we solve two boundary value problems with integral conditions for ODEs. Such problems have been intensively studied in recent years [1–3] and they occur in numerous areas of applied mathematics. In the second part of the algorithm, we solve the Goursat problem for a hyperbolic equation with parameters. In Section 3, the conditions for the existence of a unique solution to the boundary value problem with integral condition for ODEs are provided. In Section 3, the convergence of the algorithm is proved, and the conditions for the unique solvability of problem (1)–(3) are given in terms of initial data.

2. The description of the method and the algorithm

We introduce the following notations: $\mu(t) = u(t,0) - \frac{1}{2}u(0,0)$, $\lambda(x) = u(0,x) - \frac{1}{2}u(0,0)$, $\widetilde{u}(t,x)$, where the latter is a new unknown function.

We make the following replacement of desired function $u(t,x)$ in problem (1)–(3):

$u(t,x) = \widetilde{u}(t,x) + \mu(t) + \lambda(x)$, and we move to the problem:

$$\frac{\partial^2 \widetilde{u}}{\partial t \partial x} = A(t,x)\frac{\partial \widetilde{u}}{\partial x} + B(t,x)\frac{\partial \widetilde{u}}{\partial t} + C(t,x)\widetilde{u} + A(t,x)\dot{\lambda}(x) + B(t,x)\dot{\mu}(t) + C(t,x)\lambda(x) + C(t,x)\mu(t) + f(t,x), \quad (4)$$

$$\widetilde{u}(t,0) = 0, \qquad t \in [0,T], \quad (5)$$

$$\widetilde{u}(0,x) = 0, \qquad x \in [0,\omega], \quad (6)$$

$$\left[P(t)+\int_0^a K(t,\xi)d\xi\right]\mu(t)+P(t)\lambda(0)+\int_0^a K(t,\xi)\lambda(\xi)d\xi+\int_0^a K(t,\xi)\widetilde{u}(t,\xi)d\xi=\psi(t), \qquad t\in[0,T], \quad (7)$$

$$\left[S(x)+\int_0^b M(\tau,x)d\tau\right]\lambda(x)+S(x)\mu(0)+\int_0^b M(\tau,x)\mu(\tau)d\tau+\int_0^b M(\tau,x)\widetilde{u}(\tau,x)d\tau=\varphi(x), \qquad x\in[0,\omega]. \quad (8)$$

A triple of functions $(\widetilde{u}(t,x),\mu(t),\lambda(x))$ satisfying the hyperbolic equation (4), the conditions on characteristics (5) and (6), and the functional relations (7) and (8) for $\mu(0)=\lambda(0)$ is called a solution to problem (4)–(8) if the function $\widetilde{u}(t,x)\in C(\Omega,R)$ has the partial derivatives $\dfrac{\partial\widetilde{u}(t,x)}{\partial x}$, $\dfrac{\partial\widetilde{u}(t,x)}{\partial t}$, $\dfrac{\partial^2\widetilde{u}(t,x)}{\partial t\partial x}$ in $C(\Omega,R)$, and the functions $\mu(t)$ and $\lambda(x)$ are continuously differentiable on $[0,T]$ and $[0,\omega]$, respectively.

The relation $\mu(0)=\lambda(0)$ is a compatibility condition of data.

Problem (4)–(8) is equivalent to problem (1)–(3). If the function $u^*(t,x)$ is a solution to problem (1)–(3), then the triple of functions $(\widetilde{u}^*(t,x),\ \mu^*(t),\ \lambda^*(x))$, where $\widetilde{u}^*(t,x)=u^*(t,x)-\mu^*(t)-\lambda^*(x)$, $\mu^*(t)=u^*(t,0)-\frac{1}{2}u^*(0,0)$, $\lambda^*(x)=u^*(0,x)-\frac{1}{2}u^*(0,0)$, is a solution to problem (4)–(8). The converse is also true. If the triple of functions $(\widetilde{u}^{**}(t,x),\mu^{**}(t),\ \lambda^{**}(x))$ is a solution to problem (4)–(8), then the function $u^{**}(t,x)$ defined by the equality

$$u^{**}(t,x)=\widetilde{u}^{**}(t,x)+\mu^{**}(t)+\lambda^{**}(x),$$

where $u^{**}(t,0)-\frac{1}{2}u^{**}(0,0)=\mu^{**}(t)$, $u^{**}(0,x)-\frac{1}{2}u^{**}(0,0)=\lambda^{**}(x)$, is a solution to problem (1)–(3).

For fixed $\mu(t)$, $\lambda(x)$, problem (4)–(6) is the Goursat problem with respect to the function $\widetilde{u}(t,x)$ on the domain Ω. Relations (7) and (8) allow us to determine the unknown parameters $\mu(t)$, $\lambda(x)$ satisfying the condition $\mu(0)=\lambda(0)$.

By virtue of conditions (5) and (6), relations (7) at $t=0$ and (8) at $x=0$ yield

$$\left[P(0)+\int_0^a K(0,\xi)d\xi\right]\mu(0)+P(0)\lambda(0)+\int_0^a K(0,\xi)\lambda(\xi)d\xi=\psi(0), \quad (9)$$

$$\left[S(0)+\int_0^b M(\tau,0)d\tau\right]\lambda(0)+S(0)\mu(0)+\int_0^b M(\tau,0)\mu(\tau)d\tau=\varphi(0). \quad (10)$$

Taking into account $\lambda(0)=\mu(0)$, we get

$$\left[2P(0)+\int_0^a K(0,\xi)d\xi\right]\lambda(0)+\int_0^a K(0,\xi)\lambda(\xi)d\xi=\psi(0), \quad (11)$$

$$\left[2S(0)+\int_0^b M(\tau,0)d\tau\right]\mu(0)+\int_0^b M(\tau,0)\mu(\tau)d\tau=\varphi(0). \quad (12)$$

Assumptions on the data of problem (1)–(3) allow us to differentiate (7) and (8) with respect to t and x, respectively. We then obtain

$$\left[P(t)+\int_0^a K(t,\xi)d\xi\right]\dot{\mu}(t)=-\left[\dot{P}(t)+\int_0^a\frac{\partial K(t,\xi)}{\partial t}d\xi\right]\mu(t)-\int_0^a\frac{\partial K(t,\xi)}{\partial t}\widetilde{u}(t,\xi)d\xi-$$

$$-\int_0^a K(t,\xi)\frac{\partial\widetilde{u}(t,\xi)}{\partial t}d\xi - \dot{P}(t)\lambda(0) - \int_0^a \frac{\partial K(t,\xi)}{\partial t}\lambda(\xi)d\xi + \dot{\psi}(t), \qquad t\in[0,T], \tag{13}$$

$$\left[S(x)+\int_0^b M(\tau,x)d\tau\right]\dot{\lambda}(x) = -\left[\dot{S}(x)+\int_0^b \frac{\partial M(\tau,x)}{\partial x}d\tau\right]\lambda(x) - \int_0^b \frac{\partial M(\tau,x)}{\partial x}\widetilde{u}(\tau,x)d\tau -$$

$$-\int_0^b M(\tau,x)\frac{\partial\widetilde{u}(\tau,x)}{\partial x}d\tau - \dot{S}(x)\mu(0) - \int_0^b \frac{\partial M(\tau,x)}{\partial x}\mu(\tau)d\tau + \dot{\varphi}(x), \qquad x\in[0,\omega]. \tag{14}$$

We introduce new unknown functions $\widetilde{v}(t,x) = \dfrac{\partial\widetilde{u}(t,x)}{\partial x}$, $\widetilde{w}(t,x) = \dfrac{\partial\widetilde{u}(t,x)}{\partial t}$ and the following notations:

$B_1(t) = P(t) + \int_0^a K(t,\xi)d\xi$, $\;B_2(x) = S(x) + \int_0^b M(\tau,x)d\tau$, $\;C_1(t) = \dot{P}(t) + \int_0^a \dfrac{\partial K(t,\xi)}{\partial t}d\xi$,

$C_2(x) = \dot{S}(x) + \int_0^b \dfrac{\partial M(\tau,x)}{\partial x}d\tau$, $\;G_1(t,\widetilde{u},\widetilde{w}) = \int_0^a \dfrac{\partial K(t,\xi)}{\partial t}\widetilde{u}(t,\xi)d\xi + \int_0^a K(t,\xi)\widetilde{w}(t,\xi)d\xi$,

$G_2(x,\widetilde{u},\widetilde{v}) = \int_0^b \dfrac{\partial M(\tau,x)}{\partial x}\widetilde{u}(\tau,x)d\tau + \int_0^b M(\tau,x)\widetilde{v}(\tau,x)d\tau$,

$L_1(t,\lambda) = \dot{P}(t)\lambda(0) + \int_0^a \dfrac{\partial K(t,\xi)}{\partial t}\lambda(\xi)d\xi$, $\;L_2(x,\mu) = \dot{S}(x)\mu(0) + \int_0^b \dfrac{\partial M(\tau,x)}{\partial x}\mu(\tau)d\tau$.

Then equations (13) and (14) can be written in the following forms:

$$B_1(t)\dot{\mu}(t) = -C_1(t)\mu(t) - G_1(t,\widetilde{u},\widetilde{w}) - L_1(t,\lambda) + \dot{\psi}(t), \qquad t\in[0,T], \tag{15}$$

$$B_2(x)\dot{\lambda}(x) = -C_2(x)\lambda(x) - G_2(x,\widetilde{u},\widetilde{v}) - L_2(x,\mu) + \dot{\varphi}(x), \qquad x\in[0,\omega]. \tag{16}$$

Thus, we have a closed system of equations (4)–(6), (15), (12), (16), and (11) for determining unknown functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$, $\dot{\lambda}(x)$, $\lambda(x)$, $\dot{\mu}(t)$, and $\mu(t)$.

Relation (15) in conjunction with (12) presents a boundary value problem with integral condition for a differential equation with respect to $\mu(t)$, and the relation (16) in conjunction with (11) presents a boundary value problem with integral condition for a differential equation with respect to $\lambda(x)$.

The boundary value problem with integral condition (15) and (12) is equivalent to relation (7), and the boundary value problem with integral condition (16) and (11) is equivalent to relation (8) for $\mu(0)=\lambda(0)$.

If $\dot{\mu}(t)$, $\dot{\lambda}(x)$, $\mu(t)$, $\lambda(x)$ are known, we can find the functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$ from (4)–(6). Conversely, if we know the functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$, then we can find $\dot{\mu}(t)$, $\mu(t)$, $\dot{\lambda}(x)$, $\lambda(x)$ from boundary value problems (15), (12) and (16), (11). The unknowns are both $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$, and $\dot{\mu}(t)$, $\mu(t)$, $\dot{\lambda}(x)$, $\lambda(x)$. Therefore, to find a solution of problem (4)–(8), we use an iterative method: determine the triple $(\widetilde{u}^*(t,x),\ \mu^*(t),\ \lambda^*(x))$ as the limit of the sequence $(\widetilde{u}^{(m)}(t,x),\ \mu^{(m)}(t)),\ \lambda^{(m)}(x))$, $m=0,1,2,\ldots$, according to the following algorithm:

Step 0. 1) Assuming $\widetilde{u}(t,x)=0$, $\widetilde{w}(t,x)=0$, $\lambda(x)=0$, on the right-hand side of equation (15), we find initial approximations $\dot{\mu}^{(0)}(t)$, $\mu^{(0)}(t)$, $t\in[0,T]$, from the boundary value problem with integral condition (15) and (12). Assuming $\widetilde{u}(t,x)=0$, $\widetilde{v}(t,x)=0$, $\mu(t)=0$ on the right-hand side of equation (16), we find initial approximations $\dot{\lambda}^{(0)}(x)$, $\lambda^{(0)}(x)$, $x \in [0,\omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\widetilde{v}^{(0)}(t,x)$, $\widetilde{w}^{(0)}(t,x)$, $\widetilde{u}^{(0)}(t,x)$, $(t,x) \in \Omega$ from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(0)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(0)}(t)$, $\lambda(x) = \lambda^{(0)}(x)$, $\mu(t) = \mu^{(0)}(t)$.

Step 1. 1) Assuming $\widetilde{u}(t,x) = \widetilde{u}^{(0)}(t,x)$, $\widetilde{w}(t,x) = \widetilde{w}^{(0)}(t,x)$, $\lambda(x) = \lambda^{(0)}(x)$ on the right-hand side of equation (15), we find $\dot{\mu}^{(1)}(t)$, $\mu^{(1)}(t)$, $t \in [0,T]$ from the boundary value problem with integral condition (15) and (12). Assuming $\widetilde{u}(t,x) = \widetilde{u}^{(0)}(t,x)$, $\widetilde{v}(t,x) = \widetilde{v}^{(0)}(t,x)$, $\mu(t) = \mu^{(0)}(t)$ on the right-hand side of equation (16), we find $\dot{\lambda}^{(1)}(x)$, $\lambda^{(1)}(x)$, $x \in [0,\omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\widetilde{v}^{(1)}(t,x), \widetilde{w}^{(1)}(t,x), \widetilde{u}^{(1)}(t,x)$, $(t,x) \in \Omega$ from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(1)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(1)}(t)$, $\lambda(x) = \lambda^{(1)}(x)$, $\mu(t) = \mu^{(1)}(t)$, and so on.

Step m. 1) Assuming $\widetilde{u}(t,x) = \widetilde{u}^{(m-1)}(t,x)$, $\widetilde{w}(t,x) = \widetilde{w}^{(m-1)}(t,x)$, $\lambda(x) = \lambda^{(m-1)}(x)$ on the right-hand side of equation (15), we find $\dot{\mu}^{(m)}(t)$, $\mu^{(m)}(t)$, $t \in [0,T]$, from the boundary value problem with integral condition (15) and (12). Assuming $\widetilde{u}(t,x) = \widetilde{u}^{(m-1)}(t,x)$, $\widetilde{v}(t,x) = \widetilde{v}^{(m-1)}(t,x)$, $\mu(t) = \mu^{(m-1)}(t)$ on the right-hand side of equation (16), we find $\dot{\lambda}^{(m)}(x)$, $\lambda^{(m)}(x)$, $x \in [0,\omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\widetilde{v}^{(m)}(t,x)$, $\widetilde{w}^{(m)}(t,x)$, $\widetilde{u}^{(m)}(t,x)$, $(t,x) \in \Omega$, from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(m)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(m)}(t)$, $\lambda(x) = \lambda^{(m)}(x)$, $\mu(t) = \mu^{(m)}(t)$, $m = 1, 2, \ldots$.

The constructed algorithm consists of two parts: we solve the boundary value problems with integral condition for the ordinary differential equations (15), (12) and (16), (11) in the first part, and we solve the Goursat problem for hyperbolic equations with functional parameters in the second part.

3. Boundary value problems with integral conditions for the ordinary differential equations

Consider the boundary value problem with integral condition for the ordinary differential equations

$$\dot{\mu}(t) = A_1(t)\mu(t) + g_1(t), \qquad t \in [0,T], \tag{17}$$

$$\left[2S(0) + \int_0^b M(\tau,0)d\tau\right]\mu(0) + \int_0^b M(\tau,0)\mu(\tau)d\tau = \varphi(0), \tag{18}$$

where the functions $A_1(t)$ and $g_1(t)$ are continuous on $[0,T]$, the function $M(t,x)$ is continuous on Ω, and $S(0)$ and $\varphi(0)$ are some constants, $0 < b \le T$.

The function $\mu(t) \in C([0,T],R)$ having the derivative $\dot{\mu}(t) \in C([0,T],R)$ is called a solution to problem (17)–(18), if it satisfies ordinary differential equation (17) and boundary condition (18).

We also consider the boundary value problem with integral condition for the ordinary differential equation of the following type:

$$\dot{\lambda}(x) = A_2(x)\lambda(x) + g_2(x), \qquad x \in [0,\omega], \tag{19}$$

$$\left[2P(0) + \int_0^a K(0,\xi)d\xi\right]\lambda(0) + \int_0^a K(0,\xi)\lambda(\xi)d\xi = \psi(0), \tag{20}$$

where the functions $A_2(x)$ and $g_2(x)$ are continuous on $[0,\omega]$, the function $K(t,x)$ is continuous on Ω, and $P(0)$ and $\psi(0)$ are some constants, $0 < a \leq \omega$.

The function $\lambda(x) \in C([0,\omega],R)$ having the derivative $\dot{\lambda}(x) \in C([0,\omega],R)$ is called a solution to problem (19)–(20), if it satisfies ordinary differential equation (19) and boundary condition (20).

Below we give conditions for the unique solvability of boundary value problems with integral condition (17), (18) and (19), (20).

Suppose

$$a_1(t) = \int_0^t A_1(\tau)d\tau, \qquad a_2(x) = \int_0^x A_2(\xi)d\xi, \qquad \alpha_1 = \max_{t\in[0,T]} |A_1(t)|, \qquad \alpha_2 = \max_{x\in[0,\omega]} |A_2(x)|,$$

$$\tilde{a}_1 = 2S(0) + \int_0^b M(\tau,0)\big[1+e^{a_1(\tau)}\big]d\tau, \qquad \tilde{a}_2 = 2P(0) + \int_0^a K(0,\xi)\big[1+e^{a_2(\xi)}\big]d\xi.$$

Assume that $|\tilde{a}_1| \geq \delta_1 > 0$, $|\tilde{a}_2| \geq \delta_2 > 0$.

Then the solutions to problems (17), (18) and (19), (20) can be written in the following forms:

$$\mu(t) = \frac{e^{a_1(t)}}{\tilde{a}_1}\cdot\varphi(0) - \frac{e^{a_1(t)}}{\tilde{a}_1}\cdot\int_0^b M(\tau,0)e^{a_1(\tau)}\int_0^\tau e^{-a_1(\tau_1)}g_1(\tau_1)d\tau_1 d\tau + e^{a_1(t)}\int_0^t e^{-a_1(\tau)}g_1(\tau)d\tau, \quad t\in[0,T], \tag{21}$$

and

$$\lambda(x) = \frac{e^{a_2(x)}}{\tilde{a}_2}\cdot\psi(0) - \frac{e^{a_2(x)}}{\tilde{a}_2}\cdot\int_0^a K(0,\xi)e^{a_2(\xi)}\int_0^\xi e^{-a_2(\xi_1)}g_2(\xi_1)d\xi_1 d\xi + e^{a_2(x)}\int_0^x e^{-a_2(\xi)}g_2(\xi)d\xi, \quad x\in[0,\omega], \tag{22}$$

respectively. Relations (21) and (22) follow from the representation of the solution to the Cauchy problem for the ordinary differential equations according to the qualitative theory of differential equations.

The following statements are true.

Theorem 1 *Suppose that* $|\tilde{a}_1| = \left|2S(0) + \int_0^b M(\tau,0)\big[1+e^{a_1(\tau)}\big]d\tau\right| \geq \delta_1 > 0$, *where* $a_1(t) = \int_0^t A_1(\tau)d\tau$.

Then problem (17)–(18) has a unique solution $\mu^*(t) \in C([0,T],R)$ *representable in the form of (21), and the estimate*

$$\max_{t\in[0,T]} |\mu^*(t)| \leq \mathcal{K}_1 \max\Big(\max_{t\in[0,T]} |g_1(t)|, |\varphi(0)|\Big) \tag{23}$$

holds, where $\mathcal{K}_1 = \frac{1}{\delta_1}e^{\alpha_1 T}\Big[1 + \max_{t\in[0,b]} |M(t,0)|\cdot\frac{b^2}{2}e^{\alpha_1 b}\Big] + Te^{\alpha_1 T}$.

Theorem 2 *Suppose that* $|\tilde{a}_2| = \left|2P(0) + \int_0^a K(0,\xi)\big[1+e^{a_2(\xi)}\big]d\xi\right| \geq \delta_2 > 0$, *where* $a_2(x) = \int_0^x A_2(\xi)d\xi$.

Then problem (19)–(20) has a unique solution $\lambda^*(x) \in C([0,\omega],R)$ *representable in the form of (22), and the estimate*

$$\max_{x\in[0,\omega]} |\lambda^*(x)| \leq \mathcal{K}_2 \max\Big(\max_{x\in[0,\omega]} |g_2(x)|, |\psi(0)|\Big) \tag{24}$$

holds, where $\mathcal{K}_2 = \frac{1}{\delta_2}e^{\alpha_2\omega}\Big[1 + \max_{x\in[0,a]} |K(0,x)|\cdot\frac{a^2}{2}e^{\alpha_2 a}\Big] + \omega e^{\alpha_2\omega}$.

4. Conditions for convergence of the algorithm and the main result

In Section 2, an algorithm for finding a solution to problem (4)–(8), which is equivalent to problem (1)–(3), is constructed. To formulate the main result, assume that

$B_1(t) = P(t) + \int_0^a K(t,\xi)d\xi \neq 0$ for all $t \in [0,T]$ and $B_2(x) = S(x) + \int_0^b M(\tau,x)d\tau \neq 0$ for all $x \in [0,\omega]$.

We introduce the following notations:

$$\alpha = \max_{(t,x)\in\Omega} |A(t,x)|, \quad \beta = \max_{(t,x)\in\Omega} |B(t,x)|, \quad \gamma = \max_{(t,x)\in\Omega} |C(t,x)|, \quad H = \alpha+\beta+\gamma, \quad \beta_1 = \max_{t\in[0,T]} |[B_1(t)]^{-1}|,$$

$$\beta_2 = \max_{x\in[0,\omega]} |[B_2(x)]^{-1}|, \quad A_1(t) = -[B_1(t)]^{-1}C_1(t), \quad A_2(x) = -[B_2(x)]^{-1}C_2(x), \quad \kappa_1 = \max_{(t,x)\in\Omega} |K(t,x)|,$$

$$\kappa_2 = \max_{(t,x)\in\Omega} \Big|\frac{\partial K(t,x)}{\partial t}\Big|, \quad \sigma_1 = \max_{(t,x)\in\Omega} |M(t,x)|, \quad \sigma_2 = \max_{(t,x)\in\Omega} \Big|\frac{\partial M(t,x)}{\partial x}\Big|, \quad \chi_1 = \max_{t\in[0,T]} |\dot{P}(t)|, \quad \chi_2 = \max_{x\in[0,\omega]} |\dot{S}(x)|,$$

$$l_1(a) = \beta_1\Big\{a(\kappa_1+\kappa_2)\max(T,\omega)e^{H(T+\omega)} + a\kappa_2 + \chi_1\Big\}, \quad l_2(b) = \beta_2\Big\{b(\sigma_1+\sigma_2)\max(T,\omega)e^{H(T+\omega)} + b\sigma_2 + \chi_2\Big\}.$$

In Section 3, the conditions for unique solvability of boundary value problems with integral condition (17), (18) and (19), (20) are established. For fixed $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$ in each step of the algorithm we solve the boundary value problems with integral condition (15), (12) and (16), (11). For fixed $\dot{\lambda}(x)$, $\dot{\mu}(t)$, $\lambda(x)$, $\mu(t)$, we solve the Goursat problem (4), (6).

The following statement provides the conditions for the convergence of the proposed algorithm and the existence of a unique solution to problem (4)–(8).

Theorem 3 *Let:*

(1) functions $A(t,x)$, $B(t,x)$, $C(t,x)$, and $f(t,x)$ be continuous on Ω;

(2) functions $K(t,x)$ and $P(t)$, $\psi(t)$ be continuously differentiable with respect to t on Ω and $[0,T]$, respectively, and functions $M(t,x)$ and $S(x)$, $\varphi(x)$ be continuously differentiable with respect to x on Ω and $[0,\omega]$, respectively;

(3) $B_1(t) = P(t) + \int_0^a K(t,\xi)d\xi \neq 0$ for all $t \in [0,T]$, and $B_2(x) = S(x) + \int_0^b M(\tau,x)d\tau \neq 0$ for all $x \in [0,\omega]$;

(4) the following conditions be fulfilled:

$$|\tilde{a}_1| = \Big|2S(0) + \int_0^b M(\tau,0)\big[1+e^{a_1(\tau)}\big]d\tau\Big| \geq \delta_1 > 0, \qquad |\tilde{a}_2| = \Big|2P(0) + \int_0^a K(0,\xi)\big[1+e^{a_2(\xi)}\big]d\xi\Big| \geq \delta_2 > 0,$$

where $a_1(t) = -\int_0^t [B_1(\tau)]^{-1}\Big[\dot{P}(\tau) + \int_0^a \frac{\partial K(\tau,\xi)}{\partial \tau}d\xi\Big]d\tau$, $a_2(x) = -\int_0^x [B_2(\xi)]^{-1}\Big[\dot{S}(\xi) + \int_0^b \frac{\partial M(\tau,\xi)}{\partial \xi}d\tau\Big]d\xi$;

(5) the inequality $q = \max\Big(\mathcal{K}_1 l_1(a) + \mathcal{K}_2 l_2(b), (\alpha_1\mathcal{K}_1+1)l_1(a), (\alpha_2\mathcal{K}_2+1)l_2(b)\Big) < 1$ hold.

Then problem (4)–(8) has a unique solution.

Proof Let conditions (1)–(3) of Theorem 3 be fulfilled. Use the 0th step of the algorithm and consider the following boundary value problem with integral condition

$$\dot{\mu}(t) = A_1(t)\mu(t) - [B_1(t)]^{-1}\dot{\psi}(t), \qquad t \in [0,T], \tag{25}$$

$$\left[2S(0)+\int_0^b M(\tau,0)d\tau\right]\mu(0)+\int_0^b M(\tau,0)\mu(\tau)d\tau=\varphi(0). \tag{26}$$

$$\dot{\lambda}(x)=A_2(x)\lambda(x)-[B_2(x)]^{-1}\dot{\varphi}(x), \qquad x\in[0,\omega], \tag{27}$$

$$\left[2P(0)+\int_0^a K(0,\xi)d\xi\right]\lambda(0)+\int_0^a K(0,\xi)\lambda(\xi)d\xi=\psi(0). \tag{28}$$

Condition (4) including the conditions of Theorems 1 and 2 yields the unique solvability of problems (17), (18) and (19), (20). Find initial approximations $\mu^{(0)}(t)$ and $\lambda^{(0)}(x)$ from the boundary value problems (17), (18) and (19), (20). Then, similarly to the estimates (23) and (24), for the functions $\mu^{(0)}(t)$, $\lambda^{(0)}(x)$ and their derivatives $\dot{\mu}^{(0)}(t)$, $\dot{\lambda}^{(0)}(x)$ the following estimates hold:

$$\max_{t\in[0,T]}|\mu^{(0)}(t)|\le\mathcal{K}_1\max\Big(\max_{t\in[0,T]}|[B_1(t)]^{-1}\dot{\psi}(t)|,|\varphi(0)|\Big), \tag{29}$$

$$\max_{t\in[0,T]}|\dot{\mu}^{(0)}(t)|\le[\alpha_1\mathcal{K}_1+1]\max\Big(\max_{t\in[0,T]}|[B_1(t)]^{-1}\dot{\psi}(t)|,|\varphi(0)|\Big). \tag{30}$$

$$\max_{x\in[0,\omega]}|\lambda^{(0)}(x)|\le\mathcal{K}_2\max\Big(\max_{x\in[0,\omega]}|[B_2(x)]^{-1}F_1(x)|,|\psi(0)|\Big), \tag{31}$$

$$\max_{x\in[0,\omega]}|\dot{\lambda}^{(0)}(x)|\le(\alpha_2\mathcal{K}_2+1)\max\Big(\max_{x\in[0,\omega]}|[B_2(x)]^{-1}\dot{\varphi}(x)|,|\psi(0)|\Big). \tag{32}$$

Solving the Goursat problem (4)–(6) for the found values of parameters, we find $\widetilde{v}^{(0)}(t,x)$, $\widetilde{w}^{(0)}(t,x)$, $\widetilde{u}^{(0)}(t,x)$ for all $(t,x)\in\Omega$.

The following inequalities are valid:

$$|\widetilde{v}^{(0)}(t,x)|\le\max(T,\omega)e^{H(T+\omega)}\max_{(t,x)\in\Omega}|\widetilde{f}(t,x)|,$$

$$|\widetilde{w}^{(0)}(t,x)|\le\max(T,\omega)e^{H(T+\omega)}\max_{(t,x)\in\Omega}|\widetilde{f}(t,x)|,$$

$$|\widetilde{u}^{(0)}(t,x)|\le\max(T,\omega)e^{H(T+\omega)}\max_{(t,x)\in\Omega}|\widetilde{f}(t,x)|,$$

where $\widetilde{f}(t,x)=A(t,x)\dot{\lambda}^{(0)}(x)+B(t,x)\dot{\mu}^{(0)}(t)+C(t,x)\Big[\lambda^{(0)}(x)+\mu^{(0)}(t)\Big]+f(t,x)$.

Successively, we determine the functions $\mu^{(m)}(t)$, $\lambda^{(m)}(x)$, $\dot{\mu}^{(m)}(t)$, $\dot{\lambda}^{(m)}(x)$, $\widetilde{v}^{(m)}(t,x)$, $\widetilde{w}^{(m)}(t,x)$, $\widetilde{u}^{(m)}(t,x)$ on the mth step of the algorithm and obtain $\mu^{(m+1)}(t)$, $\lambda^{(m+1)}(x)$, $\dot{\mu}^{(m+1)}(t)$, $\dot{\lambda}^{(m+1)}(x)$, $\widetilde{v}^{(m+1)}(t,x)$, $\widetilde{w}^{(m+1)}(t,x)$, $\widetilde{u}^{(m+1)}(t,x)$ on the $(m+1)$th step, $m=1,2,\ldots$.

Evaluating the corresponding differences of successive approximations, we obtain

$$\max_{t\in[0,T]}|\mu^{(m+1)}(t)-\mu^{(m)}(t)|\le$$

$$\leq \mathcal{K}_1 \max_{t\in[0,T]} |[B_1(t)]^{-1}| \Big[|L_1(t, \lambda^{(m)} - \lambda^{(m-1)})| + |G_1(t, \widetilde{u}^{(m)} - \widetilde{u}^{(m-1)}, \widetilde{w}^{(m)} - \widetilde{w}^{(m-1)})| \Big], \tag{33}$$

$$\max_{x\in[0,\omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| \leq$$

$$\leq \mathcal{K}_2 \max_{x\in[0,\omega]} |[B_2(x)]^{-1}| \Big[|L_2(x, \mu^{(m)} - \mu^{(m-1)})| + |G_2(x, \widetilde{u}^{(m)} - \widetilde{u}^{(m-1)}, \widetilde{v}^{(m)} - \widetilde{v}^{(m-1)})| \Big], \tag{34}$$

$$\max_{t\in[0,T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| \leq [\alpha_1 \mathcal{K}_1 + 1] \max_{t\in[0,T]} |[B_1(t)]^{-1}| \Big[|L_1(t, \lambda^{(m)} - \lambda^{(m-1)})| +$$

$$+ |G_1(t, \widetilde{u}^{(m)} - \widetilde{u}^{(m-1)}, \widetilde{w}^{(m)} - \widetilde{w}^{(m-1)})| \Big], \tag{35}$$

$$\max_{x\in[0,\omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| \leq [\alpha_2 \mathcal{K}_2 + 1] \max_{x\in[0,\omega]} |[B_2(x)]^{-1}| \Big[|L_2(x, \mu^{(m)} - \mu^{(m-1)})| +$$

$$+ |G_2(x, \widetilde{u}^{(m)} - \widetilde{u}^{(m-1)}, \widetilde{v}^{(m)} - \widetilde{v}^{(m-1)})| \Big], \tag{36}$$

$$|\widetilde{v}^{(m+1)}(t,x) - \widetilde{v}^{(m)}(t,x)| \leq$$

$$\leq \max(T,\omega) e^{H(T+\omega)} \Big\{ \alpha \max_{x\in[0,\omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t\in[0,T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| +$$

$$+ \gamma \Big[\max_{x\in[0,\omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t\in[0,T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \Big] \Big\}, \tag{37}$$

$$|\widetilde{w}^{(m+1)}(t,x) - \widetilde{w}^{(m)}(t,x)| \leq$$

$$\leq \max(T,\omega) e^{H(T+\omega)} \Big\{ \alpha \max_{x\in[0,\omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t\in[0,T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| +$$

$$+ \gamma \Big[\max_{x\in[0,\omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t\in[0,T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \Big] \Big\}, \tag{38}$$

$$|\widetilde{u}^{(m+1)}(t,x) - \widetilde{u}^{(m)}(t,x)| \leq$$

$$\leq \max(T,\omega) e^{H(T+\omega)} \Big\{ \alpha \max_{x\in[0,\omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t\in[0,T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| +$$

$$+ \gamma \Big[\max_{x\in[0,\omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t\in[0,T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \Big] \Big\}. \tag{39}$$

Suppose

$$\Delta_{m+1} = \max\Big(\max_{x\in[0,\omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t\in[0,T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)|,$$

$$\max_{x\in[0,\omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)|, \max_{t\in[0,T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| \Big).$$

Then, from relations (33)–(36), taking into account estimations (37)–(39) and using the notation introduced, we obtain the main inequality:

$$\Delta_{m+1} \le q\Delta_m. \tag{40}$$

Condition (5) of Theorem 3 leads to the convergence of the sequence Δ_m to Δ_* as $m \to \infty$. This gives the uniform convergence of the sequences $\lambda^{(m)}(x)$, $\dot{\lambda}^{(m)}(x)$, $\mu^{(m)}(t)$, $\dot{\mu}^{(m)}(t)$ to $\lambda^*(x)$, $\dot{\lambda}^*(x)$, $\mu^*(t)$, $\dot{\mu}^*(t)$, respectively, as $m \to \infty$. The functions $\lambda^*(x)$ and $\mu^*(t)$ are continuous and continuously differentiable on $[0,\omega]$ and $[0,T]$, respectively. Based on estimates (37)–(39), we establish the uniform convergence of the sequences $\widetilde{v}^{(m)}(t,x)$, $\widetilde{w}^{(m)}(t,x)$, $\widetilde{u}^{(m)}(t,x)$ to the functions $\widetilde{v}^*(t,x)$, $\widetilde{w}^*(t,x)$, $\widetilde{u}^*(t,x)$, respectively, with respect to $(t,x) \in \Omega$. Obviously, the functions $\widetilde{u}^*(t,x)$, $\widetilde{v}^*(t,x)$, and $\widetilde{w}^*(t,x)$ are continuous on Ω. Solving the problems on the $(m+1)$th step of the algorithm and passing to the limit as $m \to \infty$, we obtain that the functions $\widetilde{u}^*(t,x), \lambda^*(x)$, $\mu^*(t)$ together with their derivatives satisfy the Goursat problem (4)–(6) and boundary value problems with integral condition (15), (12) and (16), (11).

Carry out the inverse transition from problem (15), (12) to relation (7), and pass from problem (16), (11) to relation (8). Then the triple of functions $(\widetilde{u}^*(t,x), \lambda^*(x), \mu^*(t))$ is a solution to problem (4)–(8).

Prove the uniqueness of a solution to problem (4)–(8). Let the function triples $(\widetilde{u}^*(t,x), \lambda^*(x), \mu^*(t))$ and $(\widetilde{u}^{**}(t,x), \lambda^{**}(x), \mu^{**}(t))$ be two solutions to the problem. We introduce the following notation:

$$\widetilde{\Delta} = \max\Big(\max_{x\in[0,\omega]} |\lambda^*(x) - \lambda^{**}(x)| + \max_{t\in[0,T]} |\mu^*(t) - \mu^{**}(t)|,$$

$$\max_{x\in[0,\omega]} |\dot{\lambda}^*(x) - \dot{\lambda}^{**}(x)|, \max_{t\in[0,T]} |\dot{\mu}^*(t) - \dot{\mu}^{**}(t)|\Big).$$

After calculation, analogically with (34)–(39), we get

$$\widetilde{\Delta} \le q\widetilde{\Delta}. \tag{41}$$

By condition (5) of Theorem 3, we have $q < 1$. Then inequality (41) takes place only for $\widetilde{\Delta} \equiv 0$, which implies $\lambda^*(x) = \lambda^{**}(x)$, $\mu^*(t) = \mu^{**}(t)$, and $\widetilde{u}^*(t,x) = \widetilde{u}^{**}(t,x)$. Therefore, the solution to problem (4)–(8) is unique. Theorem 3 is proved.

□

From equivalence of problem (1)–(3) to problem (4)–(8), the next assertion follows.

Theorem 4 *Let conditions (1)–(5) of Theorem 3 be fulfilled.*

Then problem (1)–(3) has a unique classical solution.

Proof Conditions (1)–(5) of Theorem 3 imply the existence of a unique solution $(\widetilde{u}^*(t,x), \lambda^*(x), \mu^*(t))$ to problem (4)–(8). According to the algorithm offered above, for each $m = 0, 1, 2, ...$, this triple is determined as the limit of the triples sequence $(\widetilde{u}^{(m)}(t,x),\ \mu^{(m)}(t)),\ \lambda^{(m)}(x))$ as $m \to \infty$.

Then the solution to problem (1)–(3), the function $u^*(t,x)$, exists and is determined by the equality

$$u^*(t,x) = \widetilde{u}^*(t,x) + \lambda^*(x) + \mu^*(t).$$

Theorem 4 is proved.

□

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